Lai Problem 5.71 page 316 PhL 4.8.12

This was a nice little logjam thrown in the road by Lai at the top of page 264. The issue is how to write the Airy equations in non-Cartesian coordinates. It is extremely non-trivial and took this 10 page effort to get it cleaned up. If I need similar equations in some other coordinate system, I can use this method. But that would have to be some other 2D coordinate system since that is where Airy really applies.

This led to major additions and alterations to tensor doc, now all finished. So most of what appears below is now somewhere in tensor doc. See red comments below.

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1. Review of vector transformation. [in tensor doc]

Our famous x-space vector expansions are

$$\mathbf{A} = \Sigma_i A^i \mathbf{u}_i \qquad \text{// axis aligned unit vectors}$$
$$\mathbf{A} = \Sigma_i A'^i \mathbf{e}_i \qquad \text{// tangent base vectors}$$
$$\mathbf{A} = \Sigma_i A'^i \mathbf{e}_i = \Sigma_i A'^i h'_i \hat{\mathbf{e}}_i = A^{(\hat{e})i} \hat{\mathbf{e}}_i \qquad \text{// unit tangent base vectors}$$
$$A^{(\hat{e})i} = A'^i h'_i \quad \text{and} \quad A'^i = h'^{-1}_i A^{(\hat{e})i}$$

where recall that

$$A'^a = R^a{}_b A^b \quad \text{and the inverse:} \quad A^a = S^a{}_b A'^b$$

We put the index on $A^{(\hat{e})i}$ in the "up" position only because we obtained $A^{(\hat{e})i}$ from A'^i and this latter thing had the index in the "up" position. We know that $A^{(\hat{e})i}$ is not a contravariant vector, but we have to put the index somewhere.

2. Relation between A^i and $A^{(\hat{e})i}$. **[in tensor doc App E (h)]**

Now, we want to find out how A^i and $A^{(\hat{e})i}$ are related to each other.

$$A^{(\hat{e})a} = A'^a h'_a = h'_a A'^a = h'_a R^a{}_b A^b = (h'_a R^a{}_b) A^b \equiv M^a{}_b A^b$$

Going the other direction

$$A^a = S^a{}_b A'^b = S^a{}_b h'^{-1}_b A^{(\hat{e})b} = (h'^{-1}_b R_b{}^a) A^{(\hat{e})b} \equiv N^a{}_b A^{(\hat{e})b}$$

Therefore we have $\mathbf{A} = A^{(\hat{e})i} \hat{\mathbf{e}}_i$

$$A^{(\hat{e})a} = M^a{}_b A^b \qquad M^a{}_b \equiv h'_a R^a{}_b$$
$$A^a = N^a{}_b A^{(\hat{e})b} \qquad N^a{}_b \equiv h'^{-1}_b R_b{}^a = h'^{-1}_b S^a{}_b$$

It seems that M and N should be inverses of each other, so let's check

$$M^a{}_b N^b{}_c = h'_a R^a{}_b h'^{-1}_c S^b{}_c = h'_a h'^{-1}_c R^a{}_b S^b{}_c = h'_a h'^{-1}_c \delta^a{}_c = \delta^a{}_c \qquad M = N^{-1}$$

Notice that we have not specified an orthogonal coordinate system for x'-space yet, these results are completely general. At this point, M and N need not be rotations, for example.

Before continuing in the general case, let us now consider polar coordinates which happens to be an orthogonal system.

3. Example: Polar Coordinates. [in tensor doc App E (h)]

Example of Polar Coordinates with 1,2 = r,θ first in developmental notation:

$$S^i{}_j = (\partial x^i / \partial x'^j)$$

$$S^1{}_1 = (\partial x/\partial r) = \cos\theta$$
$$S^1{}_2 = (\partial x/\partial \theta) = -r\sin\theta$$
$$S^2{}_1 = (\partial y/\partial r) = \sin\theta$$
$$S^2{}_2 = (\partial y/\partial \theta) = r\cos\theta$$

$$S^i{}_j = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \qquad R^i{}_j = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}$$ // Note on this just added to tensor doc.

