Notes on Lai Section 3.29 page 131 PhL 2.15.12

Recall that F = (∇**x**) is the "deformation tensor" and

$d\mathbf{x} = F\, d\mathbf{X} = (\nabla \mathbf{x})\, d\mathbf{X}$ $\qquad F_{ij} = (\nabla \mathbf{x})_{ij} = \partial_j^{(\mathbf{X})} x_i$

tells how a "short vector" d**X** moves through a flow (from time $t=t_0$ to time t=t) and ends up as short vector d**x.** Certain other tensors are derived from F, such as $C = F^T F$ and $B = FF^T$ (the right and left Cauchy's). B^{-1} and C^{-1} are also of interest.

Overview (4 pages, 2/15/12) **1**
Raw Notes: **5**

Overview (4 pages, 2/15/12)

Chronology of where F = (∇f) appears in Lai:

In **Section 2.28** we first encounter the (∇**v**) notation where **v** is **v**(**r**), and we have then

$d\mathbf{v} = (\nabla \mathbf{v})d\mathbf{r}$ $\Rightarrow$ $dv_i = (\nabla \mathbf{v})_{ij}\, dr_j = (\partial_j v_i)\, dr_j = dr_j(\partial_j v_i) = (d\mathbf{r} \bullet \nabla)\, v_i$

$\Rightarrow$ $d\mathbf{v} = (d\mathbf{r} \bullet \nabla)\, \mathbf{v}$

Object (∇**v**) makes its *next* appearance in **Section 3.3** where we have for acceleration in the Eulerian view, where **a** = **a**(**x**,t),

$\mathbf{a} = \partial \mathbf{v}/\partial t + (\nabla \mathbf{v})\mathbf{v}$

Still later in **Section 3.7** the object F = (∇**u**) = (∇**x**) and d**x** = Fd**X** first appears. Other subjects intervene, and then F *reappears* in **Section 3.18.** We can reexamine the math here for d**x** = Fd**X**:

$dx_i = (\nabla \mathbf{x})_{ij}\, dX_j \equiv (\partial^{(\mathbf{X})}{}_j x_i)\, dX_j = dX_j(\partial^{(\mathbf{X})}{}_j x_i) = (d\mathbf{X} \bullet \nabla^{(\mathbf{X})})\, x_i(\mathbf{X},t)$

Isochronous dx. Something Lai mentions but does not emphasize is this: Normally one would write

$dx_i(\mathbf{x},t) = [\partial^{(\mathbf{X})}{}_j x_i(\mathbf{X},t)]\, dX_j + [\partial_t x_i(\mathbf{X},t)]\, dt$

but in the discussion of d**x** = Fd**X,** the dt term fails to appear. The reason is that d**x** is specified as a distance between two points at the same time t, so dt = 0. If you think of the Stakgold **x**,t cylinder, this d**x** is perpendicular to the time axis. Similarly, d**X** is the separation between the same two points at $t = t_0$, so it too is perpendicular to the time axis.

Sections 1,2,3,4,5 Overview

Lai wants to be able to use two different curvilinear coordinate systems x↔x' and X↔X'. That is, at the start of the flow, we use curvilinear coordinate system X', but at the end we use x' . The systems x and X are both Cartesian. So we have this little picture

time t time t_0
$x' \leftrightarrow x \leftarrow X \leftrightarrow X'$

In Cartesian coordinates, we start off with

$$F = \Sigma_{ij} [F^{(x,X)}]_{ij} \hat{\mathbf{e}}^{(x)}{}_n \hat{\mathbf{e}}^{(X)T}{}_j \qquad \text{where then} \qquad [F^{(x,X)}]_{ij} = \hat{\mathbf{e}}^{(x)T}{}_i \; F \; \hat{\mathbf{e}}^{(X)}{}_j$$

where

$\hat{\mathbf{e}}^{(x)}{}_n$ = Cartesian unit vectors for the x system = $\hat{x}_n$ that is to say $\hat{x}$, $\hat{y}$ and $\hat{z}$
$\hat{\mathbf{e}}^{(X)}{}_n$ = Cartesian unit vectors for the X system = $\hat{X}_n$ that is to say $\hat{X}$, $\hat{Y}$ and $\hat{Z}$

From tensor doc App F (b) I know that

$$\mathbf{u}_n = \Sigma_i \, S_n{}^i \, \mathbf{e}_i$$

$$\mathbf{e}_n = \Sigma_i \, R_n{}^i \, \mathbf{u}_i$$

$$\hat{\mathbf{e}}_n = (1/h'_n) \, \Sigma_i \, R_n{}^i \, \mathbf{u}_i = (1/h'_n) \, \Sigma_i \, S^i{}_n \, \mathbf{u}_i$$

In the detail section below, I write this for the x and X systems as

$$[\hat{\mathbf{e}}^{(x')}{}_n] = (1/h^{(x')}{}_n) \, \Sigma_i \, [S^{(x')}]^i{}_n \, \hat{\mathbf{i}}$$
$$[\hat{\mathbf{e}}^{(X')}{}_n] = (1/h^{(X')}{}_n) \, \Sigma_i \, [R^{(X')}]^i{}_n \, \hat{\mathbf{I}}$$

The expansion Lai wants to do is this one:

$$F = \Sigma_{ij} [F^{(x',X')}]_{ij} \hat{\mathbf{e}}^{(x')}{}_n \hat{\mathbf{e}}^{(X')T}{}_j \qquad \text{where then} \qquad [F^{(x',X')}]_{ij} = \hat{\mathbf{e}}^{(x')T}{}_i \; F \; \hat{\mathbf{e}}^{(X')}{}_j$$

I now know things like $\hat{\mathbf{e}}^{(X')}{}_j = \hat{\mathbf{e}}^{(X')j}$ since X space is Cartesian and so up and down indices on this F object are all the same. I also know that $[F^{(x',X')}]_{ij}$ is just a set of coefficients and these coefficients are not the components of a tensor. Even if $F = (\nabla \mathbf{x})$ *were* a tensor, $[F^{(x',X')}]_{ij}$ would not be tensor components due to the hats. See tensor doc end of Section E (b) for more detail on this observation.

