Lai Chapter 8: Non-Newtonian Fluids PhL 9.23.12

PART A: LINEAR VISCOELASTIC FLUID (444) 2

8.1 Linear Maxwell Fluid (444) 2

8.2 Fluid with multiple components (450). [generalized linear Maxwell discrete] 7

8.3 Green's Function solution to our ODE's (451) 9

8.4 Continuous spectrum version (450). 11

8.5 Computation of H(λ) from φ from G* (454). 12

PART B: NONLINEAR VISCOELASTIC FLUID (456) 13

8.6 Current configuration as the reference configuration (456) 13

8.7 The Relative Deformation Gradient (457) 15

8.8 The Relative "other tensors" (459) 15

8.9 The Relative C tensor in various coordinates (460) 16

8.10 "Deformation History" and the Rivlin-Ericksen Tensors (463) 16

8.11 Relating A_1 to D; a recursion formula for the A_N (468) 17

8.12 Some tensor relations involving F_t and (∇v) and the polar decomposition (471) 20

8.13 Change of Frame Stuff (471) 23

8.14 Change of Frame for the A tensors (474) 23

8.15 Incompressible Simple Fluid (474) 23

8.16 Two specific models which reduce to Maxwell for small deformation (475) 24

8.17 Single-integral non-linear constitutive equations (478) 26

8.18 Differential-type constitutive equations for incompressible fluid (481) 28

8.19 Objective Rate of Stress (483) 29

8.20 Rate-type constitutive equations (487) 30

PART C: VISCOMETRIC FLOW FOR INCOMPRESSIBLE SIMPLE FLUID (491) 30

8.21 Viscometric Flow (491) 30

8.22 Form of T_{ij} for Viscometric Flow (493) 31

8.23 Channel Flow (495) 31

8.24 Couette Flow (497) 32

Appendix 8.1. 33

Water does behave as a Newtonian fluid as discussed in Chapter 6, but paint and polymer fluids don't. In this chapter, which has Part A,B,C, we are going to examine other Fluid Models.

In Ch 6 we found this rule for an isotropic Newtonian fluid

$$T_{ij} = -p\,\delta_{ij} + T'_{ij} \qquad T'_{ij} = \lambda\,\Delta\,\delta_{ij} + 2\mu D_{ij} \qquad \Delta = \mathrm{tr}(D) \qquad D = (\partial_i v_j + \partial_j v_i)/2$$

where μ was the constant viscosity. For incompressible,

$$\text{tr}(D) = \partial_i v_i = \text{div}(\mathbf{v}) = 0 \iff d\rho/dt = 0,$$

so the incompressible equation is simpler; D is the "rate of deformation tensor" :

$$T_{ij} = -p\,\delta_{ij} + T'_{ij} \qquad T'_{ij} = 2\mu D_{ij} \qquad D = (\partial_i v_j + \partial_j v_i)/2$$

Part A: Linear ViscoElastic Fluid (444)

8.1 Linear Maxwell Fluid (444)

Now suddenly, without any comment, the piece T' is renamed S, the "extra stress" so for a Newtonian fluid we would say

$$T_{ij} = -p\,\delta_{ij} + S_{ij} \qquad S_{ij} = 2\mu D_{ij} \qquad D = (\partial_i v_j + \partial_j v_i)/2$$

Finally, then, we are able to see the new feature involved with the "linear Maxwell fluid"

$$T_{ij} = -p\,\delta_{ij} + S_{ij} \qquad S_{ij} = 2\mu D_{ij} - \lambda \partial S_{ij}/\partial t \qquad D = (\partial_i v_j + \partial_j v_i)/2$$

where we now have some "drag" or "velocity dependent force" element added in, though it is not yet clear why that name is appropriate. First, recall that any equation like F = -kx which relates "stress" to "strain" is called a constitutive relation/equation, and this describes the nature of the medium and is separate from the equations of motion and the various other "conservation of" principles.

I think this is a "linear" Maxwell fluid because the term $\partial S_{ij}/\partial t = \partial_t S_{ij}$ is linear, and we end up with a system of linear ODE's.

Why Maxwell? I just read the entire wiki site on Maxwell, Scottish. He did so much stuff that this little fluid thing is not even mentioned: E&M, thermo, kinetic theory of gases, color photos, rings of Saturn as particles, on and on. But in 1867 Maxwell managed to work in the above little model for visco-elastic materials. Kelvin was a contemporary and did similar stuff.

Example 8.1.1.

As an elementary model of this Maxwell fluid, we study the "spring plus dashpot" situation of 8.1.3. You apply some force Lai calls S on the left. Each of the two series elements has its own displacement which here are called ε_e (spring, elastic) and ε_η (viscous), and the total displacement is $\varepsilon = \varepsilon_e + \varepsilon_\eta$. Like components in an electric circuit, each element here has a governing equation:

$S = G\varepsilon_e$ which says $F = kx$ // G and η are assumed constants

$S = \eta\, d\varepsilon_\eta/dt$ which says dashpot represents a frictional force proportional to velocity

So G is the spring constant, and η is the viscosity (the dashpot drag coefficient F = η * velocity)

The two S's are the same because these circuit elements are in series (see comment below!). If we then differentiate the geometric fact that $\varepsilon = \varepsilon_e + \varepsilon_\eta$ we get,

$$d\varepsilon/dt = d\varepsilon_e/dt + d\varepsilon_\eta/dt = (1/G)\, dS/dt + (1/\eta)\, S$$

or

$$S + (\eta/G)dS/dt = \eta\, (d\varepsilon/dt) \qquad \text{//which is 8.1.6}$$

Remember that S = force ~ "stress". If we define $\lambda = (\eta/G)$ = drag/spring, this equation says

$$S + \lambda\, dS/dt = \eta\, (d\varepsilon/dt) \qquad \text{// } \lambda \text{ has dimensions of time} \qquad \text{// 8.1.6}$$

which we compare to our Maxwell fluid model for the extra stress tensor S

$$S_{ij} + \lambda\, (\partial S_{ij}/\partial t) = 2\mu D_{ij} = \mu\, (\partial_i v_j + \partial_j v_i) \qquad \text{// partial derivative on S !!}$$

In this fairly close analogy, we can see that

$\mu \sim \eta$ = viscosity
$\lambda = (\eta/G)$ = (viscosity/elasticity) ratio = relaxation time

Somehow the RHS drag term $\eta\, (d\varepsilon/dt)$ gets replaced by $\mu\, (\partial_i v_j + \partial_j v_i)$. In a simple layer situation, adjacent layers have a relative velocity between them which causes drag: $\Delta v \sim (\partial v_x/\partial y)\, \Delta y$, so we expect the drag force to be $\eta\, \Delta v = \eta\, (\partial v_x/\partial y)\, \Delta y$. So when you make the adjustment to "per unit area" and all that, you end up with a RHS as shown. So for S we end up with a first order matrix ODE which is "driven" by a RHS inhomogeneous term which is the drag. But the dS/dt term feels the effect of both drag and elasticity through the coefficient λ.

Comment : The RHS of 8.1.6 has sort of η * velocity which we know is the usual frictional force, so the role of the $\eta\, (d\varepsilon/dt)$ term in 8.1.6 seems very clear (think S = force), and this is the viscous effect. But where is the "spring effect" coming from? If the dashpot is very viscous, $\eta >> 1$, then in effect the entire system is just the spring. In this limit 8.1.6 says $(\eta/G)dS/dt = \eta\, (d\varepsilon/dt)$ which you can integrate in effect to get $S = G\, \Delta\varepsilon$ and that is your $F = k\Delta x$, and there is your spring effect. But in general the two effects are very well mixed together, and the term viscoelastic fluid seems very appropriate.

Comment on massless spring. If you apply F_1 and F_2 to the ends of a spring, it stretches $\Delta x = (F_1+F_2)/k$ and it does this instantly. Both forces might be time dependent, so $\Delta x(t) = [F_1(t)+F_2(t)]/k$ and things are still instantaneous. If the spring had distributed mass, you might have some inertial m**a** force at different points along the spring, but it is massless so inertial force is zero, even if the spring is in the act of stretching. Therefore, at any instant of time you have $F_1(t) = -F_2(t)$. If you have both forces pulling to the left, then we negate F_2 and we then have $F_1(t) = F_2(t)$. What this says is that the spring "transmits force through itself instantly. This then means that, even in a dynamic situation, both the spring and the dashpot in our little model experience the same force.

Comment on electrical analogy: In this series circuit problem, the two voltages which you add are the two displacements, so you then have $\varepsilon = \varepsilon_e + \varepsilon_\eta$. Since the spring transmits force through itself instantly, the force S across the spring is the same as the force S across the dashpot, so S is *like* the current in a series circuit. Then Ohm's Law for the spring says V = IR or $\varepsilon_e = S\ (1/G)$ and the resistance is R = 1/G = 1/k, so G is more like a conductance. The dashpot is like a capacitor Q = CV and I = C dV/dt so get $S = \eta\ d\varepsilon_\eta/dt$ and then you can identify the capacitance with η. A stiff dashpot means large C = η, so it is hard to change the voltage across a large capacitor, meaning here that ε_η does not change, and you have just the spring. A stiff spring on the other hand means large G = k and small R and then ε_e does not change and then you have just a dashpot.

Now we are going to set up some "experiments" similar to those at the start of the elastic solid chapter which serve to clarify the meaning of these model constants. We first apply these experiments to our spring and dashpot, then later we will show how they apply to stuff like S and D for the medium.

Creep Experiment. Just apply constant force S_0 starting at t = 0 and what happens? From 8.1.6

$$S_0 = \eta\ (d\varepsilon/dt) \qquad \Rightarrow \qquad \varepsilon(t) = (S_0/\eta)\ t + \varepsilon(0)$$

At t = 0 when S_0 is applied, the spring element instantly stretches so $\varepsilon_o = S_0/G$, whereas the dashpot just sits there with $\varepsilon_\eta = 0$. How do we know these things. The first is from F = kx. The second is from

$$S_0 = \eta\ d\varepsilon_\eta/dt \qquad \Rightarrow \qquad \varepsilon_\eta(t) = (S_0/\eta)\ t \quad \text{which is 0 at } t = 0+.$$

So it takes the dashpot "time" to do something (capacitor), and at t = 0+ we have $\varepsilon_\eta(0+) = 0$. Therefore, only the spring contributes to ε(0) and we get

$$\varepsilon(t) = (S_0/\eta)\ t + S_0/G$$

So from such a creep experiment, we could in theory measure G and η for our Maxwell fluid: G coming from the jump just after t = 0, and η from the rate of movement. The spring here instantly reaches its amount of stretch $\varepsilon_e = S/G$ and that then never changes. The dashpot then just keeps moving at a constant velocity and ε grows linearly. The dashpot creeps along at constant $v = (S_0/\eta)$. Notice that this creep experiment does not tell you the value of μ.

Relaxation Experiment. Apply an impulse displacement ε_0 at time 0 and watch what happens. The spring uses up all this displacement, so we get $\varepsilon_e(t=0+) = \varepsilon_0$ and $\varepsilon_\eta(t=0+) = 0$, because the dashpot has no time to get rolling. Then $S(0+) = G\varepsilon_0$ and we have to solve our ODE which says ($\varepsilon = \varepsilon_0 =$ constant)

$$S + \lambda\ \partial S/\partial t = 0 \qquad S(0+) = G\varepsilon_0 \qquad \Rightarrow \qquad S(t) = (G\varepsilon_0)\ e^{-t/\lambda}$$

$\varepsilon_e(t) = S/G = \varepsilon_o\ e^{-t/\lambda}$ starts at ε_0 and decays to no stretch

$\varepsilon_\eta(t) = \varepsilon - \varepsilon_e = \varepsilon_0[\ 1 - e^{-t/\lambda}]$ starts at 0 and eventually uses all the ε_0

So this gives some intuitive feel for meaning of parameter λ. But we have NOT yet connected these experiments to experiments we might do with fluids. Hopefully the examples will do that. Notice that this experiment lets you measure only parameter λ.

Key idea: the λ parameter which is the elasticity/viscosity ratio has units of time and should be thought of as a characteristic time for the material, like τ = RC for a resistor capacitor circuit. I could imagine another book calling this τ, or some other symbol that more looks like a time. (RC = (1/G)η = η/G = λ).

Example 8.1.2. Simple shear flow of a Maxwell fluid (steady state flow). The stress, whatever it is, should be constant, so $\partial S/\partial t = 0$, so then $S_{ij} = 2\mu D_{ij}$ or $S = 2\mu D$. Same as Newtonian fluid.

