+My claimed errata for Lai p 431 (7.9.9) PhL 10.23.12

This was all handled in a Section 11 rewrite of frames doc, but I will keep these notes.

Something is strange here, and I may have a problem in frames doc that this reveals.

In frames doc (11.9a) I have this special case equation,

$$\dot{\mathbf{L}}^{(b)} = \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{b}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'] + \mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \qquad (11.9a)$$

where Frame S' is the one rotating and the rotation axis is though the origin of Frame S'. So Frame S' is just rotating about its own origin, and the boom **b**(t) could be doing whatever.

Is this the situation of Lai page 428? Remember that the G Rule itself does not tell you anything about the origin of rotation or the axis of rotation. So it appears that all Lai's work on pages 428-431 is completely general in this sense. In that case, I really should use my more general result

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{c}}'_S \times [\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] - [\boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S] \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S]$$
$$+ (\mathbf{r}'-\mathbf{c}') \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \; . \qquad (11.9)$$

Lai's first appearance of angular momentum is in (7.9.1) where **L** = **x** x **p** in some unspecified frame of reference. Lai says "moment about a fixed point" but he really means that point is the origin!

So let's start off by assuming that **c** = **0**. According to my Fig 11.3, this forces **c'** = **-b.** In this case, the above equation becomes

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(b)} + \dot{\mathbf{b}}_S \times [\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] - [\boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S] \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S]$$
$$+ (\mathbf{r}'+\mathbf{b}) \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \; .$$

STOP. My picture 11.3 is not very good because **(r-c)** x **v** is pretty much 0 as I have drawn it! ***** (maybe fix this soon)

I define $\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}'$ in (11.1) which seems reasonable. The **c'** label is a location in the S' system. So if you set **c** = 0, so **L** = **r** x **p** in Frame S, and *if you choose* that same point in space for your S' origin, then you are forced to have **c'** = **-b** and then $\mathbf{L}'^{(-b)} = (\mathbf{r}'+\mathbf{b}) \times \mathbf{v}'$.

But that might not be what we want to do! Maybe in S' we want to talk about $\mathbf{L}'^{(0)}$. I could then get a relationship between $\mathbf{L}^{(0)}$ and $\mathbf{L}'^{(0)}$, which are indeed two objects of possible interest. Someone working in Frame S' in general is not going to be interested in $\mathbf{L}'^{(-b)}$. How does this "fit in".

Let's first see where it leads, then later fit it in. This would allow us to ignore the last two equations in (11.2) and just set c = c' = 0 and be done with it. Then (11.9) becomes

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{b}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] + \mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \; .$$

or

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{b}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'] + \mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \ .$$

Question: Why does this look the same as (11.9a) wherein $\dot{\mathbf{L}}^{(b)}$ is a different animal??

My two Newton Laws would be

$\mathbf{N}^{(0)} = (d\mathbf{L}^{(0)}/dt)_S = \dot{\mathbf{L}}^{(0)}{}_S = \dot{\mathbf{L}}^{(0)}$ // true Newton's Rot Law in Frame S (11.10)

$\mathbf{N}'^{(0)} = (d\mathbf{L}'^{(0)}/dt)_{S'} = \dot{\mathbf{L}}'^{(0)}{}_{S'} = \dot{\mathbf{L}}'^{(0)}$ // fake Newton's Rot Law in Frame S' (11.10)

$\mathbf{N}'_{\mathbf{eff}}{}^{(0)} = \mathbf{N}^{(0)} + \mathbf{N}'^{(0)}{}_{\mathbf{fict}}$ // defining ficitious torque (11.12)

$\dot{\mathbf{L}}'^{(0)} = \dot{\mathbf{L}}^{(0)} + \mathbf{N}'^{(0)}{}_{\mathbf{fict}}$ // just from above 3 equations

$\mathbf{N}'^{(0)}{}_{\mathbf{fict}} = \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{L}}^{(0)}$

$= + \dot{\mathbf{b}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'] - \mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S]$

But this is the same as my (11.14a) and I have not assumed any special case.