The next step I do below is to expand the two "short vectors" (isochronous!) in the usual way,

$$d\mathbf{x} = \Sigma_n \, dx'^n \; h^{(x')}{}_n \, \hat{\mathbf{e}}^{(x')}{}_n$$
$$d\mathbf{X} = \Sigma_n \, dX'^n \; h^{(X')}{}_n \, \hat{\mathbf{e}}^{(X')}{}_n$$

If I insert these expansions into d**x** = F d**X** and then apply $\hat{\mathbf{e}}^{(x')}{}_m{}^T$ from the left to both sides (and here is where I use orthonormality of the $\hat{\mathbf{e}}^{(x')}{}_m$) I get

$$dx'^m = [h^{(x')}{}_m]^{-1} \Sigma_n dX'^n \, h^{(X')}{}_n [\, \hat{\mathbf{e}}^{(x')}{}_m{}^T F \, \hat{\mathbf{e}}^{(X')}{}_n]$$

in which my "object of interest" $[\, F^{(x',X')}]_{mn}$ conveniently appears on the RHS. The problem is then to solve the above equation for this object. Inserting the following into the above,

$$dx'^m = \Sigma_n (\partial x'^m/\partial X'^n) \, dX'^n \ ,$$

and then matching dX'^n coefficients on both sides, I find that

$$[\, F^{(x',X')}]_{mn} = \hat{\mathbf{e}}^{(x')}{}_m{}^T F \, \hat{\mathbf{e}}^{(X')}{}_n = [h^{(x')}{}_m / h^{(X')}{}_n] \, (\partial x'^m/\partial X'^n)$$

This then is the big result at the **end of section 5**.

Then in **section 5a** I show that

$$F \, \hat{\mathbf{e}}^{(X')}{}_n = \Sigma_m [h^{(x')}{}_m / h^{(X')}{}_n] \, (\partial x'^m/\partial X'^n)] \, \hat{\mathbf{e}}^{(x')}{}_m$$

$$F^T \, \hat{\mathbf{e}}^{(x')}{}_n = \Sigma_m [h^{(x')}{}_n / h^{(X')}{}_m] \, (\partial x'^n/\partial X'^m)] \, \hat{\mathbf{e}}^{(X')}{}_m$$

which are used below in computations of B and C.

In **Section 6** I consider the case where both systems x' and X' are cylindrical and I obtain a matrix for the object $[\, F^{(x',X')}]_{mn}$ using Maple which agrees with Lai's matrix p 132 3.29.12. This is the first part of my Maple program file **lai 3_29 cyl cyl.mws** .

In **Section 7** I then move on to compute $\mathbf{B = FF^T}$ (the left Cauchy) using those section 5a results above, and here is what I get (and of course this is symmetric, so it has only 6 unique elements)

$$[\, B^{(x',x')}]_{mn} = \Sigma_i [\{ \, [h^{(x')}{}_n / h^{(X')}{}_i] \, [h^{(x')}{}_m / h^{(X')}{}_i] \, (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

I then show another expression which is

$$[\, B^{(x',x')}]_{mn} = [F^{(x',X')} \, F^{(x',X')T}]_{mn}$$

which says you could naively compute $B = FF^T$ in this basis! But in the Maple program, I implement the long expression shown above and I get for B the same thing that Lai does on page 131.

In **Section 8** I want to find $\mathbf{B^{-1}}$. But $B^{-1} = F^{-1,T} F^{-1}$, so I first have to compute $\mathbf{F^{-1} \equiv G}$ and I find rather obviously that

$$F^{-1(X',x')}{}_{mn} = [\,\hat{e}^{(X')}{}_m{}^T\, F^{-1}\hat{e}^{(x')}{}_n] \;=\; [h^{(X')}{}_m \,/\, h^{(x')}{}_n]\,(\partial X'^m/\partial x'^n) \;=\; G^{(X',x')}{}_{mn}$$

which is similar to $[\,F^{(x',X')}]_{mn}$ stated above a few inches. Then I use the "easy method" and write

$$[B^{-1}]^{(x',x')} = G^{T(X',x')}\, G^{(X',x')}$$

and this produces a Maple result which replicates Lai's results on pages 134-135 .

In **Section 9** I then go after **C = F^{T}F** and I use the "easy method" in Maple to get

$$[\,C^{(X',X')}]_{mn} = \hat{e}^{(X')}{}_m{}^T\, F^T\; \hat{e}^{(x')}{}_i\, \hat{e}^{(x')T}{}_i\, F\, \hat{e}^{(X')}{}_n]$$

$$= [\,F^{(x',X')T}]_{mi}\,[\,F^{(x',X')}]_{in} = [\,F^{(x',X')T}\, F^{(x',X')}]_{mn}$$

and Maple then generates a C matrix which agrees with Lai page 135.

Lai goes on to do C^{-1} but I did not bother to do this in the Maple program.

I will now directly quote my last two sections:

Section 10 Conclusion. To compute all these matrices in any combination of bases, you need to first compute these two fundamental matrices in the mixed basis as shown:

$$[\,F^{(x',X')}]_{mn} = [\,\hat{e}^{(x')}{}_m{}^T\, F\, \hat{e}^{(X')}{}_n] = [h^{(x')}{}_m / h^{(X')}{}_n]\,(\partial x'^m/\partial X'^n)$$

$$F^{-1(X',x')}{}_{mn} = [\,\hat{e}^{(X')}{}_m{}^T\, F^{-1}\hat{e}^{(x')}{}_n] \;=\; [h^{(X')}{}_m \,/\, h^{(x')}{}_n]\,(\partial X'^m/\partial x'^n) \qquad // = G$$

Once you do that, you compute the other matrices in the obvious manner using simple matrix multiplication and transposing in Maple:

$B = FF^T$
$B^{-1} = F^{-1,T}\, F^{-1}$
$C = F^TF$
$C^{-1} = F^{-1}\, F^{-1,T}$

In Maple the main work is just to give all 6 coordinates appropriate names, and get the six scale factors entered properly.

Section 11. <u>The case X = Cartesian and x = cylinder</u>. I created a new Maple file lai 3_29 cart cyl and it gives all the right results.

Section 12. <u>The case X = spherical and x = spherical.</u> Done, just another Maple file.

So I did all the cases Lai did, and I could no doubt put check marks everywhere if I wanted.