Implication: You can measure parameter μ, the viscosity of a fancy Maxwell fluid, by doing some static flow experiment, just as you would do with a Newtonian fluid. For example you could set up a simple layer flow between two parallel plates and then measure the force of the liquid on either plate. Lai has failed to make this point clearly. So our "new" task for a Maxwell fluid is to find an experiment by which we can measure that ratio parameter λ. ($S_{ij} + \lambda\, \partial S_{ij}/\partial t = 2\mu D_{ij}$), and of course this has to be some kind of non-static experiment. You might say that μ is the "static flow viscosity". We have seen how to measure λ in the model, but we need soon to do it with the real fluid.

Example 8.1.3. Here we shall get the relaxation idea going with a fluid. Assume a step function here

$$u_1 = \varepsilon_o H(t)\, x_2$$

so this is a transverse (shear as in half river)) step in displacement in a 3D liquid, so we are talking shear. This is like the relaxation experiment above. Maybe you could physically do this with a fluid between two plates and you suddenly slide one of the plates so $u_1 = \varepsilon_o h$ at the upper moved plate. Probably this creates the above displacement pattern in the fluid at time t=0+.

We can compute D_{ij} by first computing $v_1 = \varepsilon_0 x_2 \delta(t)$ and then $D_{12} = (\partial v_1/\partial x_2)/2 = \varepsilon_o \delta(t)/2$ and then we have (notice that D_{12} is not a function of transverse x_2 and is thus the same at all x_2)

$$S_{12} + \lambda\, \partial S_{12}/\partial t = 2\mu D_{12} = 2\mu\varepsilon_o\, \delta(t)/2 = \mu\varepsilon_o\, \delta(t)$$

Integrate in time range (-e,e) where e is small to get

$$2eS_{12} + \lambda\, [S_{12}(e) - S_{12}(-e)] = \mu\varepsilon_0$$

or

$$\lambda\, [S_{12}(0+) - S_{12}(0-)] = \mu\varepsilon_0$$

But before our step application nothing was happening so there was not strain, so $S_{12}(0-) = 0$ and then

$$S_{12}(0+) = \mu\varepsilon_0/\lambda$$

Then for t > 0 we solve

$S_{12} + \lambda\, dS_{12}/dt = \mu\varepsilon_o\, \delta(t) = 0 \qquad \Rightarrow \qquad S_{12}(t) = S_{12}(0+)e^{-t/\lambda} = (\mu\varepsilon_0/\lambda)\, e^{-t/\lambda}$

and this shows that the shear strain S_{12} (same at all x_2) jumps to a constant value then decays away, just like the dashpot spring system where recall $S(t) = (G\varepsilon_0)\, e^{-t/\lambda}$. So the analogy says $G = \mu/\lambda$ = the springiness of the medium, since G was the spring constant. This is our first interpretation of the μ and λ parameters in the actual medium model. This is the "relaxation experiment" done for the actual medium. The stress S_{12} starts out and then decays away as the dashpot element flows. Measure initial S_{12} and that is a measure of μ/λ. Then measure decay rate and that tells you λ and then you can have μ as well.

Example 8.1.4. (448) Two infinite plates separated by h. The lower plate is fixed, the upper one does shear oscillation, so we have various BC's on the **u** displacement (no slip condition) as shown in (i) and (ii). We try out $u_x(y,t) = u_0(y/h)e^{i\omega t}$ as a solution everywhere between the plates. Then $v_x = i\omega\, u_0(y/h)e^{i\omega t}$ and then $D_{12} = (1/2)\partial v_x/\partial y = i\omega\, (u_0/2h)e^{i\omega t}$. This rate of deformation thing is independent of y, it is the same at every level. Next, we try out $S_{12} = S_0 e^{i\omega t}$ as a solution everywhere in the gap, also independent of y. We plug this form into $S_{12} + \lambda\, dS_{12}/dt = 2\mu D_{12}$ and we find that it works just fine as long as we select S_0 properly:

$S_0 = (u_0/h)G^*$ where G^* = as shown 8.123 through 8.125

so the solution is then $S_{12}(t) = (u_0/h)G^*e^{i\omega t}$ and G^* is complex indicating phase stuff. Eventually we take the real part and get

$S_{12}(t) = (u_0/h)|G^*| \cos(\omega t+\delta) \qquad \tan\delta = 1/(\lambda\omega) \qquad \lambda = \text{drag/spring}$

Not surprisingly , since D_{12} was the same at every level (indep of y), so is S_{12}.

So what do we learn from this example? We learn that the shear "extra stress" S_{12} oscillates along with that top plate, and is the same at all levels y, but has a certain phase δ relative to the driving oscillation displacement u. If we remember that λ is the relaxation time, we see that there will be no phase if $\lambda = 0$ (no drag), but as λ increases, we have δ increasing at a given ω and reaching a max of $\pi/2$ when $\lambda = \infty$ (no spring). The phase delay is the same at all y. So really this example is an experiment to measure the λ parameter.

Note added: Let's examine this plane oscillation experiment again. We assume that the shear displacement u_x at the driving plate has zero phase in (i), and then we find that the shear stress is $S_{12} = G^*\, (u_0/h)$ in 8.1.22. In practice, you drive the plate with S_{12} at zero phase, and then the displacement will be out of phase, think either way. But in practice that second way allows you to measure the displacement including its phase. The model says that

$u_o = h\, (S_0/G^*)$ that is u_o is the shear displacement just below the driving plate

If G^* is real, then displacement and force are in phase, your medium just vibrates like a purely elastic substance work goes into stored spring energy. But if G^* is imaginary, then $\pi/2$ phase shift, and dashpot dominates things and you have a heat loss going on. So the real thing G' is called **storage modulus** since

energy is stored in the springs, while G" is called the **loss modulus**. My main point is that this kind of experiment let's you measure G*, and later Lai will assume that one knows G* experimentally and wants to do other things with that data.

8.2 Fluid with multiple components (450). [generalized linear Maxwell discrete]

Something like paint is a mixture of fluids, so we need to get that into the model. So imagine that n = 1 to N lists off different "species" of materials in the fluid. This mix idea is called "generalized".

$$S_n + \lambda_n \partial_t S_n = 2\mu_n D \qquad\qquad S = \Sigma_{n=1}^{N} S_n$$

So here each species has its own private "extra stress" tensor S_n. The claim is that you have N dashpots in parallel, each one with force S_n and total force S. Each has its own spring and viscosity constant. But all the dashpots see the same total fluid velocity **v** so they have the same D_{ij} tensor. Lai and I make no pretense toward explaining exactly how the mixture behaves like parallel dashpots. OK for now.

I verify below that (8.2.3) is valid for any N mixture. I know now that the a_n and b_n exist because I know exactly how to compute them.

Digression: Maple Lai_p450.mws. Now I am going to solve the problem he poses in the text and solve it with Maple. If we apply ∂_t multiple times to the above equation, we have a whole set of equations of the form

$$\partial_t^i S_n = \lambda_n \partial_t^{i+1} S_n = 2\mu_n \partial_t^i D \qquad\qquad i = 0,1,2...$$

This is a recursion relation and on scratch I solved it to find this solution:

$$\partial_t^i S_n = (2\mu_n) \Sigma_{k=1}^{i} (-1)^{k-1} c_n^k (\partial_t^{i-k} D) + (-1)^i c_n^i S_n \qquad\qquad c_n \equiv 1/\lambda_n$$

which expresses any derivative of S_n in terms of S_n and derivatives of D.

In Maple, I make these definitions

DS(i,n) ≡ $\partial_t^i S_n$
D[r] ≡ $\partial_t^r D$
S[n] = S_n

Then the above equation reads

$$DS(i,n) = (2\mu_n) \Sigma_{k=1}^{i} (-1)^{k-1} c_n^k D[i-k] + (-1)^i c_n^i S[n] \qquad\qquad c_n \equiv 1/\lambda_n$$

```
DS := (i,n) -> 2*u[n]*sum((-1)^(k-1)*c[n]^k * D[i-k] ,k=1..i)+ (-1)^i * S[n]* c[n]^i;
```

$$DS := (i,n) \to 2\,u_n \left(\sum_{k=1}^{i} (-1)^{(k-1)}\, c_n{}^k\, D_{i-k} \right) + (-1)^i\, S_n\, c_n{}^i$$

Our task is to find coefficients a_n and b_n such that (8.2.3) is valid for some given N.

$$\Sigma_{r=1}^{N} S_r + \Sigma_{n=1}^{N} a_n \left[\Sigma_{r=1}^{N} \partial_t^n S_r\right] = \Sigma_{N=0}^{N-1} b_n \partial_t^n D$$

or

$$\Sigma_{r=1}^{N} S_r + \Sigma_{r=1}^{N} \Sigma_{n=1}^{N} a_n \, DS(n,r) = \Sigma_{N=0}^{N-1} b_n D[n]$$

or

$$\Sigma_{r=1}^{N} \left[S_r + \Sigma_{n=1}^{N} a_n \, DS(n,r) \right] = \Sigma_{N=0}^{N-1} b_n D[n]$$

```
LHS := sum(S[r] + sum(a[n]*DS(n,r),n=1..N), r=1..N);
```

$$LHS := \sum_{r=1}^{N} \left(S_r + \left(\sum_{n=1}^{N} a_n \, DS(n,r) \right) \right)$$

```
RHS := sum(b[n]*D[n], n=0..N-1);
```

$$RHS := \sum_{n=0}^{N-1} b_n D_n$$

```
Q := LHS - RHS;
```

The problem is to find a_n and b_n such that Q = 0. For N=3 we get for Q this expression

$$\begin{aligned} Q := {} & S_1 + a_1 (2 u_1 c_1 D_0 - S_1 c_1) + a_2 (2 u_1 (c_1 D_1 - c_1^2 D_0) + S_1 c_1^2) + a_3 (2 u_1 (c_1 D_2 - c_1^2 D_1 + c_1^3 D_0) - S_1 c_1^3) + S_2 \\ & + a_1 (2 u_2 c_2 D_0 - S_2 c_2) + a_2 (2 u_2 (c_2 D_1 - c_2^2 D_0) + S_2 c_2^2) + a_3 (2 u_2 (c_2 D_2 - c_2^2 D_1 + c_2^3 D_0) - S_2 c_2^3) + S_3 \\ & + a_1 (2 u_3 c_3 D_0 - S_3 c_3) + a_2 (2 u_3 (c_3 D_1 - c_3^2 D_0) + S_3 c_3^2) + a_3 (2 u_3 (c_3 D_2 - c_3^2 D_1 + c_3^3 D_0) - S_3 c_3^3) - b_0 D_0 - b_1 D_1 - b_2 D_2 \end{aligned}$$

which you see has terms for S_1, S_2, S_3 and for D_0, D_1, D_2. Since these are all independent functions, their coefficients have to separately be zero. So then we can get the coefficients this way:

```
for m from 0 to N-1 do coeff(Q,D[m],1) od;
```

$$2 a_1 u_1 c_1 - 2 a_2 u_1 c_1^2 + 2 a_3 u_1 c_1^3 + 2 a_1 u_2 c_2 - 2 a_2 u_2 c_2^2 + 2 a_3 u_2 c_2^3 + 2 a_1 u_3 c_3 - 2 a_2 u_3 c_3^2 + 2 a_3 u_3 c_3^3 - b_0$$

$$2 a_2 u_1 c_1 - 2 a_3 u_1 c_1^2 + 2 a_2 u_2 c_2 - 2 a_3 u_2 c_2^2 + 2 a_2 u_3 c_3 - 2 a_3 u_3 c_3^2 - b_1$$

$$2 a_3 u_1 c_1 + 2 a_3 u_2 c_2 + 2 a_3 u_3 c_3 - b_2$$

```
for m from 1 to N do eq[m] := coeff(Q,S[m],1)=0 od;
```

$$eq_1 := 1 - a_1 c_1 + a_2 c_1^2 - a_3 c_1^3 = 0$$

$$eq_2 := 1 - a_1 c_2 + a_2 c_2^2 - a_3 c_2^3 = 0$$

$$eq_3 := 1 - a_1 c_3 + a_2 c_3^2 - a_3 c_3^3 = 0$$

You can see that the S_n coefficients define an NxN linear problem which can be solved for the a_n. Then those a_n are inserted into the first set of equations to get the b_n. I first solve the NxN problem this way:

```
A := solve({seq(eq[m],m=1..N)},{seq(a[m],m=1..N)});
```

$$A := \{a_3 = \frac{1}{c_2 c_1 c_3}, a_2 = \frac{c_2 + c_3 + c_1}{c_1 c_3 c_2}, a_1 = \frac{c_2 c_1 + c_3 c_2 + c_3 c_1}{c_2 c_1 c_3}\}$$

```
assign(A);
```

where the assign makes the a_n accessible in Maple. I then obtain the b_n values as B[n] this way

```
for m from 0 to N-1 do B[m] := coeff(Q,D[m],1)+b[m]; od:
```