Now we want to compute the M and N objects:

$$M^a{}_b \equiv h'_a R^a{}_b = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = R_z(-\theta)$$ in my matrix rot doc

$$N^a{}_b \equiv S^a{}_b\, h'^{-1}_b = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/r \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_z(\theta)$$

4. Fact that $N^a{}_b = \{ \hat{e}_1 , \hat{e}_2, ... \}$ in general case. [in tensor doc App E (h)]

Now back to the general case. We know from Section 3 of tensor doc that

$$\mathbf{e}_n \equiv S\mathbf{e}'_n \quad \text{or} \quad \mathbf{e}_n \equiv S\mathbf{u}_n \quad \text{or} \quad h'_n\, \hat{\mathbf{e}}_n = S\, \mathbf{u}_n$$

so

$$\hat{\mathbf{e}}_n = h'^{-1}_n\, S\, \mathbf{u}_n \text{ or}$$

or

$$(\hat{\mathbf{e}}_n)^a = h'^{-1}_n\, S^a{}_b\, (\mathbf{u}_n)^b = h'^{-1}_n\, S^a{}_b\, \delta_n{}^b = h'^{-1}_n\, S^a{}_n = N^a{}_n$$

and this last says that the n^{th} column of $N^a{}_n$ is equal to $\hat{\mathbf{e}}_n$. Therefore

$$N^a{}_b = \{ \hat{\mathbf{e}}_1 , \hat{\mathbf{e}}_2, ... \}$$

So this tells us that the matrix $N^a{}_b$ is composed of the unit tangent base vectors as its columns.

5. Theorem: $\hat{e}_n$ orthogonal => $M^a{}_b$ and $N^a{}_b$ are rotation matrices. [in tensor doc App E (h)]

We want to show now that if the $\hat{\mathbf{e}}_n$ are orthogonal, then $N^a{}_b$ must be a rotation matrix. I don't know how to do that in standard notation, so let's work in developmental notation which matches the notation of ordinary linear algebra. Then

$$N^a{}_n = (\hat{\mathbf{e}}_n)^a \quad \rightarrow \quad N_{an} = (\hat{\mathbf{e}}_n)_a$$

Then consider

$$N_{an}N^T{}_{nc} = N_{an}N_{cn} = (\hat{\mathbf{e}}_n)_a\, (\hat{\mathbf{e}}_n)_c = ?$$

Suppose the $\hat{\mathbf{e}}_n$ are *orthogonal* in x-space. Then we know that

$$\hat{\mathbf{e}}_n \bullet \hat{\mathbf{e}}_m = \delta_{nm} \quad \text{or} \quad \langle \hat{e}_n | \hat{e}_m \rangle = \delta_{n,m}$$

In the orthogonal case we know that completeness can be stated this way,

$$1 = |\hat{e}_n\rangle\langle\hat{e}_n| \quad \Rightarrow \quad \langle u_a|u_c\rangle = \langle u_a|\hat{e}_n\rangle\langle\hat{e}_n|\, u_c\rangle = (\hat{\mathbf{e}}_n)_a\, (\hat{\mathbf{e}}_n)_c = \delta_{a,c}$$

We can now complete what was started above

$$N_{an}N^T{}_{nc} = N_{an}N_{cn} = (\hat{\mathbf{e}}_n)_a\, (\hat{\mathbf{e}}_n)_c = \delta_{a,c}$$

$\Rightarrow \quad NN^T = 1$

So as long as the set of basis vectors $\hat{\mathbf{e}}_n$ forms a complete orthonormal set, we find that N is real orthogonal. Meanwhile

$NN^T = 1 \quad \Rightarrow \quad \det(N)^2 = 1 \qquad \det(N) = \pm 1$

and we can just assume det(N) = 1.

Theorem just proved: If the unit tangent base vectors $\hat{\mathbf{e}}_n$ are orthogonal, then N = rotation, and of course this means that M = the inverse rotation.

6. More general tensors. [in tensor doc App E (h)]

Above we talked about vectors, and here we want to generalize the above comments to tensors of arbitrary rank. Our expansions are these (from tensor doc)

$A = \Sigma_{ijk\ldots} A^{ijk\cdots} (\mathbf{u}_i \otimes \mathbf{u}_j \otimes \mathbf{u}_k \ldots)$ $\quad A^{ijk\cdots}$ = contravariant components of A in x-space

$A = \Sigma_{ijk\ldots} A'^{ijk\cdots} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \ldots)$ $\quad A'^{ijk\cdots}$ = contravariant components of A in x'-space