Raw Notes:

1. Concerning the systems x and X. I want both these systems to be Cartesian, but they are *different* Cartesian systems with separate basis vectors. So:

$\hat{\mathbf{e}}^{(x)}{}_n$ = Cartesian unit vectors for the x system = $\hat{x}_n$ that is to say $\hat{x}$, $\hat{y}$ and $\hat{z}$

$\hat{\mathbf{e}}^{(X)}{}_n$ = Cartesian unit vectors for the X system = $\hat{X}_n$ that is to say $\hat{X}$, $\hat{Y}$ and $\hat{Z}$

These are the basis vectors that appear in the "normal" expansion of the F tensor.

$$F = \Sigma_{ij} [F^{(x,X)}]_{ij} \hat{\mathbf{e}}^{(x)}{}_n \hat{\mathbf{e}}^{(X)T}{}_j \qquad (*) \qquad // = \Sigma_{ij} [F^{(x,X)}]_{ij} \hat{x}_n \hat{X}^T{}_j$$

In my tensor doc, I never used two different coordinate systems in an expansion like this, by the way, I might want to think about that more. I would like to be able to say that

$$[F^{(x,X)}]_{ij} = \hat{\mathbf{e}}^{(x)T}{}_i \; F \; \hat{\mathbf{e}}^{(X)}{}_j \qquad (**)$$

since X is "on the right" and is the earlier time. If I close (*) from the right with $\hat{\mathbf{e}}^{(X)}{}_j$ and from the left with $\hat{\mathbf{e}}^{(x)}{}_i{}^T$, then we get (**), so I have things in the right ordering in (*). I know that in this basic pair of systems we have that

$F = (\nabla \mathbf{x})$ and $F_{ij} = (\nabla \mathbf{x})_{ij} = \partial_j{}^{(X)} x_i$ eg $F_{12} = \partial x/\partial Y$.

Comment: We are certainly allowed to consider using the *same* set of Cartesian unit vectors for the x and X systems. That is, we could select $\hat{\mathbf{e}}^{(x)}{}_n \equiv \hat{\mathbf{e}}^{(X)}{}_n$ for n = 1,2,3. *In this case* we could conclude that

$$[F^{(x,x)}]_{ij} = [F^{(x,X)}]_{ij} = [F^{(x,x)}]_{ij} = [F^{(X,x)}]_{ij} = \partial_j{}^{(X)} x_i = F_{ij}$$

But when x and X have different unit vectors, these things will not be equal. They will be related by some finite rotation matrix which brings the vectors unit vectors into alignment.

2. The system x' . Since we have x' ← x, things are in my standard order for thinking about curvilinear coordinates x'. In my tensor doc, I write the basis vectors for the x' system as $\hat{\mathbf{e}}_n$ these being the unit tangent base vectors drawn in x-space. But here I need to refer to these as $\hat{\mathbf{e}}^{(x')}{}_n$ so I know which of the four coordinate systems they belong to. Perhaps this is a weakness of my tensor doc notation where the prime indicated in which space the vectors exist. In any event, we then have

$$[\hat{\mathbf{e}}^{(x')}{}_n]^i = [\hat{\mathbf{e}}_n]^i = [\mathbf{e}_n]^i / h'_n = S^i{}_n / h'_n = R_n{}^i / h'_n$$

But we need to label things more carefully, so write

$$[\hat{\mathbf{e}}^{(x')}{}_n]^i = [S^{(x')}]^i{}_n \;/\, h^{(x')}{}_n = [R^{(x')}]_n{}^i \;/\, h^{(x')}{}_n$$

where we now label R and X and scale factors h with the appropriate labels.

Perhaps the information presented on the two lines above is not quite what we really need to know. What will be more useful is to know how to expand the $[\hat{\mathbf{e}}^{(x')}{}_n]$ in terms of the $\hat{\mathbf{e}}^{(x)}{}_n \equiv \hat{\mathbf{n}}$. We are assuming of course that x' and X' are orthogonal coordinate systems, so this is going to be a simple rotation. I addressed this same problem recently in another doc, but let's try it here again from scratch. Let's go back to the simple tangent base vector and write

$$\mathbf{e}_n = \Sigma_i a_{ni}\, \hat{\mathbf{i}} \qquad \text{see want to learn the } a_i \text{ !}$$

$$\mathbf{e}_n \bullet \hat{\mathbf{j}} = \Sigma_i a_{ni}\, \hat{\mathbf{i}} \bullet \hat{\mathbf{j}} = \Sigma_i a_{ni}\, \delta_{ij} = a_{nj}$$

But we know that from item 2 above that, in Cartesian x-space,

$$\mathbf{e}_n \bullet \hat{\mathbf{j}} = (\mathbf{e}_n)^j = S^j{}_n = R_n{}^j = a_{nj}$$

and therefore we have shown that

$$\mathbf{e}_n = \Sigma_i\, S^i{}_n\, \hat{\mathbf{i}} = \Sigma_i\, R_n{}^i\, \hat{\mathbf{i}}$$

$$\hat{\mathbf{e}}_n = \Sigma_i\, S^i{}_n\, \hat{\mathbf{i}} \,/\, h'_n = \Sigma_i\, R_n{}^i\, \hat{\mathbf{i}} \,/\, h'_n$$

So we can now translate this into our fancier notation to write

$$[\hat{\mathbf{e}}^{(x')}{}_n] = (1/\, h^{(x')}{}_n)\, \Sigma_i\, [S^{(x')}]^i{}_n\, \hat{\mathbf{i}}$$

and we can leave it at that. I can then use $\hat{\mathbf{I}}$ for the X unit vectors.

3. <u>The system X'.</u> Since we have X ← X', we have to talk here about the "inverse" transformation. But I know that means swapping R and S, so I think this is the right answer for this case

$$[\hat{\mathbf{e}}^{(X')}{}_n]^i = [R^{(X')}]^i{}_n \;/\, h^{(X')}{}_n = [S^{(X')}]_n{}^i \;/\, h^{(X')}{}_n$$

where you have to pay attention to whether the parenthetical letter is upper or lower case! I then expect the expansion corresponding to the above to be

$$[\hat{\mathbf{e}}^{(X')}{}_n] = (1/\, h^{(X')}{}_n)\, \Sigma_i\, [R^{(X')}]^i{}_n\, \hat{\mathbf{I}}$$

4. <u>Here is the thing we want to compute:</u>

$$[\, F^{(x',X')}]_{ij} = \hat{\mathbf{e}}^{(x')T}{}_i\; F\; \hat{\mathbf{e}}^{(X')}{}_j$$

and this is the most general case, as desired.