where you can see that I cancel off the b_n's appearing in the first set of equations above. Then I print out the solution a_n and b_n

```
for m from 1 to N do c[m] := 1/lambda[m]; od:
for m from 1 to N do simplify(a[m]) od;
```

$$\lambda_3+\lambda_1+\lambda_2$$

$$\lambda_3\lambda_1+\lambda_2\lambda_1+\lambda_3\lambda_2$$

$$\lambda_2\lambda_1\lambda_3$$

```
for m from 0 to N-1 do simplify(B[m]) od;
```

$$2u_1+2u_2+2u_3$$

$$2\lambda_3u_1+2\lambda_2u_1+2\lambda_1u_2+2\lambda_3u_2+2\lambda_1u_3+2\lambda_2u_3$$

$$2\lambda_2\lambda_3u_1+2\lambda_1\lambda_3u_2+2\lambda_2\lambda_1u_3$$

and this agrees with the result quoted on p 451. Here are my results for N = 2,

```
for m from 1 to N do simplify(a[m]) od;
```

$$\lambda_2+\lambda_1$$

$$\lambda_1\lambda_2$$

```
for m from 0 to N-1 do simplify(B[m]) od;
```

$$2u_1+2u_2$$

$$2\lambda_2u_1+2\lambda_1u_2$$

which agrees with (8.26). Just for fun here are the N=4 results

```
for m from 1 to N do simplify(a[m]) od;
```

$$\lambda_4+\lambda_3+\lambda_2+\lambda_1$$

$$\lambda_4\lambda_2+\lambda_3\lambda_2+\lambda_2\lambda_1+\lambda_4\lambda_3+\lambda_4\lambda_1+\lambda_3\lambda_1$$

$$\lambda_2\lambda_4\lambda_3+\lambda_1\lambda_4\lambda_3+\lambda_4\lambda_2\lambda_1+\lambda_3\lambda_2\lambda_1$$

$$\lambda_3\lambda_2\lambda_1\lambda_4$$

```
for m from 0 to N-1 do simplify(B[m]) od;
```

$$2u_2+2u_3+2u_4+2u_1$$

$$2\lambda_4u_2+2\lambda_3u_2+2\lambda_1u_2+2\lambda_2u_3+2\lambda_4u_3+2\lambda_1u_3+2\lambda_2u_4+2\lambda_3u_4+2\lambda_1u_4+2\lambda_2u_1+2\lambda_4u_1+2\lambda_3u_1$$

$$2\lambda_4\lambda_3u_1+2\lambda_4\lambda_2u_1+2\lambda_2\lambda_3u_1+2\lambda_4\lambda_3u_2+2\lambda_4\lambda_1u_2+2\lambda_1\lambda_3u_2+2\lambda_4\lambda_2u_3+2\lambda_4\lambda_1u_3+2\lambda_1\lambda_2u_3+2\lambda_2\lambda_3u_4+2\lambda_1\lambda_3u_4+2\lambda_1\lambda_2u_4$$

$$2\lambda_4\lambda_2\lambda_3u_1+2\lambda_4\lambda_1\lambda_3u_2+2\lambda_4\lambda_1\lambda_2u_3+2\lambda_1\lambda_2\lambda_3u_4$$

8.3 Green's Function solution to our ODE's (451)

Now forget the mixture stuff of the last section, and go back to the original single-component ODE which is the constitutive relation for a linear Maxwell fluid:

$$S + \lambda \partial_t S = 2\mu D$$

We are now going to solve this to get S as a function of D using the Green's Function method which I have spent a year studying lately! The associated Green's Function problem is this

$$(1 + \lambda \partial_t) G(t,0) = \delta(t-0)$$

Claim that the solution is

$$G(t,0) = (1/\lambda)\, e^{-t/\lambda}\, H(t-0)$$

so

$$G(t,t') = (1/\lambda)\, e^{-(t-t')/\lambda}\, H(t-t') \qquad \text{// the propagator}$$

Proof:

$$\partial_t G(t,0) = (-1/\lambda)\, G + (1/\lambda)\, e^{-t/\lambda}\, \delta(t-0) = -G/\lambda + \delta(t)/\lambda$$

Then

$$(1 + \lambda \partial_t) G(t,0) = G + \lambda\, [-G/\lambda + \delta(t)/\lambda] = \delta(t) \qquad \text{so yes, is solution}$$

Then the inhomo solution is then given by integrating the Green's against the RHS,

$$S(t) = \int_{-\infty}^{\infty} dt'\, G(t,t')\, [2\mu D(t')] = \int_{-\infty}^{\infty} dt'\, [(1/\lambda)\, e^{-(t-t')/\lambda}\, H(t-t')]\, [2\mu D(t')]$$

$$= \int_{-\infty}^{t} dt'\, [(1/\lambda)\, e^{-(t-t')/\lambda}\, [2\mu D(t')] = (2\mu/\lambda) \int_{-\infty}^{t} dt'\, e^{-(t-t')/\lambda}\, D(t')$$

$$= 2 \int_{-\infty}^{t} dt'\, [\, (\mu/\lambda) e^{-(t-t')/\lambda}]\, D(t') = 2 \int_{-\infty}^{t} dt'\, [\, \varphi(t-t')]\, D(t') = (8.3.1)$$

This is the solution !!

I also did this the other way Lai shows, as a sort of verification. (ignoring BC's !)

- SO, this looks more like a standard-issue "constitutive relation" where stress S is on the left side and some function of strain/deformation is on the right side (in this case, rate of deformation D). Written in this integral form, we don't have a $\partial_t S$ term sitting in a constitutive relation.

- the "coefficient" that you might find in a usual stress/strain constitutive relation is now that function φ, and you see that, unlike the simpler cases, the stress at current time t gets a contribution from D at each point in the past time (history), and the coefficient is then a function of both times as t-t'. This coefficient

φ is called the **stress relaxation function**. The larger the difference in the times t - t', the less influence of a past point in time. Each past point in time "emits" a propagator to the present, and older propagators die out while more recent once are still active. So φ is a history weight function, and it could have been called a memory function for that reason, but that term is saved for below when E enters the scene.

Now having done all this for a one-component fluid, we add back in the "mixture" idea. We have,

$$S_n + \lambda_n \partial_t S_n = 2\mu_n D$$

Then solution is

$$S_n = 2 \int_{-\infty}^{t} dt' \, [(\mu_n/\lambda_n) e^{-(t-t')/\lambda_n}] \, D(t')$$

and then sum to get

$$S = 2 \int_{-\infty}^{t} dt' \, \Sigma_n [(\mu_n/\lambda_n) e^{-(t-t')/\lambda_n}] \, D(t') = 2 \int_{-\infty}^{t} dt' \, \Sigma_n [\varphi_n (t-t')] \, D(t') \qquad (*)$$

with φ(t) as shown in 8.3.4. So now φ has a contribution for each component of the mixture.

8.4 Continuous spectrum version (450).

In the discrete mix problem we had

$$\varphi(t) = \Sigma_{n=1}^{N} (\mu_n/\lambda_n) \, e^{-t/\lambda_n}$$ = the "stress relaxation function"

Looking just above, you see that φ(t-t') is a convolution form propagator! The driver is the rate of deformation tensor D, and the output is the stress S. Equation (*) is similar to earlier constitutive relations, but here we are not relating stress to deformation, but rather to *rate of* deformation! The propagator is obtained in the mixed medium by simple addition over the components. I think Stakgold would refer to this propagator as a "fundy solution". This is always the Green's function with no boundary conditions, the free-space propagator. Stak never did this with such a simple differential equation.

Imagine a large number of components in the mixture so this becomes a continuum, and then the propagator is an integral. Each component in the mix has some value of λ.

$$\varphi(t) = \int_0^{\infty} [\mu(\lambda)/\lambda] \, e^{-t/\lambda} d\lambda$$ μ(λ) he calls H(λ) [H not Heaviside as in 8.1.17]

Seems a bad choice for μ(λ) but OK. Notice that λ must have dimensions of time! We need to know the ratio (μ_n/λ_n) for each component, and this is H(λ)/λ, the "relaxation spectrum".

One idea now is to assume small displacement so you can replace D with ∂E/∂t as page 205. D is the rate of deformation tensor, E is the infinitesimal strain tensor that we used a lot in elastic solids work. If we replace D in this way, we get various results quote p 452-453. We start with $S = 2 \int \varphi \, D$ were φ is a

convolution kernel. Parts gets this to be $S = -2 \int ds\, E\, \varphi'$ where now E is the kernel and $\varphi' = f$. Remember that φ is called the relaxation function, and its derivative f is called the memory function. I derived everything.

Comment: Look at this form

$$S_{ij}(t) = 2 \int_{-\infty}^{t} \varphi'(t-t')\, E_{ij}(t')dt' \qquad f(t-t') \equiv \varphi'(t-t') \quad = \textbf{memory function}$$

What you see here is that the stress at current time t is determined by the strain E at all earlier times! This is just coming from the infinitesimal relation $D = \partial_t E$ and parts integration. The weight ascribed to various times in the history is provided by $f = \varphi'$. The current stress "remembers" the entire history of the strain E, and the weight function as a function of historical time t is called "the memory function". It all makes perfect sense.

Dimensions of things: $D = v/L = sec^{-1}$; S = stress = force/area = ma/area = $kg/(m\text{-}sec^2) = nt/m^2$.
μD = stress = nt/m^2. Look at p 452 to see that $S = \varphi\, D\, dt = \varphi$, so φ must also be nt/m^2. Then 8.4.1 says that $\varphi = H/\lambda\, e^{-t/\lambda} d\lambda = H$ in units, so H is also stress. μ = stress/D = stress * time. = nt/m^2 * sec so I think $Hd\lambda$ has the same units as μ which is stress*time. G* from 8.1.21 is stress as well, which is also clear from 8.1.28, so here is a little summary of units:

$S = \mu D = H = \varphi = G^*$ = stress units = nt/m^2 $\qquad \lambda$ = sec $\qquad D = sec^{-1} \qquad \mu$ = stress * sec

Example 8.4.1. Here we are back to the oscillating plate, but now instead of using the ODE 8.1.2 to get S, we use 8.4.12's integral form instead. Either way D_{12} is exactly the same. We now find that G* is given as the integral 8.4.13 in terms of the stress relaxation function $\varphi(t)$, but then we write *that* as its integral over $H(\lambda)$ which is the $\mu(\lambda)$ object from p 452 (not sure how to interpret this thing H), so then we end up with 8.4.14 where G' and G" are given as integrals of H, reminds me of dispersion relations. Remember that $G^* = (h/u_o)\, S_{12}$ from the oscillating plate experiment and it has phase.

8.5 Computation of H(λ) from φ from G* (454).

We assume we have done some oscillating plate experiment and we have data for $G^*(\omega)$, and we have at our disposal 8.4.15 and 16. I bet there is a nice way of inverting those equations, but instead Lai describes a numerical method for finding a reasonable $H(\lambda)$ that makes those integrals work. λ is a time variable. It is an iterative method with a 1968 paper reference of Tanner. Again, we have here a method for measuring the relaxation spectrum $H(\lambda)/\lambda$ for any complicated mixed fluid having a continuum of components. This method in theory should work for a finite number of components, and delta functions will then appear for H. Remember that we are constantly looking for experiments that can measure model parameters. Above we found μ, then we found λ for single-component fluid, and now $H(\lambda)$ for a fluid with many components.

Example 8.5.1 (455)

We are given some G* curves for synovial fluid of young, old, and arthritic as plots page 455. Both storage and loss generally seem to increase with ω in the 10 Hz range as shown. We are supposed to do the numerical method just described and obtain plots of $H(\tau)$ and then $\varphi(\tau)$ from 8.4.1. Only the latter plots are shown on page 456. I really don't know how to interpret the conclusions because I have only just learned about $\varphi(\tau)$.

Part B: Nonlinear Viscoelastic Fluid (456)

This section has no opening comments at all and just dives into the reference version of things, so we are not told what the term "non-linear" refers to. We have seen that the Maxwell type fluids always have a constitutive equation of the form S = KD where K is a linear operator as for example in 8.3.1 p 451. In Hooke's law we thing of F = kx as being "linear", stress is linear function of the strain. I guess in fluids, the "strain" thing is the rate of deformation tensor D, so if S = KD, then your fluid model is linear. The very first "model" discussed in Part B is way down at page 474, where S is said to be a functional of C_t. Then finally on page 475 we see the first ACTUAL model which has the [C_t-I] form. Is this non-linear because it is not linear in D or E (except for small deformations)? Or is it non-linear because linear would mean linear in F_t but C_t is quadratic in F_t? I think C_t is the "acting" strain object, so I think it is the fact that the -I is in there in [C_t-I] that makes it non-linear.