We rewrite the second one in terms of unit tangent base vectors

$A = \Sigma_{ijk\ldots} h'_i h'_j h'_k \ldots\ldots A'^{ijk\cdots} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \ldots)$

$= A^{(\hat{e})ijk\cdots} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \otimes \hat{\mathbf{e}}_k \ldots)$ where $A^{(\hat{e})ijk\cdots} \equiv h'_i h'_j h'_k \ldots\ldots A'^{ijk\cdots}$

Then we write

$A'^{ijk\cdots} = R^i{}_{i'} R^j{}_{j'} R^k{}_{k'} \ldots\ldots A^{i'j'k'\cdots}$

and so multiplying both sides by $h'_i h'_j h'_k \ldots$ we get

$A^{(\hat{e})ijk\cdots} = h'_i R^i{}_{i'} h'_j R^j{}_{j'} h'_k R^k{}_{k'} \ldots\ldots A^{i'j'k'\cdots}$

$= M^i{}_{i'} M^j{}_{j'} M^k{}_{k'} \ldots\ldots A^{i'j'k'\cdots}$

Going the other way, we use the fact that N is the inverse of M to write

$A^{ijk\cdots} = N^i{}_{i'} N^j{}_{j'} N^k{}_{k'} \ldots\ldots A^{(\hat{e})\,i'j'k'\cdots}$

We showed in the last section that if the $\hat{\mathbf{e}}_i$ are orthogonal, then M and N are rotation matrices.

7. Rank-2 tensors. **[in tensor doc App E (h)]**

In the special case of rank-2 tensors we can write

$$A^{(\hat{e})ij} = M^{i}{}_{i'}\, M^{j}{}_{j'}\, A^{i'j'}$$
$$A^{ij} = N^{i}{}_{i'}\, N^{j}{}_{j'}\, A^{(\hat{e})\, i'j'}$$

If we now convert this to developmental notation, we have

$$A^{(\hat{e})}{}_{ij} = M_{ii'}\, M_{jj'}\, A_{i'j'}$$
$$A_{ij} = N_{ii'}\, N_{jj'}\, A^{(\hat{e})}{}_{i'j'}$$

which then allows us to write

$$A^{(\hat{e})}{}_{ij} = M_{ii'}\, A_{i'j'}\, M^{T}{}_{j'j}$$
$$A_{ij} = N_{ii'}\, A^{(\hat{e})}{}_{i'j'}\, N^{T}{}_{j'j}$$

or

$$A^{(\hat{e})} = MA\ M^{T}$$
$$A = NA^{(\hat{e})}N^{T}$$

If the $\hat{e}_i$ are orthogonal, then M and N are rotations, so we can write these as

$$A^{(\hat{e})} = MA\ M^{-1} = MA\ N$$
$$A = NA^{(\hat{e})}N^{-1} = NA^{(\hat{e})}M$$

Example: In polar coordinates, we can write

$$\begin{pmatrix} A_{rr} & A_{r\theta} \\ A_{\theta r} & A_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Thus, we have proven the first part of Problem 5.71. It took a lot of fiddling!

Comment #1: I might want to add the above to tensor doc, and maybe reconsider my θ,r ordering once again. It has caused me endless trouble. **********

Comment #2: Even in the very simple 2x2 case of polar coordinates, things are complicated! The object A_{rr} for example is some linear combination of ALL the A_{ij} .

8. Rest of Problem 5.71 Setup. **[in tensor doc App E (h)]**

For plane strain and plane stress applications in Cartesian coordinates, we used the Airy function in this manner:

$T_{11} = \partial_2^2\varphi \qquad T_{22} = \partial_1^2\varphi \qquad T_{12} = T_{21} = -\partial_1\partial_2\varphi$

Therefore we can write our polar coordinates stress tensor situation as follows

$$\begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_2^2\varphi & -\partial_1\partial_2\varphi \\ -\partial_1\partial_2\varphi & \partial_1^2\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where ∂_n refers to x_n.

The following stuff is NOT in tensor doc!