One way to write the above is then as follows:

$$[\,F^{(x',X')}]_{ij} \;=\; \hat{\mathbf{e}}^{(x')T}{}_i \;F\; \hat{\mathbf{e}}^{(X')}{}_j$$

$$=\; \{\;(1/h^{(x')}{}_i)\,\Sigma_k\,[S^{(x')}]^k{}_i\;\hat{\mathbf{k}}^T\;\}\;F\;\{(1/h^{(X')}{}_j)\,\Sigma_s\,[R^{(X')}]^s{}_j\;\hat{\mathbf{S}}\}$$

$$=\; \{\;(1/h^{(x')}{}_i)\,\Sigma_k\,[S^{(x')}]^k{}_i\;\}\;\{(1/h^{(X')}{}_j)\,\Sigma_s\,[R^{(X')}]^s{}_j\;\hat{\mathbf{k}}^T\;F\;\hat{\mathbf{S}}\}$$

$$=\; \{\;(1/h^{(x')}{}_i)\,\Sigma_k\,[S^{(x')}]^k{}_i\;\}\;\{(1/h^{(X')}{}_j)\,\Sigma_s\,[R^{(X')}]^s{}_j\;F_{ks}$$

I think this answer is correct, but it is not in the form that we want. The form here is a linear combination of the Cartesian F_{ks} and of course it is a big mess. We want a result directly in terms of x' and X' derivatives!

5. Computing the differential d**x** . We know from our tensor doc vector expansions that we can write

$$d\mathbf{x} \;=\Sigma_n\, dx'^n\;\mathbf{e}_n \;=\Sigma_n\, dx'^n\;h'_n\;\hat{\mathbf{e}}_n \qquad\qquad \text{// valid since d}\mathbf{x}\text{ is here a "short vector"}$$

which we translate into

$$d\mathbf{x} \;=\Sigma_n\, dx'^n\;h^{(x')}{}_n\;\hat{\mathbf{e}}^{(x')}{}_n$$

Similarly it must be that

$$d\mathbf{X} \;=\Sigma_n\, dX'^n\;h^{(X')}{}_n\;\hat{\mathbf{e}}^{(X')}{}_n$$

On the other hand we know that

$$d\mathbf{x} = F\;d\mathbf{X}$$

Inserting the above, we get

$$\Sigma_n\, dx'^n\;h^{(x')}{}_n\;\hat{\mathbf{e}}^{(x')}{}_n \;=F\;\Sigma_n\, dX'^n\;h^{(X')}{}_n\;\hat{\mathbf{e}}^{(X')}{}_n$$

Now apply $\hat{\mathbf{e}}^{(x')}{}_m{}^T$ from the left to both sides to get

$$\Sigma_n\, dx'^n\;h^{(x')}{}_n\;\hat{\mathbf{e}}^{(x')}{}_m{}^T\,\hat{\mathbf{e}}^{(x')}{}_n \;=\; \hat{\mathbf{e}}^{(x')}{}_m{}^T\,F\,\Sigma_n\, dX'^n\;h^{(X')}{}_n\;\hat{\mathbf{e}}^{(X')}{}_n$$

$$\Sigma_n\, dx'^n\;h^{(x')}{}_n\;\delta_{m,n} \;=\Sigma_n\, dX'^n\;h^{(X')}{}_n\;\hat{\mathbf{e}}^{(x')}{}_m{}^T\,F\,\hat{\mathbf{e}}^{(X')}{}_n$$

$$dx'^m\;h^{(x')}{}_m \;=\; \Sigma_n\, dX'^n\;h^{(X')}{}_n\,[\;\hat{\mathbf{e}}^{(x')}{}_m{}^T\,F\,\hat{\mathbf{e}}^{(X')}{}_n]$$

$$dx'^m = [h^{(x')}{}_m]^{-1} \Sigma_n dX'^n h^{(X')}{}_n [\hat{e}^{(x')}{}_m{}^T F \hat{e}^{(X')}{}_n]$$

On the other hand, we know that $\mathbf{x}' = \mathbf{x}'(\mathbf{X}')$ so that

$$dx'^m = \Sigma_n (\partial x'^m/\partial X'^n)\, dX'^n$$

Comparing the above we get

$$(\partial x'^m/\partial X'^n) = [h^{(x')}{}_m]^{-1} h^{(X')}{}_n [\hat{e}^{(x')}{}_m{}^T F \hat{e}^{(X')}{}_n]$$

or

$$[F^{(x',X')}]_{mn} = \hat{e}^{(x')}{}_m{}^T F \hat{e}^{(X')}{}_n = [h^{(x')}{}_m / h^{(X')}{}_n] (\partial x'^m/\partial X'^n)$$

and (finally) this is the result I have been seeking!

<u>5a. Some expansions of F acting on unit vectors.</u>

A. Consider

$$F \hat{e}^{(X')}{}_n = \Sigma_i a_{ni} \hat{e}^{(x')}{}_i = \Sigma_m a_{nm} \hat{e}^{(x')}{}_m$$

Apply $\hat{e}^{(x')T}{}_m$ to both sides to get

$$\hat{e}^{(x')T}{}_m F \hat{e}^{(X')}{}_n = \Sigma_i a_{ni} \hat{e}^{(x')T}{}_m \hat{e}^{(x')}{}_i = a_{nm}$$

Therefore

$$F \hat{e}^{(X')}{}_n = \Sigma_m a_{nm} \hat{e}^{(x')}{}_m = \Sigma_m [\hat{e}^{(x')T}{}_m F \hat{e}^{(X')}{}_n] \hat{e}^{(x')}{}_m$$

$$= \Sigma_m [h^{(x')}{}_m / h^{(X')}{}_n] (\partial x'^m/\partial X'^n)] \hat{e}^{(x')}{}_m$$

B. Consider

$$F^T \hat{e}^{(x')}{}_n = \Sigma_i b_{ni} \hat{e}^{(X')}{}_i = \Sigma_m b_{nm} \hat{e}^{(X')}{}_m$$

Apply $\hat{e}^{(X')T}{}_m$ to both sides to get

$$\hat{e}^{(X')T}{}_m F^T \hat{e}^{(x')}{}_n = \Sigma_i b_{ni} \hat{e}^{(X')T}{}_m \hat{e}^{(X')}{}_i = b_{nm}$$