Question: How are F_t and D related?

Answer: On page 469 we see how D and A_1 are related, and we know $A_1 = C_t'$ page 463, so roughly this says that $2D = A_1 = C_t'$ where $C_t = F_t^T F_t$ so we then have $2D = (d_t[F_t^T F_t])$ in effect. This then is how the "rate of deformation tensor" is related to the "relative deformation tensor". We don't for example have the result $D = d_t F_t$.

8.6 Current configuration as the reference configuration (456)

When I first started here, I was very confused. Having written Appendix J, it all seems very simple, but I will leave my past scribblings here between marks.

Suddenly we have a whole new Lagrange/Euler functional form world to ponder. Recall that long ago Lai talked about (see Ch 3 notes)

$\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ $\quad$ $\mathbf{X} = \mathbf{x}(\mathbf{X},t_0)$ $\quad$ t_0 = the reference time

Lagrangian $\quad$ $\mathbf{v} = \hat{\mathbf{v}}(\mathbf{X},t)$

Eulerian $\quad$ $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x},t)$

Now we have some all new stuff:

$\mathbf{x} = \mathbf{x}'_t(\mathbf{x},t)$

$\mathbf{x}' = \mathbf{x}'_t(\mathbf{x},\tau)$ $\quad$ $\mathbf{x}$ = where particle is at time t $\quad$ t = the reference time

$\mathbf{x}'$ = where particle is at time τ

I think the analogy is this

	reference time and position
$\mathbf{x} = \mathbf{x}(\mathbf{X},t) = \mathbf{x}_{t0}(X,t)$	$\mathbf{X} = \mathbf{x}(\mathbf{X},t_0)$
$\mathbf{x}' = \mathbf{x}'_t(\mathbf{x},\tau)$	$\mathbf{x} = \mathbf{x}'_t(\mathbf{x},t)$

Lagrangian $\mathbf{v} = \hat{\mathbf{v}}(\mathbf{X},t)$

So we seem to have a mapping of all the names:

reference time	t_0	→	t
reference position	$\mathbf{X}$	→	$\mathbf{x}$
non-reference time	t	→	τ
non-reference position	$\mathbf{x}$	→	$\mathbf{x}'_t$
current time	t	=	t

Consider a particle which has position $\mathbf{x}$ at time t. Where was or will be that same particle at time τ? The answer is that it is at $\mathbf{x}' = \mathbf{x}'_t(\mathbf{x},\tau)$. So this $\mathbf{x}'$ is position away from time t and position $\mathbf{x}$.

Consider a particle which has position $\mathbf{X}$ at time t_0. Where was or will be that same particle at time t? The answer is that it is at $\mathbf{x} = \mathbf{x}_{t0}(\mathbf{X},t)$. So this $\mathbf{x}$ is position away from time t_0 and position $\mathbf{X}$.

So yes, I think it is all just a name change. Before we had $d\mathbf{x} = F\, d\mathbf{X}$ (p 86) where F was the deformation gradient. The dumbbell $d\mathbf{X}$ moved to $d\mathbf{x}$. Now we are going to have $d\mathbf{x}' = F_t\, d\mathbf{x}$, it is all the same thing with the names changed. The subscript t on F_t reminds you of the reference time when you have dumbbell $d\mathbf{x}$. And this will be the "relative" deformation gradient. I think it helps a little to think of τ as being in the future relative to t if we are talking forward flow.

When we said $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$, this thing shows where the particle actually goes in space, so as time goes on, this is going to map out a pathline, not a streamline. In the new notation, $\mathbf{x}' = \mathbf{x}'_t(\mathbf{x},\tau)$ is again the pathline for the particle that starts at position $\mathbf{x}$ at time t.

The Eulerian velocity would at some future point $\mathbf{x}'$ and time τ,

$\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}',\tau)$

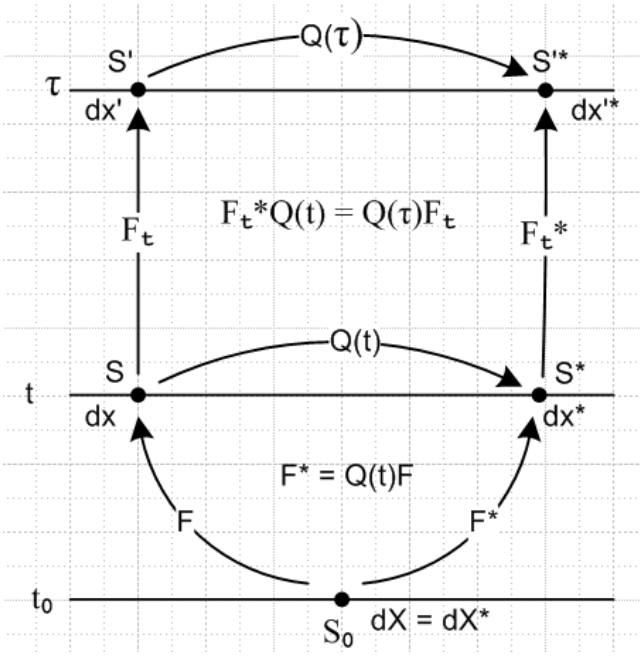
As a derivative, this would be

$\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}',\tau) = d\mathbf{x}'/d\boldsymbol{\tau}$ // he writes $d\mathbf{x}'_t/d\boldsymbol{\tau}$

Example 8.6.1. (457). We have a steady state velocity field at time t. Thus, $v(\mathbf{x}') = v(x)$.

OK, I think I am making a mountain out of a molehill here, but Lai is so silent. The example seems pretty reasonable. The idea is that "now" always means $\tau = t$. Sometime in future or past has some other τ.

Here is the picture in Appendix J and a few comments:



$d\mathbf{x} = F(\mathbf{x},t)\, d\mathbf{X}$		$dx_i = F_{ij} dX_j$	// Lai p 105 (3.18.13)
$F_{ij} = (\partial x_i / \partial X_j)$		or $\mathbf{F} = (\nabla \mathbf{x})$	// Lai p 105 (3.18.4)
$F^* = Q(t)\, F$	or	$F^*(\mathbf{x}^*,t) = Q(t)\, F(\mathbf{x},t)$	// Lai p 336 (5.56.21)
$d\mathbf{x}' = F_t(\mathbf{x},\tau)\, d\mathbf{x}$		// compare to $d\mathbf{x} = F(\mathbf{x},t)\, d\mathbf{X}$ from the lower left	
$(F_t)_{ij} = (\partial x'_i / \partial x_j)$		or $F_t = (\nabla \mathbf{x}')$	

My Appendix explains all the Subscript t relative objects, but it does not (at this time) say why this is of any practical use.

8.7 The Relative Deformation Gradient (457)

Fine, we just replace $d\mathbf{x} = F\, d\mathbf{X}$ which was referenced to time t_0 and X with $d\mathbf{x}' = F_t\, d\mathbf{x}$ which has reference t and **x**. So we immediately translate earlier results for Cartesian in both systems, Cylindrical in both systems, and Spherical in both systems, all from my huge effort with section 3.29 long ago. The translation rule for something like (8.7.8) is s→s' and $s_0 \rightarrow$ s compared with the Section 3.29 notation. Notice that, in this entire section and the next. we never see the *arguments* of F_t which are $F_t(\mathbf{x},\tau)$.

8.8 The Relative "other tensors" (459)

The relative polar decomposition is stated with its $F_t = R_t U_t = V_t R_t$ and various relations. Then we get the left and right Cauchy-Greens which are now B_t and C_t . Remember that F^{-1} just means you do that

s↔s' swap for all the coordinates. Then it is easy to write C_t^{-1} (Finger) and B_t^{-1}(Piola). All the time, the t label indicates the reference time. If you actually GO to that time, then all tensors are unity since there is no movement from reference time to current time! [all this is in Appendix J]

Example 8.8.1. This reprises results found earlier for dumbbell dot products.

Text following this example are more reprises.

8.9 The Relative C tensor in various coordinates (460)

In this section Lai does some actual examples of the relative tensors, some in curvilinear coordinates, which is all based on Section 3.29 which, luckily, I did completely.

A. Both x and x' Cartesian coordinates, we just translate some section 3.29 results

Example 8.9.1. For a specific simple "pathline equation" (which means $\mathbf{x}' = \mathbf{x}_t'(\mathbf{x},\tau)$) we compute first F_t and C_t, and then inverting the equation to get $\mathbf{x} = \mathbf{x}_\tau(\mathbf{x}', t)$ we can get F_t^{-1} and C_t^{-1}. In this example, we have a simple shear velocity field $v_1(x_2)$ with all else 0. We earlier found the pathline for this example, and so that pathline is then used to find F_t where $k = \partial_2 v$. This example will be used later in Example 8.16.1.

B. Both x and x' Cylindricals, compute a few C_{ij} elements, more Section 3.29 translation.

C. Both x and x' Sphericals, compute a few C_{ij} elements, more Section 3.29 translation.

8.10 "Deformation History" and the Rivlin-Ericksen Tensors (463)

Here for the first time we see some relative tensor arguments, and we see $C_t(\mathbf{x},\tau)$ and of course the arguments are the same for ALL the subscript-t deformation related tensors.

Fancy words for simple object.

1. Any of our tensors, including $C_t(\mathbf{x},\tau)$, can be tracked backwards in time toward $\tau = -\infty$, and thus any tensor gives a "history of the deformation". Recall that F, C and B are all called "deformation tensors", with adjective nothing, right Cauchy-Green and left Cauchy-Green. So this notion of "history" seems pretty obvious.

2. If you Taylor expand $C(\tau)$ around $C(t)$, you get derivatives as coefficients and these get the fancy name Rivlin-Ericksen tensors and are called A_n. This derivative idea seems very simple.

The examples are drills in computing actual tensors in different situations. No new theory here.

Example 8.10.1 . Compute C and A_n for the little $v_1 = v(x_2)$ we have had going this chapter.

Example 8.10.2. For a pipe flow in cylindricals with $v_z = v(r)$, compute pathline, C and A_n.

Example 8.10.3. For Couette flow in cylindricals with $v_\theta = v(r)$, compute pathline, C and A_n.

Example 8.10.4. For a sink flow in sphericals $v_r = =-a/r^2$, compute pathline, C and A_n.

These are all just exercises in turning the crank, I do appreciate their presence in the book.

8.11 Relating A_1 to D; a recursion formula for the A_N (468)

Note Added: I have done all this now in full detail in "RE tensors A_n.doc". Leaving notes below.

There is some subtle notation in use here after equation (8.11.4). Remember that our Euler functions now all look like $c_t(\mathbf{x},\tau)$. I think what Lai has in mind is setting $\mathbf{x}$ to some fixed value, call it $\mathbf{x}_1$. Consider then a function $\alpha(\mathbf{x},\tau)$. Suppose we set $\mathbf{x} = \mathbf{x}_1$ = a constant vector, and then define a function only of τ,

$$g(\tau) \equiv \alpha(\mathbf{x}_1,\tau) \qquad => \qquad \partial_\tau g = d_\tau g \qquad (1)$$

We can certainly write

$$[\partial_\tau \alpha(\mathbf{x},\tau)]_{\mathbf{x}=\mathbf{x}1} = \partial_\tau \alpha(\mathbf{x}_1,\tau) \qquad (2)$$

because that *partial* derivative only hits the τ argument and ignores the first argument, so we can set $\mathbf{x}$ to $\mathbf{x}_1$ either before or after doing the derivative. Therefore, from (1) and (2) we get

$$d_\tau g = d_\tau \alpha(\mathbf{x}_1,\tau) = \partial_\tau \alpha(\mathbf{x}_1,\tau) = [\partial_\tau \alpha(\mathbf{x},\tau)]_{\mathbf{x}=\mathbf{x}1} \qquad (3)$$

Setting $\alpha = (ds'^2)$ this becomes, // note that ds'^2 really means $(ds')^2$

$$d_\tau[(ds'^2)(\mathbf{x}_1,\tau)] = \partial_\tau\{(ds'^2)(\mathbf{x}_1,\tau)\} = [\partial_\tau\{(ds'^2)(\mathbf{x},\tau)\}]_{\mathbf{x}=\mathbf{x}1} \qquad (4)$$

Meanwhile, we have

$(ds'^2)(\mathbf{x}_1,\tau) = d\mathbf{x} \bullet C_t(\mathbf{x}_1,\tau)\, d\mathbf{x}$ = a function of τ, where $d\mathbf{x}$ = some constant dumbbell value where the tail end of $d\mathbf{x}$ lies at location $\mathbf{x}_1$

$$d_\tau[(ds'^2)(\mathbf{x}_1,\tau)] = d\mathbf{x} \bullet [d_\tau C_t(\mathbf{x}_1,\tau)]\, d\mathbf{x} \qquad (5)$$

So now write LHS(5) = RHS(4) = RHS(5) to get

$$d_\tau[(ds'^2)(\mathbf{x}_1,\tau)] = [\partial_\tau\{(ds'^2)(\mathbf{x},\tau)\}]_{\mathbf{x}=\mathbf{x}1} = d\mathbf{x} \bullet [d_\tau C_t(\mathbf{x}_1,\tau)]\, d\mathbf{x} \qquad (6)$$

and suppressing arguments and setting d = D, this becomes

$$D_\tau(ds'^2) = [\partial_\tau(ds'^2)]_{\mathbf{x}=\mathbf{x}1} = d\mathbf{x} \bullet [d_\tau C_t]\, d\mathbf{x} \qquad (7)$$

and this is p 468 equation A. The entire argument above can be repeated with $\partial_t \rightarrow \partial_t^N$ and also with $d_t \rightarrow d_t^N$. The result is then

$$D_\tau^N(ds'^2) = [\partial_\tau^N(ds'^2)]_{x=x1} = d\mathbf{x} \bullet [d_\tau^N C_t]\, d\mathbf{x} \qquad (8) = // (8.11.5)$$

where Lai does not use my value $\mathbf{x_1}$ but just says "x_i-fixed" which means exactly the same thing.