9. Now we have to compute ∂_1 and ∂_2. [results only in tensor doc App E (h)]

$x_1 = r\cos\theta \qquad \theta = \tan^{-1}(x_2/x_1)$
$x_2 = r\sin\theta \qquad r = [x_1^2 + x_2^2]^{1/2}$

$\partial_1 = \partial/\partial x_1 = \partial r/\partial x_1\, \partial_r + \partial\theta/\partial x_1 \partial_\theta$
$\partial_2 = \partial/\partial x_2 = \partial r/\partial x_2\, \partial_r + \partial\theta/\partial x_2 \partial_\theta$

$\partial r/\partial x_1 = (1/2)r^{-1}2x_1 = x_1/r = \cos\theta$
$\partial r/\partial x_2 = (1/2)r^{-1}2x_2 = x_2/r = \sin\theta$

$\partial\theta/\partial x_1 = 1/(1+(x_2/x_1)^2)\,(-x_2/x_1^2) = x_1^2/(x_2^2+x_1^2)\,(-x_2/x_1^2) = -x_2/r^2 = -\sin\theta/r$
$\partial\theta/\partial x_2 = 1/(1+(x_2/x_1)^2)\,(1/x_1) = x_1^2/(x_2^2+x_1^2)\,(1/x_1) = x_1/r^2 = \cos\theta/r$

Then

$\partial_1 = \cos\theta\, \partial_r - (\sin\theta/r)\partial_\theta$
$\partial_2 = \sin\theta\, \partial_r + (\cos\theta/r)\partial_\theta$

Question: Is there some simple way to make Maple do the above work, now that I have done it? The answer is a resounding YES. Here is how you do it:

```
with(PDEtools,dchange);
```

$$[dchange]$$

```
tr := {x = r*cos(theta), y=r*sin(theta)};
```

$$tr := \{x = r\cos(\theta), y = r\sin(\theta)\}$$

```
dfdx := dchange(tr,diff(f(x,y),x)): expand(%);
```

$$\cos(\theta)\left(\frac{\partial}{\partial r}f(r,\theta)\right) - \frac{\sin(\theta)\left(\frac{\partial}{\partial \theta}f(r,\theta)\right)}{r}$$

```
dfdy := dchange(tr,diff(f(x,y),y)): expand(%);
```

$$\sin(\theta)\left(\frac{\partial}{\partial r}f(r,\theta)\right) + \frac{\cos(\theta)\left(\frac{\partial}{\partial \theta}f(r,\theta)\right)}{r}$$

This should be put into my User Guide right now!!! DONE.

10. Finish solving Problem 5.71. **[results only in tensor doc App E (h)]**

Our situation so far is this:

$$\begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_2{}^2\varphi & -\partial_1\partial_2\varphi \\ -\partial_1\partial_2\varphi & \partial_1{}^2\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = M\,T\,M^T$$

$$\partial_1 = \cos\theta\,\partial_r - (\sin\theta/r)\partial_\theta$$
$$\partial_2 = \sin\theta\,\partial_r + (\cos\theta/r)\partial_\theta$$

Maple is going to know about these last two lines for ∂_1 for ∂_2 from the transformations tr I enter below. Here then is how I got Maple to do everything (see Lai5_71b.mws). First, we implement the above matrix equation: (the Cartesian matrix is T^{ij} while TP = $T^{(\hat{e})ij}$, but we work in dev not so all indices are down and say then we start with T_{ij} and we are computing TP = $T^{(\hat{e})}{}_{ij}$)

```
> restart;
> with(linalg):
Warning, new definition for norm
Warning, new definition for trace
> with(PDEtools, dchange);
```

$$[dchange]$$

```
> M := matrix(2,2,[cos(theta),sin(theta),-sin(theta),cos(theta)]);
```

$$M := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

```
> T := matrix(2,2);
```

$$T := \mathrm{array}(1..2, 1..2, [\,])$$

```
> TP := evalm(M &* T &* transpose(M)):simplify(%);
```

$$\begin{bmatrix} T_{1,1}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,1} + \cos(\theta)\sin(\theta)T_{1,2} + T_{2,2} - \cos(\theta)^2 T_{2,2}, -\cos(\theta)\sin(\theta)T_{1,1} - T_{2,1} + \cos(\theta)^2 T_{2,1} + T_{1,2}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,2} \\ -\cos(\theta)\sin(\theta)T_{1,1} + \cos(\theta)^2 T_{2,1} - T_{1,2} + T_{1,2}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,2}, T_{1,1} - T_{1,1}\cos(\theta)^2 - \sin(\theta)\cos(\theta)T_{2,1} - \cos(\theta)\sin(\theta)T_{1,2} + \cos(\theta)^2 T_{2,2} \end{bmatrix}$$