Therefore

$$F^T \hat{e}^{(x')}{}_n = \Sigma_m b_{nm} \hat{e}^{(X')}{}_m = \Sigma_m [\hat{e}^{(X')T}{}_m F^T \hat{e}^{(x')}{}_n] \hat{e}^{(X')}{}_m$$

$$= \Sigma_m \, [\hat{\mathbf{e}}^{(x')T}{}_n \, F \, \hat{\mathbf{e}}^{(X')}{}_m] \, \hat{\mathbf{e}}^{(X')}{}_m$$

$$= \Sigma_m \, [h^{(x')}{}_n / h^{(X')}{}_m] \, (\partial x'^n / \partial X'^m)] \, \hat{\mathbf{e}}^{(X')}{}_m$$

So we can summarize now our two little expansions:

$$F \, \hat{\mathbf{e}}^{(X')}{}_n = \Sigma_m \, [h^{(x')}{}_m / h^{(X')}{}_n] \, (\partial x'^m / \partial X'^n)] \, \hat{\mathbf{e}}^{(x')}{}_m$$

$$F^T \, \hat{\mathbf{e}}^{(x')}{}_n = \Sigma_m \, [h^{(x')}{}_n / h^{(X')}{}_m] \, (\partial x'^n / \partial X'^m)] \, \hat{\mathbf{e}}^{(X')}{}_m$$

Notice that if one set of basis vectors appears on the left, the other appears on the right. Then we avoid the need for any explicit rotation matrices.

6. Application: Suppose both x' and X' systems are cylindricals with r,θ,z and R,Θ,Z. We know that

$$h^{(x')}{}_1 = h^{(x')}{}_r = 1$$
$$h^{(x')}{}_2 = h^{(x')}{}_\theta = r$$
$$h^{(x')}{}_3 = h^{(x')}{}_\varphi = 1$$

$$h^{(X')}{}_1 = h^{(X')}{}_R = 1$$
$$h^{(X')}{}_2 = h^{(X')}{}_\Theta = R$$
$$h^{(X')}{}_3 = h^{(X')}{}_Z = 1$$

Then we have

$F_{11} = (\partial r/\partial R)$ $F_{12} = (1/R)\,(\partial r/\partial \Theta)$ $F_{13} = (\partial r/\partial Z)$

$F_{21} = r(\partial \theta/\partial R)$ $F_{22} = (r/R)\,(\partial \theta/\partial \Theta)$ $F_{23} = r\,(\partial \theta/\partial Z)$

$F_{31} = (\partial z/\partial R)$ $F_{32} = (1/R)\,(\partial z/\partial \Theta)$ $F_{33} = (\partial z/\partial Z)$

If we now replace $R = r_0$ and $\Theta = \theta_0$ and $Z = z_0$, the above 3x3 matrix agrees with Lai p 132 3.29.12. So now I know how to express the F matrix in any orthogonal coordinate system pair I want. The result is really quite simple. // I then did a Maple program to do this and got

$$F := \begin{bmatrix} \frac{\partial}{\partial r_0} r & \frac{\frac{\partial}{\partial \theta_0} r}{r_0} & \frac{\partial}{\partial z_0} r \\ r\left(\frac{\partial}{\partial r_0}\theta\right) & \frac{r\left(\frac{\partial}{\partial \theta_0}\theta\right)}{r_0} & r\left(\frac{\partial}{\partial z_0}\theta\right) \\ \frac{\partial}{\partial r_0} z & \frac{\frac{\partial}{\partial \theta_0} z}{r_0} & \frac{\partial}{\partial z_0} z \end{bmatrix}$$

I just noted the two colors which seems odd.

7. Try matrix $B = FF^T$. We can write

$$[B^{(x',x')}]_{mn} = [\hat{e}^{(x')}{}_m{}^T B \hat{e}^{(x')}{}_n] = [\hat{e}^{(x')}{}_m{}^T F F^T \hat{e}^{(x')}{}_n]$$

Now quote expansion derived in 5a (B) above,

$$F^T \hat{e}^{(x')}{}_n = \Sigma_m [h^{(x')}{}_n / h^{(X')}{}_m] (\partial x'^n/\partial X'^m)] \hat{e}^{(X')}{}_m = \Sigma_i [h^{(x')}{}_n / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)] \hat{e}^{(X')}{}_i$$

Install this to get

$$[B^{(x',x')}]_{mn} = [\hat{e}^{(x')}{}_m{}^T F F^T \hat{e}^{(x')}{}_n]$$

$$= [\hat{e}^{(x')}{}_m{}^T F \{ \Sigma_i [h^{(x')}{}_n / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)] \hat{e}^{(X')}{}_i]$$

$$= [\hat{e}^{(x')}{}_m{}^T \{ \Sigma_i [h^{(x')}{}_n / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)] F \hat{e}^{(X')}{}_i]$$

Now quote the expansion derived in 5a (A) above

$$F \hat{e}^{(X')}{}_n = \Sigma_m [h^{(x')}{}_m / h^{(X')}{}_n] (\partial x'^m/\partial X'^n)] \hat{e}^{(x')}{}_m = \Sigma_j [h^{(x')}{}_j / h^{(X')}{}_n] (\partial x'^j/\partial X'^n)] \hat{e}^{(x')}{}_j$$

or

$$F \hat{e}^{(X')}{}_i = \Sigma_j [h^{(x')}{}_j / h^{(X')}{}_i] (\partial x'^j/\partial X'^i)] \hat{e}^{(x')}{}_j$$

Then the above becomes

$$[B^{(x',x')}]_{mn} = [\hat{e}^{(x')}{}_m{}^T \{ \Sigma_i [h^{(x')}{}_n / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)] F \hat{e}^{(X')}{}_i]$$

$$= [\hat{e}^{(x')}{}_m{}^T \{ \Sigma_i [h^{(x')}{}_n / h^{(X')}{}_i] (\partial x'^n/\partial X'^i)] \{ \Sigma_j [h^{(x')}{}_j / h^{(X')}{}_i] (\partial x'^j/\partial X'^i)] \hat{e}^{(x')}{}_j \}]$$

$$= \Sigma_{ij}[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)]\{\ [h^{(x')}{}_j / h^{(X')}{}_i]\ (\partial x'^j/\partial X'^i)]\ \hat{e}^{(x')}{}_m{}^T\ \hat{e}^{(x')}{}_j\ \}]$$

$$= \Sigma_{ij}[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)]\{\ [h^{(x')}{}_j / h^{(X')}{}_i]\ (\partial x'^j/\partial X'^i)]\ \delta_{m,j}]$$

$$= \Sigma_i[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)]\{\ [h^{(x')}{}_m / h^{(X')}{}_i]\ (\partial x'^m/\partial X'^i)]$$

$$= \Sigma_i[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ [h^{(x')}{}_m / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