When we set $\tau = t$ (t = reference time), then

$$(ds'^2)(\mathbf{x_1},t) = d\mathbf{x} \bullet C_t(\mathbf{x_1},t)\, d\mathbf{x} = d\mathbf{x} \bullet I\, d\mathbf{x} = d\mathbf{x} \bullet d\mathbf{x} = (ds)^2$$

So now we arrive at the last line on page 468 which is not clear to me. First, it has three typos which I have fixed. But how now do be bring in the object $d_t(ds)^2$? In what sense is $(ds)^2$ a function of t ? Using the above development, you would say $(ds)^2 = d\mathbf{x} \bullet d\mathbf{x}$ = a constant.

He wants to claim this. Start with (8) above and set $\tau = t$ wherever τ appears. So let's put the arguments back into (8) so we can see them,

$$D_\tau^N(ds'^2(\mathbf{x_1},\tau)) = [\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{x=x1} = d\mathbf{x} \bullet [d_\tau^N C_t(\mathbf{x_1},\tau)]\, d\mathbf{x}$$

Now do an honest setting of $\tau = t$,

$$\{D_\tau^N(ds'^2(\mathbf{x_1},\tau))\}^{\tau=t} = [\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{x=x1}{}^{\tau=t} = d\mathbf{x} \bullet [d_\tau^N C_t(\mathbf{x_1},\tau)]^{\tau=t}\, d\mathbf{x}$$

In the last term, $C_t(\mathbf{x_1},\tau)$ is just some $h(\tau)$, so we can replace d_τ by ∂_τ there. Then (8.10.2) says

Then using (8.10.2) this says

$$\{D_\tau^N(ds'^2(\mathbf{x_1},\tau))\}^{\tau=t} = [\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{x=x1}{}^{\tau=t} = d\mathbf{x} \bullet A_N\, d\mathbf{x}$$

But now Lai wants to claim that the LHS's of this equation are in fact $D_t^N\,(ds^2)$. That is the part I am unable to show. I agree that $ds'^2(\mathbf{x_1},\tau=t) = ds^2$. Well, consider the middle term above,

$$[\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{x=x1}{}^{\tau=t} = (\ \partial_\tau^N(ds'^2(\mathbf{x_1},\tau))\ )^{\tau=t} = (\ \partial_r^N(ds'^2(\mathbf{x_1},r))\ )^{r=t}$$

where I have set the dummy variable to r. Now consider from regular calculus,

$$[d_r f(r)]^{r=t} = f'(r)^{r=t} = f'(t) = d_t f(t)$$

So I guess we then can write the above line with an extra = at the end

$$[\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{x=x1}{}^{\tau=t} = (\ \partial_\tau^N(ds'^2(\mathbf{x_1},\tau))\ )^{\tau=t} = (\ \partial_r^N(ds'^2(\mathbf{x_1},r))\ )^{r=t} = \partial_t^N(ds'^2(\mathbf{x_1},t))$$

But again, $\mathbf{x_1}$ is fixed, so $ds'^2(\mathbf{x_1},t)$ is a function only of t, so could add another $= d_t^N(ds'^2(\mathbf{x_1},t))$. So at this point we have

$$[\partial_\tau^N(ds'^2(\mathbf{x},\tau))]_{\mathbf{x}=\mathbf{x1}\,\tau=t} = d_t^N(ds'^2(\mathbf{x_1},t)) = d_t^N(ds^2)$$

Now it is true that $ds'^2(\mathbf{x_1},t) = ds^2$ as shown above which is $d\mathbf{x} \bullet d\mathbf{x}$, so add extra = above.

Now I think we have to change our footing in order to tie back to earlier in the book. In the new footing, **x** = a reference point in space. But in the old footing, **X** = reference point and **x** moves in time. In this old footing, we had $d\mathbf{x} \bullet d\mathbf{x}$ being a function of time, so we have $ds^2(t)$ and this derivative $d_t^N(ds^2) \neq 0$. This footing change is indeed confusing, I will just have to "buy it" for the moment. My counter argument is that in the new footing, ds^2 = constant because $d\mathbf{x}$ = constant.

But this tensor D really is an animal from the old footing world so maybe that makes it OK.

Status: OK, I will accept 8.11.6 for now. Yes, when this is combined with the much earlier result relating ds^2/dt to the D tensor, yes, we get 8.11.8 for A_1 . Let's review the path

$F_t \rightarrow \quad C_t = F_t^T F_t \qquad \partial_t C_t = A_1$

$D = (\nabla v)^S$ = rate of deformation. **result is $A_1 = 2\,D$**

Both objects being related involve the time change for deformation. Then if you want to get into the acceleration of deformation, so to speak, you can relate A_2 to $d_t D$ as in 8.11.9.

Example 8.11.1. We take our spherical sink flow example, we compute $\nabla\mathbf{v}$ and then $(\nabla\mathbf{v})^S = D$ and from it A_1 as in 8.11.12. Notice there is no explicit time dependence in A_1 so that $\partial_t A_1 = 0$.

We are then going to compute A_2 using the formula 8.11.2, and then $D_t A_1 = (\nabla\mathbf{A_1})\,\mathbf{v}$ if I have written it right (see top line of page 470). This is *the first time* in this book we have encountered an object like $(\nabla\mathbf{A_1})$ which is the gradient of a rank-2 tensor. This is not an object I treated in tensor doc. Let's go back to

$$DA_{ij}/Dt = \partial A_{ij}/\partial t + \partial A_{ij}/\partial x_k * \partial x_k/dt = \partial A_{ij}/\partial t + [\text{object}]_{ijk} * v_k$$

so this is the thing we need to compute

$[\text{object}]_{ijk} = [\nabla A]_{ijk}$ // $= \partial_k A_{ij} = A_{ij,k}$ in Cartesian coordinates only

so we have this same "reversal" of the last index, I could of course (and may) power up my tensor doc machinery to compute this object. Maybe I would treat $\partial_n A_{ijk\ldots}$ as a general case. There is nothing mysterious about this, it is just turn the crank. Lai in fact turns this crank in his usual cases of Polar, Cylindrical and Spherical coordinates in his Appendix 8.1, which is the concluding tour de force section of the book. He also does divA at the same time, but I have already done that.

Recall that $A_{ij,k}$ is the form of the result in Cartesian coordinates only. Looking at tensor doc, I would start with this from my examples section

$T^{ab}{}_{;\alpha} \equiv \partial_\alpha T^{ab} + \Gamma^{a}{}_{\alpha n} T^{nb} + \Gamma^{b}{}_{\alpha n} T^{an}$. // and Γ = 0 for Cartesians

$T'^{ab}{}_{;\alpha} \equiv \partial'_\alpha T'^{ab} + \Gamma'^{a}{}_{\alpha n} T'^{nb} + \Gamma'^{b}{}_{\alpha n} T'^{an}$. // prime means curvilinear coordinates

and that is basically it! Most of my tensor doc work expresses Γ in terms of the R type object.

There is of course *one detail* here: the above T'^{ab} is a true contravariant tensor, whereas in Lai we always use "unit vector tensors". I fiddled on this for a while and managed to confuse myself so I moved those confused notes into a separate doc because I want to stay on message right now regarding Chapter 8.

So Lai uses his appendix results to go ahead and compute $(\nabla \mathbf{A}_1)$ and thereby to compute $D_t A_1$ and thereby to compute A_2 as shown in (8.11.12). But we just computed these same A_2 in Example 8.10.4 by computing C_t and then differentiating it wrt τ, and his answers found here agree with those results.

Comments: This section 8.11 was a bit "rocky" for me. I get the main point, but I had trouble with the fine details. At some point I will need to clean this up.

8.12 Some tensor relations involving F_t and (∇v) and the polar decomposition (471)

We did a lot with the various "deformation tensors", but we never did time derivatives of them. This is the famous math that makes me very uncomfortable, so I will do all the lines.

(a) Pondering equations (8.12.1)

We start with

$$d\mathbf{x}'(\tau) = \mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - x'_t(\mathbf{x},\tau)$$

Even this simple equation is confusing, even after 470 pages of reading Lai, so I have to laboriously wrote out all the words. A Particle starts out at location **x** at time t. The position of that particle is $\mathbf{x}'_t(\mathbf{x},t) = \mathbf{x}$. *Another* particle is nearby at the same time t, and it has position $\mathbf{x}'_t(\mathbf{x}+d\mathbf{x},t) = \mathbf{x}+d\mathbf{x}$. The separation of these two particles at time t is $\mathbf{x}+d\mathbf{x} - \mathbf{x} = d\mathbf{x}$. Now we look at some time $\tau > t$ in the future, perhaps far in the future. The particle what was at position **x** is now at position $\mathbf{x}'_t(\mathbf{x},\tau)$, wherever that might be. And the particle that was at $\mathbf{x} + d\mathbf{x}$ is now at $\mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau)$. Thus, at time τ, the distance between these two particles is $\mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}'_t(\mathbf{x},\tau)$. We might call this distance $d\mathbf{x}'(\mathbf{x},\tau)$. So we then have

$$d\mathbf{x}'_t(\mathbf{x},\tau) = \mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}'_t(\mathbf{x},\tau)$$

so I disagree with the first equation in (8.12.1) where we see $d\mathbf{x}'(\tau)$ because that does not show the dependence on parameters x and t. Now the general dumbbell rule goes on to say

$$\mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}'_t(\mathbf{x},\tau) = F_t(\mathbf{x},\tau)\, d\mathbf{x}.$$

So here then is my reading of (8.12.1)

$$d\mathbf{x}'_t(\mathbf{x},\tau) = \mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - \mathbf{x}'_t(\mathbf{x},\tau) = F_t(\mathbf{x},\tau)\, d\mathbf{x} \qquad \text{// (8.21.1)}$$

(b) Pondering the three equations p471 A

We are supposed to apply d_τ to the above line and add more stuff. Question: is there any τ dependence "through" the x argument of $d\mathbf{x}'_t(\mathbf{x},\tau)$? That's like asking in the old world if there was any t dependence of **X**. We would have said then "no, because X is a fixed material location which is our reference. ". So we must say the same thing here, "no, because x is a fixed material location which is our reference.". Therefore we can say

$$d_\tau\,[d\mathbf{x}'_t(\mathbf{x},\tau)] \;=\; d_\tau\,[\mathbf{x}'_t(\mathbf{x}+\mathbf{dx},\tau)] - d_\tau\,[\mathbf{x}'_t(\mathbf{x},\tau)] \;=\; d_\tau\,[F_t(\mathbf{x},\tau)\,d\mathbf{x}] \qquad (1)$$

Now we pause to ask about $d_\tau\,[\mathbf{x}'_t(\mathbf{x},\tau)]$. Since **x** is again a reference, this is

$$d_\tau\,[\mathbf{x}'_t(\mathbf{x},\tau)] \;=[\,\mathbf{x}'_t(\mathbf{x},\tau+d\tau) - \mathbf{x}'_t(\mathbf{x},\tau)]/d\tau \;=\; \mathbf{v}'(\mathbf{x},\tau) \qquad (2)$$

Notice in passing the convention that Lai only puts that t subscript on the position vector, not other vectors like velocity. I am not sure why that is. But OK, and then using (2) twice in (1) we get

$$d_\tau\,[d\mathbf{x}'_t(\mathbf{x},\tau)] \;=\; \mathbf{v}'(\mathbf{x}+\mathbf{d}x,\tau) - \mathbf{v}'(\mathbf{x},\tau) \;=\; (d_\tau F_t(\mathbf{x},\tau))\,d\mathbf{x} \qquad (3)$$