STOP. Let's attempt a rewrite of Section 11 as per below:

(b) Expression of $\mathbf{L}^{(0)}$ and $\dot{\mathbf{L}}^{(0)}$ in terms of Frame S' objects

We replicate the picture presented in Section 1 (h),

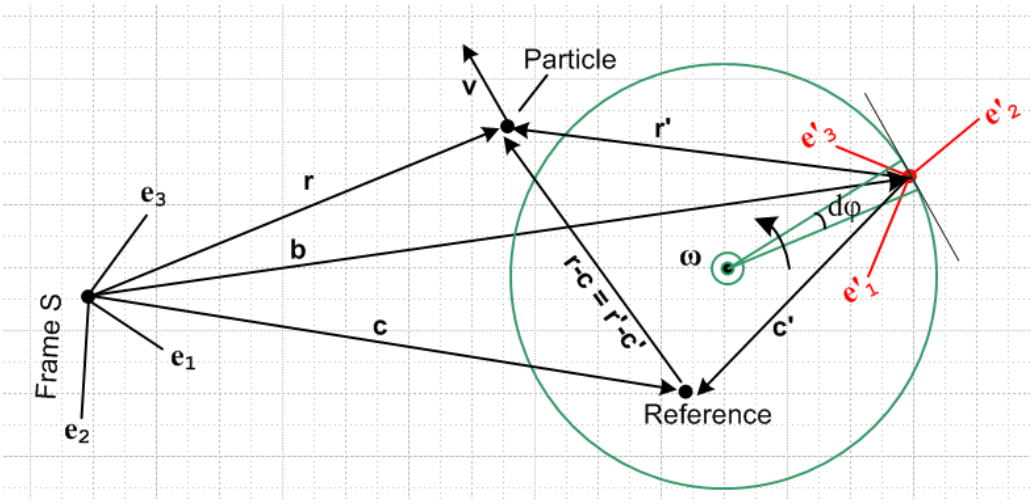

Fig 11.3

We again set mass m = 1 so momentum $\mathbf{p} = m\mathbf{v} = \mathbf{v}$. We assume that our objects of interest are $\mathbf{L}^{(0)}$ and $\mathbf{L}'^{(0)}$, so that $\mathbf{L}^{(0)}$ is defined relative to the origin of Frame S, and $\mathbf{L}'^{(0)}$ relative to the origin of Frame S'. We then have

$$\mathbf{L}^{(0)} = \mathbf{r} \times \mathbf{v}$$

$$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times \mathbf{a} - \mathbf{v} \times \mathbf{v} = \mathbf{r} \times \mathbf{a} \qquad // = \partial_S \mathbf{L}^{(0)}$$

$$\mathbf{L}'^{(0)} = \mathbf{r}' \times \mathbf{v}'$$

$$\dot{\mathbf{L}}'^{(0)} = \mathbf{r}' \times \mathbf{a}' - \mathbf{v}' \times \mathbf{v}' = \mathbf{r}' \times \mathbf{a}' \qquad // = \partial_{S'} \mathbf{L}'^{(0)} \qquad (1.36) \qquad (11.1)$$

We know from Sections 6 and 7 how $\mathbf{r},\mathbf{v},\mathbf{a}$ and $\mathbf{r}',\mathbf{v}',\mathbf{a}'$ are related,

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad (6.1)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad (6.6a)$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad (7.6a) \qquad (11.3)$$

So, we first write $\mathbf{L}^{(0)}$ in terms of Frame S' objects as follows,

$$\mathbf{L}^{(0)} = \mathbf{r} \times \mathbf{v} = (\mathbf{r}'+\mathbf{b}) \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S]$$

$$= \mathbf{r}' \times \mathbf{v}' + \mathbf{r}' \times [\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] + \mathbf{b} \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S]$$

$$= \mathbf{L}'^{(0)} + \mathbf{r}' \times [\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] + \mathbf{b} \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S]$$

Next,

$$\dot{\mathbf{L}}'^{(0)} = \mathbf{r}' \times \mathbf{a}' = (\mathbf{r}-\mathbf{b}) \times [\mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S]$$

$$= \mathbf{r} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{r}' \times [-\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S]$$

$$= \dot{\mathbf{L}}^{(0)} - \mathbf{b} \times \mathbf{a} + \mathbf{r}' \times [-\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S]$$

$$= \dot{\mathbf{L}}^{(0)} - \mathbf{b} \times \mathbf{a} + \mathbf{r}' \times \mathbf{F}'_{fict}$$

We then write,

$\mathbf{N}^{(0)} = (d\mathbf{L}^{(0)}/dt)_S = \dot{\mathbf{L}}^{(0)}{}_S = \dot{\mathbf{L}}^{(0)}$ // true Newton's Rot Law in Frame S (11.10)