So here is our big result:

$$[\ B^{(x',x')}]_{mn} = \Sigma_i[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ [h^{(x')}{}_m / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

For example

$$[\ B^{(x',x')}]_{mn} = \Sigma_i[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ [h^{(x')}{}_m / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

$$[\ B^{(x',x')}]_{11} = \Sigma_i[\{\ 1/ h^{(X')}{}_i]\ [1/ h^{(X')}{}_i]\ (\partial x'^1/\partial X'^i)(\partial x'^1/\partial X'^i)$$

$$= (\partial x'^1/\partial X'^1)^2 + (1/R)^2\ (\partial x'^1/\partial X'^2)^2 + (\partial x'^1/\partial X'^3)^2$$

$$= (\partial r/\partial R)^2 + (1/R)^2\ (\partial r/\partial \Theta)^2 + (\partial r/\partial Z)^2 \qquad \text{// agrees with 3.29.19}$$

$$[\ B^{(x',x')}]_{mn} = \Sigma_i[\{\ [h^{(x')}{}_n / h^{(X')}{}_i]\ [h^{(x')}{}_m / h^{(X')}{}_i]\ (\partial x'^n/\partial X'^i)(\partial x'^m/\partial X'^i)$$

$$[\ B^{(x',x')}]_{12} = \Sigma_i[\{\ [h^{(x')}{}_2 / h^{(X')}{}_i]\ [h^{(x')}{}_1 / h^{(X')}{}_i]\ (\partial x'^2/\partial X'^i)(\partial x'^1/\partial X'^i)$$

$$= r(\partial x'^2/\partial X'^1)(\partial x'^1/\partial X'^1) + (r/R^2)\ (\partial x'^2/\partial X'^2)(\partial x'^1/\partial X'^2) + r\ (\partial x'^2/\partial X'^3)(\partial x'^1/\partial X'^3)$$

$$= r\ (\partial\theta/\partial R)(\partial r/\partial R) + (r/R^2)\ (\partial\theta/\partial\Theta)\ (\partial r/\partial\Theta) + r\ (\partial\theta/\partial Z)\ (\partial r/\partial Z) \quad \text{// agrees with 3.29.20}$$

Here is another derivation:

$$[\ B^{(x',x')}]_{mn} = [\ \hat{e}^{(x')}{}_m{}^T\ B\ \hat{e}^{(x')}{}_n] \qquad = [\ \hat{e}^{(x')}{}_m{}^T\ F\ F^T\ \hat{e}^{(x')}{}_n]$$

Now insert completeness,

$$= [\ \hat{e}^{(x')}{}_m{}^T\ F\ \hat{e}^{(X')}{}_i\ \hat{e}^{(X')T}{}_i\ F^T\ \hat{e}^{(x')}{}_n]$$

$$= [\hat{e}^{(x')}{}_m{}^T\ F\ \hat{e}^{(X')}{}_i][\ \hat{e}^{(X')T}{}_i\ F^T\ \hat{e}^{(x')}{}_n]$$

$$= \Sigma_i[\hat{e}^{(x')}{}_m{}^T\ F\ \hat{e}^{(X')}{}_i][\ \hat{e}^{(x')T}{}_n\ F\ \hat{e}^{(X')}{}_i]$$

$$= \Sigma_i F^{(x',X')}{}_{mi}\ F^{(x',X')}{}_{ni}$$

$$= \Sigma_i F^{(x',X')}{}_{mi}\ F^{(x',X')T}{}_{in}$$

$= [F^{(x',X')}\, F^{(x',X')T}]_{mn}$

So this is " B with respect to x ".

If I throw my formula into Maple, here is the result for matrix $B^{(x',x')}$:

```
B := matrix(3,3,B_);
```

$$B := \begin{bmatrix} \left(\frac{\partial}{\partial r_0} r\right)^2 + \frac{\left(\frac{\partial}{\partial \theta_0} r\right)^2}{r_0^2} + \left(\frac{\partial}{\partial z_0} r\right)^2 & r\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial r_0} r\right) + \frac{r\left(\frac{\partial}{\partial \theta_0}\theta\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0^2} + r\left(\frac{\partial}{\partial z_0}\theta\right)\left(\frac{\partial}{\partial z_0} r\right) & \left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial r_0} r\right) + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0^2} + \left(\frac{\partial}{\partial z_0} z\right)\left(\frac{\partial}{\partial z_0} r\right) \\ r\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial r_0} r\right) + \frac{r\left(\frac{\partial}{\partial \theta_0}\theta\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0^2} + r\left(\frac{\partial}{\partial z_0}\theta\right)\left(\frac{\partial}{\partial z_0} r\right) & r^2\left(\frac{\partial}{\partial r_0}\theta\right)^2 + \frac{r^2\left(\frac{\partial}{\partial \theta_0}\theta\right)^2}{r_0^2} + r^2\left(\frac{\partial}{\partial z_0}\theta\right)^2 & r\left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial r_0}\theta\right) + \frac{r\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial \theta_0}\theta\right)}{r_0^2} + r\left(\frac{\partial}{\partial z_0} z\right)\left(\frac{\partial}{\partial z_0}\theta\right) \\ \left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial r_0} r\right) + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0^2} + \left(\frac{\partial}{\partial z_0} z\right)\left(\frac{\partial}{\partial z_0} r\right) & r\left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial r_0}\theta\right) + \frac{r\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial \theta_0}\theta\right)}{r_0^2} + r\left(\frac{\partial}{\partial z_0} z\right)\left(\frac{\partial}{\partial z_0}\theta\right) & \left(\frac{\partial}{\partial r_0} z\right)^2 + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)^2}{r_0^2} + \left(\frac{\partial}{\partial z_0} z\right)^2 \end{bmatrix}$$

$B = FF^T$ is symmetric, so one need only state 6 of the above entries, which is what Lai does. The 6 entries above agree with Lai's quoted results on page 133.