So far then we have three pieces of equations A. The missing piece is this

$$d\mathbf{v}'_i \equiv \mathbf{v}'_i(\mathbf{x}+\mathbf{d}x,\tau) - \mathbf{v}'_i(\mathbf{x},\tau) \;= \partial\mathbf{v}'_i(\mathbf{x},\tau)/\partial x_k * dx_k \;=\; \partial\mathbf{v}'_i/\partial x_k * dx_k \qquad (4)$$

This $d\mathbf{v}'_i$ is only non-zero due to the first arguments of the two terms being different, and that is what justifies the expression on the right. Now we need to think of $\partial\mathbf{v}'_i/\partial x_k$ as a matrix $(\nabla\mathbf{v}')_{ik}$ and then we are allowed to write the above as

$$d\mathbf{v}'_i \equiv \mathbf{v}'_i(\mathbf{x}+\mathbf{d}x,\tau) - \mathbf{v}'_i(\mathbf{x},\tau) \;= \partial\mathbf{v}'_i(\mathbf{x},\tau)/\partial x_k * dx_k \;=\; \partial\mathbf{v}'_i/\partial x_k * dx_k$$

$$= (\nabla_{\mathbf{x}}\mathbf{v}')_{ik}\,dx_k \;=\; [(\nabla_{\mathbf{x}}\mathbf{v}')\,d\mathbf{x}]_i$$

Therefore removing component index i we get

$$d_\tau\,[d\mathbf{x}'_t(\mathbf{x},\tau)] \;\equiv\; \mathbf{v}'(\mathbf{x}+\mathbf{d}x,\tau) - \mathbf{v}'(\mathbf{x},\tau) = \;[(\nabla_{\mathbf{x}}\mathbf{v}')\,d\mathbf{x}] \;=\; (d_\tau F_t(\mathbf{x},\tau))\,d\mathbf{x} \qquad \text{// p 471 A}$$

The last equality above says matrix = matrix, or, re-displaying the arguments of v'.

$$d_\tau F_t(\mathbf{x},\tau) \;= \nabla_{\mathbf{x}}[\mathbf{v}'(\mathbf{x},\tau)] \qquad \text{// (8.12.2)}$$

Now I arrive at another "standard point of confusion". Suppose we evaluate both sides of the above equation at $\tau = t$. We must then write

$$[d_\tau F_t(\mathbf{x},\tau)]^{\tau=t} = \{\,\nabla_{\mathbf{x}}[\mathbf{v}'(\mathbf{x},\tau)]\,\}^{\tau=t}$$

On the RHS, we can set $\tau = t$ either before or after applying $\nabla_{\mathbf{x}}$, so we can write RHS = $\nabla_{\mathbf{x}}[\mathbf{v}'(\mathbf{x},t)]$. But since we are at the reference place and time, we want to say that $\mathbf{v}'(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t)$ where we have a convention I think of representing any object at x and t without a prime. We do this for **x**, for example. This may be something I missed. In this case, the above says

$$[d_\tau F_t(\mathbf{x},\tau)]^{\tau=t} = \nabla_{\mathbf{x}}[\mathbf{v}(\mathbf{x},t)]$$

Now we examine the LHS. I had this situation earlier, and I said that $[\ f'(x)]^{x=x_1} = f'(x_1)$ in our usual notation of single variable calculus. So $[f'(\tau)]^{\tau=t} = f'(t)$. This is the derivative evaluated at t. Now I guess you might write this as $[\ df(\tau)/d\tau]^{\tau=t}$. But you could also write it as $f'(t) = df/dt$ and be done with it. But this would be $d_t f$. Therefore, thinking of the LHS F_t as f, we have $d_t F_t$ and we are done with it. Then we have

$$d_t F_t(\mathbf{x},t) = (\nabla_{\mathbf{x}}[\mathbf{v}(\mathbf{x},t)]) = (\nabla_{\mathbf{x}}\mathbf{v}) \qquad \text{// (8.12.3)}$$

I guess any time you have a d_τ acting on something and then you set $\tau = t$, you can do what I just did.

Comment: We now have in 8.12.3 some information on the total time derivative of one of the deformation tensors, namely F_t. We are able to relate it to the $(\nabla_{\mathbf{x}}\mathbf{v})$ tensor. F_t has no dimensions, so both sides of the above are sec^{-1}. Let's go back now to an earlier equation which I am happy now to write as

$$d\mathbf{x}'(\tau) = \mathbf{x}'_t(\mathbf{x}+d\mathbf{x},\tau) - x'_t(\mathbf{x},\tau) = (\nabla_{\mathbf{x}}\mathbf{x}'_t(\mathbf{x},\tau))\, d\mathbf{x} = F_t(\mathbf{x},\tau)\, d\mathbf{x} \qquad \text{// (8.7.1,2)}$$

so that

$$F_t(\mathbf{x},\tau) = (\nabla_{\mathbf{x}}\mathbf{x}'_t(\mathbf{x},\tau))$$

Now what happens here if we evaluate at $\tau = t$? We get

$$F_t(\mathbf{x},t) = (\nabla_{\mathbf{x}}\mathbf{x}'_t(\mathbf{x},t))$$

but since we are at reference x and t, the LHS is just the unity matrix. And RHS = $(\nabla_{\mathbf{x}}\mathbf{x})$ = unity as well. So we don't have anything "interesting" at this no-derivative level. But we could write this pair

$$F_t(\mathbf{x},\tau) = (\nabla_{\mathbf{x}}\mathbf{x}'_t(\mathbf{x},\tau))$$
$$d_\tau F_t(\mathbf{x},\tau) = (\nabla_{\mathbf{x}}[\mathbf{v}'(\mathbf{x},\tau)])$$

or

$$F_t = (\nabla_{\mathbf{x}}\mathbf{x}'_t) \qquad \text{at } \mathbf{x},\tau$$
$$d_\tau F_t = (\nabla_{\mathbf{x}}\mathbf{v}')$$

so there is some parallel structure here.

(c) the symmetry question. I am happy with (8.12.6), but how does Lai arrive at his symmetry claims here? As I review polar decomposition.doc in the Lai support folder, I see that in that decomposition, U is always symmetric! I forgot that fact. So consider

$$Q_{ab} = d_t U_{ab} = d_t U_{ba} = Q_{ba}$$

so of course $d_t U_{ab}$ is symmetric. now if you write for matrices that A = B + C and you know that C is symmetric, I think you then know that B is antisymmetric.

Proof: You know you can decompose $A = A_s + A_a$ in a unique manner:

$$A^s_{ab} = (A_{ab} + A_{ba})/2 \qquad \text{etc}$$

Therefore if A = B + C and C is symmetric, it must be that $C = A^s$ and therefore $B = A^a$ QED.

(d) finish I have derived (8.12.5) and I want to continue down this page 471. We now think of (8.12.3)'s LHS in the sense I had it above.

$$LHS = [d_\tau F_t(\mathbf{x},\tau)]^{\tau=t}$$

The object here is (8.12.5) if you evaluate each term at $\tau = t$, and THAT is where (8.12.6) comes from. Note that U and R become unity when $\tau = t$, so this (8.12.6) has a lot in it. From (c) above I now agree with the symmetry comments . But I do know that $D + W = \nabla v$ because we have used that a zillion times. And so finally I have 8.12.7 derived.

8.13 Change of Frame Stuff (471)

I now have to go reread previous sections on this subject starting on p 334 in Chapter 5. A thorough review is essential here! // I have now done this, with lots of notes taken in "essay on covariance of...". So I am now ready to start reading this section of Lai. We are going to see the same discussion but reframed in terms of the various "relative" tensors.

With Appendix J I think I am now happy on this change of frame stuff, BUT I have to redo things because things are still not optimally presented.

8.14 Change of Frame for the A tensors (474)

This tiny section just shows that the A tensors are objective relative to Q(t), nothing more or less.

8.15 Incompressible Simple Fluid (474)

OK, we are starting to make sense. From Appendix J, we found that C_t was objective (and B_t was not). Therefore, a covariant model for some fluid might be (8.15.2) which is somewhat similar to our Maxwell fluid model (constitutive relation),

$$S(t) = 2 \int_{-\infty}^{t} dt' \, [\, (\mu/\lambda) e^{-(t-t')/\lambda}] \; D(t')$$

where all of history plays a role. Both S and D are objective, though I am not sure Lai has said so, but that does make the above equation covariant. Now in this section, we could imagine a similar thing

$$S(t) = 2 \int_{-\infty}^{t} d\tau \, [\text{some kernel}] \; C_t(\tau)$$

where the weight function or kernel I guess should be a rotational scalar. We know that C_t is objective and so is S, so we have covariance for a model of this form.

But Lai wants to be even more general and in (8.15.2) he just says that S is a "**functional** of the history of C_t" which is fine by me. I think he means a linear functional, but maybe not. In Stak you could have either kind of functional. The word **"simple"** means that the functional is a functional of C_t which means of F_t in effect, and not of the gradient of F_t or multiple gradients thereof! He requires that the functional of C_t be objective in (8.15.5). Just as we did with f = f* in our isotropic example in App J, here we assume H = H* and again we are isotropic. So in our model with isotropic, we have to make sure that (8.15.5) is satisfied, which is again similar to our App J thing, so I expect soon to see lower powers of C_t appearing somewhere.

It is obvious that our linear Maxwell fluid fits into this general model, as does Newtonian.

At this point come some comments I will need to return to later. // Comments now make sense.

8.16 Two specific models which reduce to Maxwell for small deformation (475)

Here we go back to our solution form for $S + \lambda \partial_t S = 2\mu D$ which was $S = 2 \int_0^\infty ds \, f(s) \, E(t-s)$ in the case of infinitesimal displacement, and this is now (8.16.1). He shows (and I follow) that E can be related to our new objective friend C_t as shown, so we get two "models" on page 475. They are both the same in the case that C_t is such that E is small. Remember that $f(s) = \varphi'$ was called a "memory function".

Now, why are these two new models non-linear? Well consider

$Y = CX$ // linear

$Y = CX + D$ // non-linear

so I guess it is the presence of two terms that makes these models non-linear, the way a Galilean transformation is non-linear. By the way, here Lai uses "frame indifferent" exactly to mean covariant, so he is applying this to an equation, not to a tensor, so I need to repair things. // Repair done.

Example 8.16.1. We have our usual super-simple shear velocity field $v_1 = kx_2$. This fits into the more general form of Example 8.9.1 (p 461) and so we already know F_t and C_t and C_t^{-1}. We then plug C_t into our first "model" (8.16.3) and we thereby obtain constitutive relations for S in terms of an integral of the memory weight function f(s) and no other unknowns. For example, we find that

$S_{12} = -k \int_0^\infty ds\, f(s)\, s \equiv \tau(k)$ = shear stress function (makes sense)

Some of the other elements of S_{ij} are also computed on page 476. The differences of diagonal elements are given special σ names. Notice that for some f(s), the various ds integrals are just numbers. Back in our Newtonian flow with this same example, we found $T_{12} = \mu k$ where μ was the viscosity. Here in our fancier flow, we are finding that $T_{12} = S_{12} = [\int_0^\infty ds\, f(s)\, s]\, k$. So comparison shows that the integral over ds, a number, is in fact the viscosity in this fancier model, called the apparent viscosity. In this particular model, τ(k) is linear in k, but the viscosity is still a constant, just as it was a constant in the Newtonian fluid.

He then asks us in a problem to repeat the above work for the other of the two models. The results are exactly the same with $f_1 \rightarrow f_2$.

Now recall from our discrete mixture Maxwell fluid we had

$$\varphi(t) = \Sigma_{n=1}^{N} (\mu_n/\lambda_n)\, e^{-t/\lambda n} \qquad \Rightarrow f(t) = -\Sigma_{n=1}^{N} (\mu_n/\lambda_n^2)\, e^{-t/\lambda n}$$

and for single component then $f(t) = -(\mu/\lambda^2)e^{-t/\lambda}$. What happens if we use that for f(s) here in our two new models? We are cribbing this form of the memory function from the Maxwell linear fluid model and applying it here to our new C_t models. Well, Tanner and Simmons (I think 1967 from below) have a certain "network model" using this f(s), but they cut it off in some way I don't care about right now.

Tanner, R. I. (1968). *Trans. Soc. Rheol.*, **12**, 155.
Tanner, R. I. and Pipkin, A. C. (1969). *Trans. Soc. Rheol.*, **13**, 471.
Tanner, R. I. and Simmons, J. M. (1967). *Chem. Eng. Sci.*, **67**, 1803.
Tanner, R. I. and Walters, K. (1998). *Rheology*: *An historical perspective*. Elsevier, Amsterdam.