Next, I tell it what the T_{ij} are in terms of the Airy function φ ,

```
> T[1,1] := diff(phi(x,y),y,y):
> T[2,2] := diff(phi(x,y),x,x):
> T[1,2] := -diff(phi(x,y),x,y):
> T[2,1] := -diff(phi(x,y),x,y):
> evalm(T);
```

$$\begin{bmatrix} \frac{\partial^2}{\partial y^2}\phi(x,y) & -\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right) \\ -\left(\frac{\partial^2}{\partial y\,\partial x}\phi(x,y)\right) & \frac{\partial^2}{\partial x^2}\phi(x,y) \end{bmatrix}$$

I then use dchange to handle all these messy derivatives. Applying dchange just to matrix T_{ij} gives pretty messy results, for example here is $\partial_y^2\varphi = [\sin\theta\,\partial_r + (\cos\theta/r)\partial_\theta]^2\,\varphi$:

```
tr := {x = r*cos(theta),y=r*sin(theta)};
```

$$tr := \{x = r\cos(\theta), y = r\sin(\theta)\}$$

```
Tx := matrix(2,2);
```

$$Tx := \text{array}(1 .. 2, 1 .. 2, [\ ])$$

```
dchange(tr, T[1,1]):simplify(%):Tx[1,1] := expand(%);
```

$$Tx_{1,1} = \left(\frac{\partial^2}{\partial r^2}\phi(r,\theta)\right) - \left(\frac{\partial^2}{\partial r^2}\phi(r,\theta)\right)\cos(\theta)^2 - 2\frac{\sin(\theta)\cos(\theta)\left(\frac{\partial}{\partial\theta}\phi(r,\theta)\right)}{r^2} + 2\frac{\sin(\theta)\cos(\theta)\left(\frac{\partial^2}{\partial\theta\,\partial r}\phi(r,\theta)\right)}{r} + \frac{\cos(\theta)^2\left(\frac{\partial}{\partial r}\phi(r,\theta)\right)}{r} + \frac{\cos(\theta)^2\left(\frac{\partial^2}{\partial\theta^2}\phi(r,\theta)\right)}{r^2}$$

where there are 5 different types of terms! But if we go ahead and do this and then compute the TP matrix which includes T_{rr} (and which matrix I call TPx), we get

```
TPx := matrix(2,2);
```

$$TPx := \text{array}(1 .. 2, 1 .. 2, [\ ])$$

```
dchange(tr, TP[1,1]):simplify(%):TPx[1,1] := expand(%);
```

$$TPx_{1,1} = \frac{\frac{\partial}{\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial^2}{\partial\theta^2}\phi(r,\theta)}{r^2}$$

```
dchange(tr, TP[1,2]):simplify(%):TPx[1,2] := expand(%);
```

$$TPx_{1,2} = -\frac{\frac{\partial^2}{\partial\theta\,\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial}{\partial\theta}\phi(r,\theta)}{r^2}$$

```
dchange(tr, TP[2,2]):simplify(%):TPx[2,2] := expand(%);
```

$$TPx_{2,2} = \frac{\partial^2}{\partial r^2}\phi(r,\theta)$$

and this agrees with the results Lai shows on in Problem 5.70 (p 316) and on page 264 top. So there are lots and lots of terms, but there is a huge amount of cancellation. I then finished up in Maple like this:

```
TPx[2,1] := TPx[1,2];
```

$$TPx_{2,1} := -\frac{\frac{\partial^2}{\partial\theta\,\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial}{\partial\theta}\phi(r,\theta)}{r^2}$$

```
evalm(TPx);
```

$$\begin{bmatrix} \frac{\frac{\partial}{\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial^2}{\partial\theta^2}\phi(r,\theta)}{r^2} & -\frac{\frac{\partial^2}{\partial\theta\,\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial}{\partial\theta}\phi(r,\theta)}{r^2} \\ -\frac{\frac{\partial^2}{\partial\theta\,\partial r}\phi(r,\theta)}{r} + \frac{\frac{\partial}{\partial\theta}\phi(r,\theta)}{r^2} & \frac{\partial^2}{\partial r^2}\phi(r,\theta) \end{bmatrix}$$