8. Try matrix B^{-1} with respect to x.

Remember that $B = FF^T$ where $d\mathbf{x} = F\, d\mathbf{X}$. Obviously $B^{-1} = F^{-1,T}\, F^{-1}$. To make a connection with this F^{-1} matrix, we can consider the reverse blob flow with $d\mathbf{X} = F^{-1} d\mathbf{x}$. We found in section 5 above that

$$d\mathbf{x} = \Sigma_n\, dx'^n\, h^{(x')}{}_n\, \hat{\mathbf{e}}^{(x')}{}_n$$
$$d\mathbf{X} = \Sigma_n\, dX'^n\, h^{(X')}{}_n\, \hat{\mathbf{e}}^{(X')}{}_n$$

which we can insert to get

$$\Sigma_n\, dX'^n\, h^{(X')}{}_n\, \hat{\mathbf{e}}^{(X')}{}_n = F^{-1}\, \{\, \Sigma_n\, dx'^n\, h^{(x')}{}_n\, \hat{\mathbf{e}}^{(x')}{}_n \,\}$$

Apply $\hat{\mathbf{e}}^{(X')}{}_m{}^T$ from the left to both sides to get

$$dX'^m\, h^{(X')}{}_m = \Sigma_n\, dx'^n\, h^{(x')}{}_n\, [\, \hat{\mathbf{e}}^{(X')}{}_m{}^T\, F^{-1} \hat{\mathbf{e}}^{(x')}{}_n]$$

or

$$dX'^m = [h^{(X')}{}_m]^{-1}\, \Sigma_n\, dx'^n\, h^{(x')}{}_n\, [\, \hat{\mathbf{e}}^{(X')}{}_m{}^T\, F^{-1} \hat{\mathbf{e}}^{(x')}{}_n]$$

This agrees with the earlier section 5 result just doing $x \leftrightarrow X$ and $F \rightarrow F^{-1}$.

On the other hand, we know that $\mathbf{X}' = \mathbf{X}'(\mathbf{x}')$ so that

$$dX'^m = \Sigma_n\, (\partial X'^m / \partial x'^n)\, dx'^n$$

Comparing the above we get

$$(\partial X'^m/\partial x'^n) = [h^{(X')}{}_m]^{-1} h^{(x')}{}_n \, [\hat{e}^{(X')}{}_m{}^T F^{-1} \hat{e}^{(x')}{}_n]$$

and again the comment is the same as just made above. Thus

$$F^{-1(X',x')}{}_{mn} = [\hat{e}^{(X')}{}_m{}^T F^{-1} \hat{e}^{(x')}{}_n] \;=\; [h^{(X')}{}_m \,/\, h^{(x')}{}_n] \, (\partial X'^m/\partial x'^n)$$

So this is a matrix I could write out if I wanted. But from above

$$B^{-1} = F^{-1,T} F^{-1} = G^T G \qquad \text{// just define } G = F^{-1} \text{ to make things easy}$$

$$G^{(X',x')}{}_{mn} = [\hat{e}^{(X')}{}_m{}^T G \, \hat{e}^{(x')}{}_n] \;=\; [h^{(X')}{}_m \,/\, h^{(x')}{}_n] \, (\partial X'^m/\partial x'^n)$$

So we then have

$$[B^{-1}]^{(x',x')}{}_{mn} = \hat{e}^{(x')}{}_m{}^T B^{-1} \, \hat{e}^{(x')}{}_n = \hat{e}^{(x')}{}_m{}^T G^T G \, \hat{e}^{(x')}{}_n$$

$$= \hat{e}^{(x')}{}_m{}^T G^T \hat{e}^{(x')}{}_i \, \hat{e}^{(x')}{}_i{}^T G \, \hat{e}^{(x')}{}_n$$

$$= G^{T(X',x')}{}_{mi} \, G^{(X',x')}{}_{in} \;\Rightarrow$$

$$[B^{-1}]^{(x',x')} = G^{T(X',x')} \, G^{(X',x')}$$

OK, I guess it is Maple Time again. Here is $G = F^{-1(X',x')}{}_{mn}$,

$$G := \begin{bmatrix} \frac{\partial}{\partial r} r_0 & \frac{\frac{\partial}{\partial \theta} r_0}{r} & \frac{\partial}{\partial z} r_0 \\ r_0 \left(\frac{\partial}{\partial r} \theta_0 \right) & \frac{r_0 \left(\frac{\partial}{\partial \theta} \theta_0 \right)}{r} & r_0 \left(\frac{\partial}{\partial z} \theta_0 \right) \\ \frac{\partial}{\partial r} z_0 & \frac{\frac{\partial}{\partial \theta} z_0}{r} & \frac{\partial}{\partial z} z_0 \end{bmatrix}$$

and then from $B^{-1} = G^T G$ we find that B^{-1} is given by

$$BINV := \begin{bmatrix} \left(\frac{\partial}{\partial r} r_0\right)^2 + r_0^2\left(\frac{\partial}{\partial r}\theta_0\right)^2 + \left(\frac{\partial}{\partial r} z_0\right)^2 & \frac{\left(\frac{\partial}{\partial r} r_0\right)\left(\frac{\partial}{\partial \theta} r_0\right)}{r} + \frac{r_0^2\left(\frac{\partial}{\partial r}\theta_0\right)\left(\frac{\partial}{\partial \theta}\theta_0\right)}{r} + \frac{\left(\frac{\partial}{\partial r} z_0\right)\left(\frac{\partial}{\partial \theta} z_0\right)}{r} & \left(\frac{\partial}{\partial r} r_0\right)\left(\frac{\partial}{\partial z} r_0\right) + r_0^2\left(\frac{\partial}{\partial r}\theta_0\right)\left(\frac{\partial}{\partial z}\theta_0\right) + \left(\frac{\partial}{\partial r} z_0\right)\left(\frac{\partial}{\partial z} z_0\right) \\ \frac{\left(\frac{\partial}{\partial r} r_0\right)\left(\frac{\partial}{\partial \theta} r_0\right)}{r} + \frac{r_0^2\left(\frac{\partial}{\partial r}\theta_0\right)\left(\frac{\partial}{\partial \theta}\theta_0\right)}{r} + \frac{\left(\frac{\partial}{\partial r} z_0\right)\left(\frac{\partial}{\partial \theta} z_0\right)}{r} & \frac{\left(\frac{\partial}{\partial \theta} r_0\right)^2}{r^2} + \frac{r_0^2\left(\frac{\partial}{\partial \theta}\theta_0\right)^2}{r^2} + \frac{\left(\frac{\partial}{\partial \theta} z_0\right)^2}{r^2} & \frac{\left(\frac{\partial}{\partial \theta} r_0\right)\left(\frac{\partial}{\partial z} r_0\right)}{r} + \frac{r_0^2\left(\frac{\partial}{\partial \theta}\theta_0\right)\left(\frac{\partial}{\partial z}\theta_0\right)}{r} + \frac{\left(\frac{\partial}{\partial \theta} z_0\right)\left(\frac{\partial}{\partial z} z_0\right)}{r} \\ \left(\frac{\partial}{\partial r} r_0\right)\left(\frac{\partial}{\partial z} r_0\right) + r_0^2\left(\frac{\partial}{\partial r}\theta_0\right)\left(\frac{\partial}{\partial z}\theta_0\right) + \left(\frac{\partial}{\partial r} z_0\right)\left(\frac{\partial}{\partial z} z_0\right) & \frac{\left(\frac{\partial}{\partial \theta} r_0\right)\left(\frac{\partial}{\partial z} r_0\right)}{r} + \frac{r_0^2\left(\frac{\partial}{\partial \theta}\theta_0\right)\left(\frac{\partial}{\partial z}\theta_0\right)}{r} + \frac{\left(\frac{\partial}{\partial \theta} z_0\right)\left(\frac{\partial}{\partial z} z_0\right)}{r} & \left(\frac{\partial}{\partial z} r_0\right)^2 + r_0^2\left(\frac{\partial}{\partial z}\theta_0\right)^2 + \left(\frac{\partial}{\partial z} z_0\right)^2 \end{bmatrix}$$