So basically this section has introduced two non-linear "new models" both of which reduce to the linear Maxwell fluid model for small displacements, and both of which are "covariant". [seems similar to my covariant method of finding differential operators.] The idea is that maybe these models might describe some actual fluids in the real world. Tanner and Simmons made use of one of these models. I would certainly call this "an advanced topic", but for me it is another example of constructing a covariant model! I like how it reduces to a known simple model in a limit. Sort of like relativity reduces to Newtonian in a limit.

Notice that these two new models are examples of functionals of C_t, but they are not the form I conjectured in my Section 8.15 notes above.

Example 8.16.2. A routine exercise I already did

Example 8.16.3. In App J I review the case of T = f(B) where B was objective, and I showed that if f is polynomial in B, then it has to be at most second degree with objective scalar coefficients. Here that derivation is repeated for a generic F(A) which is some polynomial in some *symmetric* matrix A. If F is an isotropic function, then we know that the coefficients are scalars.

He then says we can apply this in the case $F(C_t)$ since C_t is symmetric. Again, any function of C_t which can be fitted by a polynomial and which is isotropic can only be a quadratic in C_t with scalar coefficients which are the invariants of C_t. Doubtless he will be using this fact soon (as in, next section). Notice that he uses the form I, A, A^{-1} as the most general form, but when he applies this to $A = C_t$, he omits the I term *with no comment*! Suppose an I term were present? That would imply an "extra stress S" that had the form scalar * δ_{ij} and such a term would by like T_{ij} = scalar function * δ_{ij} and I guess the idea is that we would just absorb such a term into the pressure term $T_{ij} = -p\,\delta_{ij} + S_{ij}$. I don't think he has ever noted that pressure p is a scalar, I guess that is obvious. So that then is the reason for just the two terms appearing in 8.16.20.

8.17 Single-integral non-linear constitutive equations (478)

For models in this class, the specific form of the "simple" model is shown in (8.17.1). Why is this model non-linear?

Answer : Well, the model first term is $S = FC_t$ where F is an integral operator. The first term in 8.17.1 is linear because then, for example, $F(\alpha C_t) = \alpha C_t$. The second term even by itself is non-linear since $S = G\,C_t^{-1}$, so both terms present is non-linear. In the analogy to Hooke's law F = kx being linear, the object playing the role of x is C_t and the object playing the role of F is S.

The BKZ Fluid (1963). We start with the general form 8.17.1 whose generalness was explained two paragraphs above in these notes. The two scalar coefficients are then written as derivatives of a potential U (similar I think to the Airy functions idea). We then have a specific expression for U which includes these ingredients: scalar invariants of C_t, c(s), β(s) where these two add up to φ(s)/s where φ(s) is the kernel function which appears in the $S = \varphi D$ relation 8.3.1, and later in the $S = 2\varphi E$ relation.

Lai now considers in this BKZ context the usual simple shear flow situation $v_1 = kx_2$ for which we computed C_t and C_t^{-1} earlier in the book, so this lets us then compute the invariants I_1 and I_2. What happened to I_3? For incompressible, Lai claims $\det(C_t) = 1$ and this follows from $\det(F_t) = 0$ and this goes back to the idea that det(F) is the Jacobian of our transformation, and if the volume does not change, det(F) = 1, as on page 130 and as in my tensor doc. That is why only I_1 and I_2 appear in the general formulas.

OK, now look at the form 8.17.3 for U. Right at that point we can compute the two derivatives to get f_1 and f_2 but for some reason Lai does not do that at that point. So I have Maple do these right here:

```
U := -(9/2)*b*ln((I1+I2+3)/9)- 24*(b-c)*ln((I1+15)/(I2+15))- c*(I1-3);
```

$$U := -\frac{9}{2} b \ln\left(\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}\right) - 24\,(b-c) \ln\left(\frac{I1+15}{I2+15}\right) - c\,(I1-3)$$

```
f1 := -2*diff(U,I2);
```

$$f1 := \frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I2+15}$$

```
f2 := 2*(diff(U,I1));
```

$$f2 := -\frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I1+15} - 2\,c$$

where I leave the dots off b and c. He has computed I1+I2+3 and I1+15. I have verified his results and he lost a factor of 2 in the last term as you see above. I go on to compute

```
S12 := f1*(-k*(t-s))+f2*(k*(t-s));
```

$$S12 := -\left(\frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I2+15}\right) k\,(t-s) + \left(-\frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I1+15} - 2\,c\right) k\,(t-s)$$

```
t := 0;
```

$$t := 0$$

```
S12;
```

$$\left(\frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I2+15}\right) k\,s - \left(-\frac{b}{\frac{1}{9} I1 + \frac{1}{9} I2 + \frac{1}{3}} - 48\,\frac{b-c}{I1+15} - 2\,c\right) k\,s$$

```
I1 := 3 + k^2*s^2;
```

$$I1 := 3 + k^2 s^2$$

```
I2 := I1;
```

$$I2 := 3 + k^2 s^2$$

```
S12;
```

$$\left(\frac{b}{1 + \frac{2}{9} k^2 s^2} - 48\,\frac{b-c}{18 + k^2 s^2}\right) k\,s - \left(-\frac{b}{1 + \frac{2}{9} k^2 s^2} - 48\,\frac{b-c}{18 + k^2 s^2} - 2\,c\right) k\,s$$

so he has quietly assumed that t = 0 in 8.17.1 so we are computing $S_{12}(t=0)$ which is fine. In the last expression above, the terms with the 48 cancel, and the b terms add, so the result according to me is

$$S_{12} = 18bks/(9+2k^2s^2) + 2cks = 2k\,[\,9bs/(9+2k^2s^2) + cs\,]$$

which agrees except I get a minus sign overall which arises from $C_t(t\text{-}s) = C_t(-s) = +ks$ for the C_{12}. Next, we have $\mu = S_{12}/k$ as on page 476 and then I agree with his μ_{app} except for sign. But 8.17.10 which I agree with says that μ is + integral of c(s) and probably c(s) is positive.

So I will have a complicated errata report here.

Review of BKZ: For some unknown reason, the $f_{1,2}$ memory functions are derived from some kind of potential U which has a strange form shown. If you then "do the math" for the usual simple shear flow situation, you end up with S_{12} and μ_{app} as shown in 8.17.9 and μ_∞ as he shows. The key idea here is that for the first time, we have $\mu_{app} = \mu_{app}(k)$, so viscosity varies with the shear speed parameter k.

Example 8.17.1. We are now going to apply the BKZ theory to synovial fluid. The claim is that at high shear rates (large k), synovial fluid basically has no viscosity at all, so from 8.17.10 we set c = 0 and get corresponding $\mu_\infty = 0$. This my sign error noted plays not role in this term at least. Next, he writes the function $\mu_{app}(k)$ as an integral of $H(\tau)$ using the definition of φ, and I agree with all this part (a) stuff. Now earlier we obtained $H(\tau)$ for our three age people in a table on page 456. So we can insert $H(\tau)$ into our equation (ii) here and numerically compute μ_{app}. Thus, the BKZ model gives the curves shown for the apparent viscosity of synovial fluid of people of three ages. For the young at heart, you have more viscosity at low shear rates than for the old. Lai wrote a paper on this subject, reference on page 455.

So to recap, we have a general-form covariant equation 8.17.1 for an incompressible fluid. It is claimed that this is the most general possible "simple", "single integral" and incompressible fluid equation. Then the BKZ model somehow comes up with some f_1 and f_2 functions using theory that Lai does not present, and the result is an expression for apparent viscosity and it is found to be a function of shear rate k. I thihk Lai has done a nice job here, one page on the BKZ theory, and one page applying it to synovial fluid.

8.18 Differential-type constitutive equations for incompressible fluid (481)

Instead of saying that S = integral(C_t and C_t^{-1}) as in 8.17.1, we want now to say $S = f(A_i)$. The idea is that maybe we can approximate C_t for times close to time t using the first N derivatives. Well, my first question is: do these derivatives have information about C_t^{-1} as well as C_t?

$$C_tC_t^{-1} = 1 \Rightarrow \quad d_t(C_tC_t^{-1}) = 0 \Rightarrow (d_tC_t)\,C_t^{-1} + C_t(d_tC_t^{-1}) = 0$$

$$\Rightarrow (d_tC_t^{-1}) = -\,(d_tC_t)/C_t^2 \qquad \text{// for first derivative}$$

Then what about the next derivative?

$$(d_tC_t)\,C_t^{-1} + C_t(d_tC_t^{-1}) = 0$$

$$\Rightarrow (d^2{}_tC_t)\,C_t^{-1} + (d_tC_t)\,(d_tC_t^{-1}) + (d_tC_t)\,(d_tC_t^{-1}) + C_t(d^2{}_tC_t^{-1}) = 0$$

$$\Rightarrow (d^2{}_tC_t)\,C_t^{-1} + 2(d_tC_t)\,(d_tC_t^{-1}) + C_t(d^2{}_tC_t^{-1}) = 0$$

So OK, then we can relate $d^2{}_tC_t^{-1}$ to all the other stuff. So the answer is this: knowledge of C_t expanded near $\tau = t$ does in fact include knowledge of C_t^{-1}expanded about the same point. Lai might have commented on this but kept quiet. The details are a bit messy, I could do something if I wanted.

So in theory if functions are smooth, then $S = f(A_i)$ with all derivatives is as good as 8.17.1.

How now does "incompressible" enter this model? Before it just means $I_3 = 1$ so we could forget about I_3 in 8.17.1. Well he showed on page 469 that $A_1 = 2D$, so incompressible means $\Delta = tr(D) = 0$, so here it means that $tr(A_1) = 0$, fine.

We then get a little cookbook of models:

(1) Keep the first N of the A_i, that is Rivlin-Ericksen fluid of complexity N

(2) Keep A_1, A_1^2 and A_2 , that is a second order fluid. Newtonian has only A_1= 2D for incompressible, so this is apparently "the next step up". Not obvious why these two extra terms are used, but I guess they both have the same "smallness" in $\varepsilon = (\tau - t)$.

Example 8.18.1. Back to our usual shearing example. He computes A_1, A_1^2 and A_2. He gets the corresponding expression for T_{ij}. We see here normal stress adjustments away from just -p. He then quotes the two σ_i objects. You can do an experiment and measure things and deduce the three constants which are μ_1, μ_2 and μ^3. Since $T_{12} = \mu_1 k = S_{12}$, this second order model has constant viscosity μ_1 so that is one of the three constants. Recall $2\mu D$ in the Newtonian equation.

Example 8.18.2. Again the simple shearing flow. Lai shows that A_i for $i \geq 3$ are all zero in this simple situation, though I would have to check why that is. He then computes all the lower A_i and their powers, and allows for cross terms like $A_2A_1^2$. So here he is saying that you might think of 8.18.3 as being a polynomial in the A_i. He does a term A_1^3 so it is not true that powers only up to 2. I guess basically he computes all possible non-zero A_i poly-like terms and then gives each one a coefficient. There are then eight different constants μ_i. Certain things are even and odd in k.

8.19 Objective Rate of Stress (483)

OK, I am familiar with this section from appendix J. The first observation is that d_tT is not covariant, if T transforms as a rank 2 tensor when the transforming rotations Q(t) are time dependent. Recall that you added a note showing that v = dx/dt is not a vector if R= R(t) in tensor doc. So you cannot have d_tT sitting in any covariant equation. We are led then to the Jaumann and two Oldroyd alternates to T, call them T', so that then d_tT' is in fact covariant and transforms as a rank-2 tensor.

Example 8.19.1. Here we start with a simple stress T_{11} = constant, other = 0. We then observe this from a rotating frame $Q(t) = R_z(\omega t)$ and we get T* as shown. He then computes d_tT^* and we of course find that in the * frame, $d_tT^* \neq 0$, whereas in the original frame $d_tT = 0$. So the point here is that the rate of stress change differs when measured in a fixed frame or in a rotating frame, the quantity d_tT cannot be a ran-2 tensor. I think now he wants to show that the "corotational stress" is the same in these two frames. He is going to sue 8.19.10 to get this corotational stress, but that requires knowledge of the spin tensor W. He uses a strange equation we had earlier in 5.56.20 which allows him to compute $(\nabla^* v^*)$ in the starred frame. I think in the rest frame since we have a static situation with uniaxial stress, v = 0 and of course $\nabla v = 0$ everywhere. Then once you compute $(\nabla^* v^*)$, you know W from the antisym part, as in (v). Then we compute that the Jaumann tensor is in fact 0 in the rotating * frame. Of course the Q(t) we use there is the same as the Q(t) relating the frames, so we are just undoing the rotation, it seems a bit circular. But

interesting to see how it works going through the ∇v pathway with W. We know the Jaumann stress is a rank-2 tensor, so if it is 0 in one frame, it has to be zero in the other frame.