$\mathbf{N}'^{(0)} = (d\mathbf{L}'^{(0)}/dt)_{S'} = \dot{\mathbf{L}}'^{(0)}{}_{S'} = \dot{\mathbf{L}}'^{(0)}$ // fake Newton's Rot Law in Frame S' (11.10)

$\mathbf{N}'_{eff}{}^{(0)} = \mathbf{N}^{(0)} + \mathbf{N}'^{(0)}{}_{fict}$ // defining ficitious torque (11.12)

$\dot{\mathbf{L}}'^{(0)} = \dot{\mathbf{L}}^{(0)} + \mathbf{N}'^{(0)}{}_{fict}$ // just from above 3 equations

$$\mathbf{N}'^{(0)}{}_{fict} = \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{L}}^{(0)}$$

$$= -\mathbf{b} \times \mathbf{a} + \mathbf{r}' \times [-\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S]$$

THIS then is what Lai p 431 is talking about, but he has assumed that $\mathbf{b} = 0$ which is his $\mathbf{R}_0 = 0$.

He would write $\mathbf{L}'^{(0)} = \mathbf{r}' \times \mathbf{v}'$ I am pretty sure, so that is why 0 is the reference in both cases.

Now, how can I isolate that first term to show it is really there?

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S$$

If we grossly set $\boldsymbol{\omega} = \dot{\boldsymbol{\omega}} = 0$ what would that mean? That means that the orientation of S' says fixed and its S' origin moves around in some manner described by b and its two derivatives. Then we have

$$\mathbf{a} = \mathbf{a}' + \ddot{\mathbf{b}}_S$$

$$\mathbf{N}'^{(0)}{}_{fict} = -\mathbf{b} \times \mathbf{a} - \mathbf{r}' \times \ddot{\mathbf{b}}_S$$

$$= -\mathbf{b} \times \mathbf{a}' - \mathbf{b} \times \ddot{\mathbf{b}}_S + \mathbf{r}' \times [-\ddot{\mathbf{b}}_S] = -\mathbf{b} \times \mathbf{a}' - (\mathbf{r}'+\mathbf{b}) \times \ddot{\mathbf{b}}_S = -\mathbf{b} \times \mathbf{a}' - \mathbf{r} \times \ddot{\mathbf{b}}_S$$

$= \dot{\mathbf{L}}'^{(0)} - \dot{\mathbf{L}}^{(0)}$

How interpret? Try this way instead

$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times \mathbf{a}$

$\dot{\mathbf{L}}'^{(0)} = \mathbf{r}' \times \mathbf{a}'$

$\mathbf{N}'^{(0)}_{\mathtt{fict}} = \mathbf{r}' \times \mathbf{a}' - \mathbf{r} \times \mathbf{a}$

$= \mathbf{r}' \times \mathbf{a}' - \mathbf{r} \times (\mathbf{a}' + \ddot{\mathbf{b}}_S) = (\mathbf{r}' - \mathbf{r}) \times \mathbf{a}' - \mathbf{r} \times \ddot{\mathbf{b}}_S = -\mathbf{b} \times \mathbf{a}' - \mathbf{r} \times \ddot{\mathbf{b}}_S$ agrees

So suppose a particle is at rest in Frame S'. Then $\mathbf{a}' = 0$ and we just have

$\mathbf{N}'^{(0)}_{\mathtt{fict}} = -\mathbf{r} \times \ddot{\mathbf{b}}_S$

$\dot{\mathbf{L}}'^{(0)} = 0$

$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times \mathbf{a} = \mathbf{r} \times [\mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] = \mathbf{r} \times \ddot{\mathbf{b}}_S$

So in this case of axis staying aligned, the S' torque is 0, but the S torque is not, and we have to account for this.

Once again. Let particle be at rest in frame S', then we have $\mathbf{a}' = \mathbf{v}' = 0$ and so

$\dot{\mathbf{L}}'^{(0)} = 0$

$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times \mathbf{a} = \mathbf{r} \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S]$

Suppose in addition the axes of S' don't rotate relative to S, then $\boldsymbol{\omega} = 0$ and we get

$\dot{\mathbf{L}}'^{(0)} = 0$

$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times \mathbf{a} = \mathbf{r} \times \ddot{\mathbf{b}}_S$

Then the particle in frame S has the $\dot{\mathbf{L}}^{(0)}$ shown because in Frame S it has $v = \dot{\mathbf{b}}_S$ and $\mathbf{L}^{(0)} = \mathbf{r} \times \mathbf{v}$.