It too is symmetric and Lai gives the 6 distinct elements on pages 134-135 and they agree with the above.

9. Try matrix C with respect to X. This is going to be just more of the same.

$$[C^{(X',X')}]_{mn} = [\hat{e}^{(X')}{}_m{}^T \, C \, \hat{e}^{(X')}{}_n] = \hat{e}^{(X')}{}_m{}^T \, F^T F \, \hat{e}^{(X')}{}_n]$$

$$= \hat{e}^{(X')}{}_m{}^T \, F^T \, \hat{e}^{(x')}{}_i \, \hat{e}^{(x')T}{}_i \, F \, \hat{e}^{(X')}{}_n] = F^T F \quad \text{where } F = [F^{(x,X)}]_{ij} = \hat{e}^{(x)T}{}_i \, F \, \hat{e}^{(X)}{}_j$$

So just tell Maple to compute F^TF in this mixed basis. We get

$$C := \begin{bmatrix} \left(\frac{\partial}{\partial r_0} r\right)^2 + r^2\left(\frac{\partial}{\partial r_0}\theta\right)^2 + \left(\frac{\partial}{\partial r_0} z\right)^2 & \frac{\left(\frac{\partial}{\partial r_0} r\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0} + \frac{r^2\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial \theta_0}\theta\right)}{r_0} + \frac{\left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial \theta_0} z\right)}{r_0} & \left(\frac{\partial}{\partial r_0} r\right)\left(\frac{\partial}{\partial z_0} r\right) + r^2\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial z_0}\theta\right) + \left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial z_0} z\right) \\ \frac{\left(\frac{\partial}{\partial r_0} r\right)\left(\frac{\partial}{\partial \theta_0} r\right)}{r_0} + \frac{r^2\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial \theta_0}\theta\right)}{r_0} + \frac{\left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial \theta_0} z\right)}{r_0} & \frac{\left(\frac{\partial}{\partial \theta_0} r\right)^2}{r_0^2} + \frac{r^2\left(\frac{\partial}{\partial \theta_0}\theta\right)^2}{r_0^2} + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)^2}{r_0^2} & \frac{\left(\frac{\partial}{\partial \theta_0} r\right)\left(\frac{\partial}{\partial z_0} r\right)}{r_0} + \frac{r^2\left(\frac{\partial}{\partial \theta_0}\theta\right)\left(\frac{\partial}{\partial z_0}\theta\right)}{r_0} + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial z_0} z\right)}{r_0} \\ \left(\frac{\partial}{\partial r_0} r\right)\left(\frac{\partial}{\partial z_0} r\right) + r^2\left(\frac{\partial}{\partial r_0}\theta\right)\left(\frac{\partial}{\partial z_0}\theta\right) + \left(\frac{\partial}{\partial r_0} z\right)\left(\frac{\partial}{\partial z_0} z\right) & \frac{\left(\frac{\partial}{\partial \theta_0} r\right)\left(\frac{\partial}{\partial z_0} r\right)}{r_0} + \frac{r^2\left(\frac{\partial}{\partial \theta_0}\theta\right)\left(\frac{\partial}{\partial z_0}\theta\right)}{r_0} + \frac{\left(\frac{\partial}{\partial \theta_0} z\right)\left(\frac{\partial}{\partial z_0} z\right)}{r_0} & \left(\frac{\partial}{\partial z_0} r\right)^2 + r^2\left(\frac{\partial}{\partial z_0}\theta\right)^2 + \left(\frac{\partial}{\partial z_0} z\right)^2 \end{bmatrix}$$

10. Conclusion. To compute all these matrices in any combination of bases, you need to first compute these two fundamental matrices in the mixed basis as shown:

$$[F^{(x',X')}]_{mn} = [\hat{e}^{(x')}{}_m{}^T \, F \, \hat{e}^{(X')}{}_n] = [h^{(x')}{}_m / h^{(X')}{}_n] \, (\partial x'^m / \partial X'^n)$$

$$F^{-1(X',x')}{}_{mn} = [\hat{e}^{(X')}{}_m{}^T \, F^{-1} \hat{e}^{(x')}{}_n] = [h^{(X')}{}_m / h^{(x')}{}_n] \, (\partial X'^m / \partial x'^n)$$

Once you do that, you compute the other matrices in the obvious manner using simple matrix multiplication in Maple:

$B = FF^T$
$B^{-1} = F^{-1,T} \, F^{-1}$
$C = F^TF$
$C^{-1} = F^{-1} \, F^{-1,T}$

In Maple the main work is just to give all 6 coordinates appropriate names, and get the six scale factors entered properly.

11. The case X = Cartesian and x = cylinder. I created a new Maple file lai 3_29 cart cyl and it gives all the right results.

12. The case X = spherical and x = spherical. Done, just another Maple file.

So I did all the cases Lai did, and I could no doubt put check marks everywhere if I wanted.