8.20 Rate-type constitutive equations (487)

The game here is to play with all our covariant derivatives and use them as building blocks in fancier equations. The "general idea" is 8.20.2 where the overstar means any of the three covariant derivative types. Two stars means a second time derivative. These extra terms "could be present" from a covariant point of view for some fluid.

A. The convected Maxwell fluid. This is a sort of "correction" to our original Maxwell fluid which we recall had a dS/dt term in the equation, which we now realize is non-covariant. So maybe that theory only works for slow-flowing fluids or some such. We correct it by using the Jaumann time derivative of S as shown in 8.20.4. In Example 8.20.1 we do our usual simple shear flow and we compute the Jaumann thing as in 8.20.7. The big deal is that if we then compute S_{12} we get a viscosity that varies with k (I think we saw this only once before). For small (kλ) we get the result of the regular Maxwell fluid where μ is the constant viscosity.

B. The Jeffrey Fluid. This is same as Maxwell, but we add in a $\overset{o}{D}$ time derivative on the RHS, so the old λ is now λ_1 and the new term has λ_2. In Example 8.20.2, since we already know W, we compute $\overset{o}{D}$ and $\overset{o}{S}$ as shown. We then use the right equation in 8.20.12 to solve for S including S_{12}, and once again of course we have $\mu_{apparent}$ being a function of k. Simple Maxwell is recovered now when $k\lambda_1$ is small and when $\lambda_1 = \lambda_2$. So this model allows some "extra normal stress".

C. Oldroyd 3-constant fluid (fluid A). Exact same as Jeffrey, but we use Oldroyd upper instead of Jaumann

> that given in eqns. (8.9) the upper convected derivative. The symbol △ over the tensor being differentiated is usually employed for the lower convected derivative and the symbol ▽ for the upper convected derivative.

So think of the triangle base as lower when down, and upper when up. It turns out that this fluid A thing gives a constant μ. He should have made this a problem perhaps.

D. Oldroyd 4-constant fluid. We stick with Oldroyd's upper convected stuff, we add that $\overset{\nabla}{D}$ extra term with λ_2 as was done in Jeffrey above, and we add a totally new term $\mu_0(\mathrm{tr}S)D$. This is patently covariant, and seems to add a viscosity which increases as S increases, ie, when there is more S and thus more deformation. In this model μ approaches a constant at both low k and high k. This one is made a problem.

Part C: Viscometric flow for incompressible simple fluid (491)

8.21 Viscometric Flow (491)

I suspect that for even very exotic fluids, the "space" is filled all the time with fluid, so to first order approximation the fluid is incompressible. That is why almost all the fancy models we are examining in this chapter are incompressible models. They are of course simpler models for that reason.

So what is viscometric flow? First ingredient is that we keep only the A_1 and A_2 terms in the C_t expansion as in 8.21.1. The second item seems very strange to me. There must be some frame in which the A_1 and A_2 have the very simple forms shown. This frame is allowed to change with position. It is easy to show, given the simple N matrix shown, that 8.21.3 is true. At first blush, it seems hard to imagine that you would ever get this simple form, but now he is going to show that it happens all the time.

Example 8.21.1. Not the simple shear flow, but a generalization of it called unidirectional flow, $v_1 = v(x_2)$. However, back on page 464 we already computed A_1 and A_2 for this kind of flow, and they have exactly the form required to be called "viscometric flow". So you might say this is just "unidirectional flow". Those basis vectors $\mathbf{n}_i$ are just the $\mathbf{e}_i$ and are the same everywhere.

Example 8.21.2. Cylindricals, and flow is $v_z = v(r)$, so this is similar to saying $v_1 = v(x_2)$ but of course is not the same. Now we have to compute $A_{1,2}$ in cylindricals. But this was done as an exercise on page 465 which is replicated into 8.21.7, where $k(r) = dv(r)/dr$. . These A_i do not have the right form in the $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ basis in which they are computed, but a trivial reordering so that 1,2,3 = z,r,θ gets it into standard form. Therefore our assumed velocity field is "in voscometric form". It happens now that the matrix elements are functions of position.

Example 8.21.3. This is the Couette concentric cylinders situation where $v_\theta = v(r)$. We compute the A_i for this case as well on page 466 where now $k(r) = dv(r)/dr - v/r$. This time we do 1,2,3 = θ,r,z and we are then in standard viscometric form as in 8.21.13.

So three very basic flows all fit into the viscometric pigeonhole.

8.22 Form of T_{ij} for Viscometric Flow (493)

Lai first restates things, then uses the phrase "the objectivity condition" for "the covariance condition". Lai next defines Q to be a simple z reflection matrix. I think he just shows that our special T is invariant under Q. The result is that for this viscometric flow, a lot of the T_{ij} are 0. With our usual σ definitions as in 8.22.15 one can summarize the T_{ij} of viscometric flow as in 8.22.16. The problem in this case is reduced to the three functions $\tau(k)$, $\sigma_{1,2}(k)$. So you do one experiment, find these functions, and then you know them for any other viscometric flow. τ is odd in k, the σ_i are even. Newtonian has $\tau = \mu k$ and the others 0 (that is, the T_{ii} are not altered). You can then try power series in small k for the three functions to model a fluid.

8.23 Channel Flow (495)

Scene is $v_1 = v(x_2)$ between static parallel plates which we know is viscometric. This flow between such parallel plates (separation is h) is called "channel flow". A new k is defined here, namely $k = dv(x_2)/dx_2$ so whatever v is, this is the slope in the x_2 direction. $v = kx_2$ would be a special case with a constant k. I

think Lai makes an ansatz here which is that the S_{ij} are functions only of this k thing. That was the case for the $v = kx_2$ flow. So we know the general viscometric forms in 8.23.3. Next, we seek a static solution for this problem, which means EOM says $\partial_j T_{ij} = 0$. Note $k = k(x_2)$. We get some simple ODE's for p, τ, S_{22}.

It is trivial then to obtain 8.23.5 which then says that $-\partial_1 p = f$, a constant. This is the constant pressure gradient in anywhere between the plates, so the pressure is dropping off linearly as we move to the right in our little flat pipe. I have verified all equations on page 496-7 top. The problem is to concoct an experiment that lets you measure $\tau(k)$ for the fluid. We find that $\tau(k) = -f\, x_2$ so $k = \tau^{-1}(-fx_2)$ which we know is odd, so $k = -\tau^{-1}(fx_2) = -\gamma(fx_2)$ where γ is a new name for function τ^{-1}. The plan is first to find the total flow Q (per unit cross section I guess) and we get 8.23.14. Fiddling gives 8.23.17 and this lays out the required experiment. For a set of pressure gradient values f_i measure Q. Then using this data, plot f^2Q versus f. Differentiate this plot to get a new plot which is $\partial_f(f^2Q)$ for different values f_i. Then you get points for function $\gamma(fh/2)$ asd shown in 8.23.17.

So, if we set up this channel flow experiment, we have a way to measure function $\gamma(S) = \tau^{-1}(S)$ and from that we deduce $\tau(k)$. This is one of the three viscometric functions of interest.

Example 8.23.1. We want to use the model just described and reduce the results to the Newtonian fluid limit and see if we replicate earlier results of page 370. We start with knowledge that $\tau(k) = \mu k$ and at once we know that $\gamma(S) = S/\mu$. So we compute Q from 8.23.14 and lo and behold the result agrees with that found on page 370. This is just a good check of signs and factors of 2, etc.

8.24 Couette Flow (497)

The channel flow is perhaps impractical since you have to make it very wide. We should be able to construct a concentric cylinders Couette experiment as an alternate method of finding $\gamma(S)$, and that is the goal of this section, where things are going to be more complicated.

Right off the bat we know the $A_{1,2}$ matrices from p 493 (we just did this). Lai replaces $v_\theta(r)$ by $r\omega(r)$ so $\omega(r)$ is our function of interest. We showed Couette is viscometric. The general form 8.22.16 is supposedly a covariant equation, so we can write it in terms of the cylindrical unit vectors to get 8.24.4 based on 8.22.14, I agree. Again, we just want to find our three functions $\sigma_{1,2}$ and τ.

As before, we write the equations of motion, but we do this in cylindricals as shown on page 170, a long time ago! For example, $\partial_\theta T_{r\theta} = 0$ because $T_{r\theta} = \tau(k(r))$ and does not depend on θ.

The middle of our three equations tells us that $T_{r\theta} = \text{constant}/r^2$ and constant $= M/2\pi$ where M is the torque per unit height applied to the fluid from the end caps of the cylinder. We apply a constant torque onto this Couette apparatus to make it go. It just turns out that M is the same at all r. The equations fly and we end up with 8.24.16 and 17. Lai notes that if we knew $\gamma(S)$ ahead of time, we could use these equations to find $\omega(r)$ assuming we are driving R_1 inner at some fixed Ω_1, and then of course we know $\omega(R_2)$ which is Ω_2. But we want to work backwards somehow to deduce $\gamma(S)$.

But first, a little digression. We now use the first of the three EOM as shown in 8.24.18. Notice how σ_1 naturally appears due to the form of this EOM, now maybe we know why σ_1 is a quantity of interest. Lai then shows the strange fact that, since σ_1 is positive in all known materials, the second term in 8.24.20 can outweigh the first, causing the radial force to be larger on the *inner* cylinder, whereas you would think centrifugal would make the opposite be true.

Now on page 500 we resume the goal of deducing $\gamma(S)$ from this experiment. The first task is to plot $\Gamma(S_1)$ versus S_1 , where $S_1 = M/(2\pi R_1^2)$ and Γ is the difference of γ's shown. We have to do this plot using 8.24.23. This means that you use lots of different M values, and for each you measure $\Delta\Omega$. You have a meter that measures M. In practice you apply some M, and this gives some Ω_2 hence $\Delta\Omega$. It seems reasonable that this gives you a plot of $\Gamma(S_1)$ and you don't have to do anything unpleasant like vary R_i. Lai then shows that $\gamma(S)$ is given by 8.24.26 so you can use your $\Gamma(S)$ data to get $\gamma(S)$. S detail here is that it is assumed that $\gamma(0) = 0$, and I think that follows from $\tau(0) = 0$ since there is no stress when there is no shear action at some radius. So that is the answer to how you do an experiment to get γ.

If you use a very small gap so $\Delta R << R$, we get a simpler result 8.24.28. Again, you apply M, measure $\Delta\Omega$ and work out points on this graph. Here is a commercial Couette machine.



1. Purpose & characteristics

Rigchina produces a range of **true Couette coaxial cylinder rotational viscometers**. The test fluid is contained in the annular space or shear gap between the cylinders. Rotation of the outer cylinder at known velocities is accomplished through precision gearing. The viscous drag exerted by the fluid creates a torque on the **inner cylinder or bob**. This torque is transmitted to a precision spring where its deflection is measured and then related to the test conditions and instrument constants. This system permits the true simulation of most significant flow process conditions encountered in industrial processing.

Direct Indicating Viscometers combine accuracy with simplicity of design, and are recommended for evaluating materials that are Bingham plastics. These instruments are equipped with factory installed R1 Rotor Sleeve, B1 Bob, F1 Torsion Spring, and a stainless steel sample cup for testing according to American Petroleum Institute Specification RP 13B. Other rotor-bob combinations and/or torsion springs can be substituted to extend the torque measuring range or to increase the sensitivity of the torque measurement. Shear stress is read directly from a calibrated scale. Plastic viscosity and yield point of a fluid can be determined easily by making two simple subtractions from the observed data when the instrument is used with the R1-B1 combination and the standard F1 torsion spring.

It applies to viscosity measurement of drilling fluid in all the big oil fields, scientific research institutions and laboratories and it also applies to geology, chemical industry, coal, wine making and etc.

Appendix 8.1.

A. Polar coordinates.

B. Cylindrical and Sphericals. Finally Lai is forced to use the affine connection Γ, but he calls is $\Gamma_{\mathbf{ijk}}$ with all indices down. The rank 2 tensor of interest is T. He does things "his way" as I think I outlined in one place in tensor doc (in fact, right in the ∇v section!). The solution for cylindricals is stated by Lai in the table on page 504. As shown in separate notes, I think $\Gamma^{\mathbf{Lai}}$ might not be the same as the real affine connection because he is dealing with unit basis vectors. But OK, eventually I could verify everything in this appendix, but for now I will just let it be. It's not like Lai is likely to apply this to something critically important before the book ends.

Task: I could try to work up (∇T) as an extension to my tensor doc work, of course valid in all coordinate systems, then I would have to verify this against the massive Lai results on page 504. This could be a big job, but maybe not. I would rather do meta notes for Chapter 8 instead.