12. Is there some simpler way? **[I don't think so]**

Well, if you want to write ANY expression for T_{rr} you have to somehow express T_{rr} in terms of the Cartesian animals, and that was provided above by

$$\begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

There IS no other way that I know to relate T_{rr} to the four Cartesian T_{ij} objects. This then gave

$$\begin{bmatrix} T_{1,1}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,1} + \cos(\theta)\sin(\theta)T_{1,2} + T_{2,2} - \cos(\theta)^2 T_{2,2}, -\cos(\theta)\sin(\theta)T_{1,1} - T_{2,1} + \cos(\theta)^2 T_{2,1} + T_{1,2}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,2} \\ -\cos(\theta)\sin(\theta)T_{1,1} + \cos(\theta)^2 T_{2,1} - T_{1,2} + T_{1,2}\cos(\theta)^2 + \sin(\theta)\cos(\theta)T_{2,2}, T_{1,1} - T_{1,1}\cos(\theta)^2 - \sin(\theta)\cos(\theta)T_{2,1} - \cos(\theta)\sin(\theta)T_{1,2} + \cos(\theta)^2 T_{2,2} \end{bmatrix}$$

so you can see for T_{rr} there are essentially 3 terms since the two T's are equal.

THEN, having found an expression for T_{rr} you install the Airy guys

$$\begin{pmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_2^2\varphi & -\partial_1\partial_2\varphi \\ -\partial_1\partial_2\varphi & \partial_1^2\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

I just don't know how else you could do this! Yes, you can show as Lai does on page 264 that the resulting forms satisfy the two polar static Navier equations, and then just start your theory from that point. I never tested that out, by the way, which is problem 5.70 (I have not done it, but could), using Maple to compute everything.

So no, I don't think there is any simpler way.

Appendix A : orthog and completeness for "non orthog" system. **[tensor doc App E (g)]**

Suppose $\hat{\mathbf{e}}_n$ are complete but non-orthogonal. How do you write completeness in this case?

Well, the claim is that for any vector **A** you can expand as follows

$\mathbf{A} = A_n \hat{\mathbf{e}}_n \qquad |A\rangle = A_n |\hat{\mathbf{e}}_n\rangle$

Then

$\mathbf{A} \bullet \hat{\mathbf{e}}_m = A_n \hat{\mathbf{e}}_n \bullet \hat{\mathbf{e}}_m = A_n g'_{nm}$ // really should be g-bar in dev not

$\mathbf{A} \bullet \hat{\mathbf{e}}_m \; (g'^{-1})_{mk} = A_n g'_{nm} (g'^{-1})_{mk} = A_n \delta_{nk} = A_k$

so

$A_k = (\mathbf{A} \bullet \hat{\mathbf{e}}_m) (g'^{-1})_{mk}$

so at least I can write something for the coefficient.

Is it possible that completeness still says

$1 = |\hat{e}_n\rangle\langle\hat{e}_n| \qquad ?$

If we apply this to $|\hat{\mathbf{e}}_m\rangle$ we would get

$|\hat{\mathbf{e}}_m\rangle = |\hat{e}_n\rangle\langle\hat{e}_n|\hat{\mathbf{e}}_m\rangle = |\hat{e}_n\rangle g'_{nm}$

Then close left with $\langle\hat{e}_m|$ to get

$\langle\hat{e}_m|\hat{\mathbf{e}}_m\rangle = \langle\hat{e}_m|\hat{e}_n\rangle g'_{nm}$

$1 = \langle\hat{e}_m|\hat{e}_n\rangle g'_{nm} = (g'_{nm})^2$

which is **wrong**. So conjecture that completeness looks like this instead

$1 = |\hat{e}_n\rangle (g'^{-1})_{nm} \langle\hat{e}_m|$

Now repeat the above steps

$|\hat{\mathbf{e}}_m\rangle = |\hat{e}_n\rangle (g'^{-1})_{nk} \langle\hat{e}_k|\hat{\mathbf{e}}_m\rangle = |\hat{e}_n\rangle (g'^{-1})_{nk} g'_{km} = |\hat{e}_n\rangle \delta_{nm} = |\hat{\mathbf{e}}_m\rangle$

and then we are happy again.

Here is another way to say all this: In all cases (orthog or not) we know that

$\langle\hat{e}^n|\hat{e}_m\rangle = \delta^n{}_m \qquad \hat{\mathbf{e}}^n \bullet \hat{\mathbf{e}}_m = \delta^n{}_m$

$1 = |\hat{e}_n\rangle \langle\hat{e}^n|$ // this is correct