Lai p 129 and tensor doc PhL 3.16.12

There is some subtle stuff going on here with regard to the "non-covariant" application of tensor doc ideas to the Lai continuum mechanics application. It needs to be cleared up.

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1. The Vector Area Ratio on Lai p 129.

Here is how tensor doc would derive the key result 3.27.10 for an area patch which lies on the famous N-piped spanned by the tangent base vectors. We start by gathering facts, and we use dev notation:

First, from Appendix A,

$$\mathbf{E_k} \equiv \det(R)\,(-1)^{k-1}\,\{\mathbf{e_1} \times \mathbf{e_2} \times \ldots\ldots \times \mathbf{e_N} \;//\; \mathbf{e_k} \text{ missing}\}$$

so

$$\{(-1)^{k-1}\,\mathbf{e_1} \times \mathbf{e_2} \times \ldots\ldots \times \mathbf{e_N} \;//\; \mathbf{e_k} \text{ missing}\} = \det(S)\,\mathbf{E_k}$$

Second, from Section 8 (c),

$$d\mathbf{A}^n = \det(S)\,\mathbf{E}_n\,(\Pi_{i\neq n} dx'_i)$$

Combining the first two items says

$$d\mathbf{A}^n = \{(-1)^{k-1}\,\mathbf{e_1} \times \mathbf{e_2} \times \ldots\ldots \times \mathbf{e_N} \;//\; \mathbf{e_k} \text{ missing}\}\,(\Pi_{i\neq n} dx'_i)$$

which we could regard really as the starting point, the N-piped area definition.

Third, from Section 6 (a) when g = 1,

$$(\mathbf{E}_n)_i = R_{ni}$$

Theorem 1: $S^T\,\mathbf{E_k} = \mathbf{e'_k}$

Proof: Look at the components:

$$(S^T)_{ij}\,(E_k)_j = (\mathbf{e'}_k)_i \qquad \Rightarrow \qquad S_{ji}R_{kj} = \delta_{k,i}$$

$\Rightarrow\ R_{kj}S_{ji} = \delta_{k,i} \qquad \Rightarrow \qquad (RS)_{ki} = \delta_{k,i}$

So just reverse these steps and Theorem 1 is proved.

Application of Theorem 1.

$d\mathbf{A}^n = \det(S)\ (\Pi_{i\neq n} dx'_i)\ \mathbf{E}_n$

$S^T\ d\mathbf{A}^n = \det(S)\ (\Pi_{i\neq n} dx'_i)\ S^T\ \mathbf{E}_n \quad = \det(S)\ (\Pi_{i\neq n} dx'_i)\ (\mathbf{e}'_n)$

In the Lai application, in what tensor doc calls the x'-space Cartesian-view,

$d\mathbf{A}'^n\ = (\Pi_{i\neq n} dx'_i)\ (\mathbf{e}'_n) \qquad \Rightarrow \qquad dA'^n\ = (\Pi_{i\neq n} dx'_i)$

and thus we have shown that

$S^T\ d\mathbf{A}^n = \ \det(S)\ d\mathbf{A}'^n \qquad \Rightarrow \qquad d\mathbf{A}^n = dA'^n \det(S)\ \ (S^T)^{-1}\ \mathbf{e}'_n$

and this is basically 3.27.10 generalized from n=3 to general n, and generalized to N dimensions as well, and expressed in tensor doc notations.

In the "inverse flow model" of matching Lai to tensor doc, we have S = F according to Section 5 (o)

continuum mechanics		this document	(Inverse Flow)	
$\mathbf{X}$, $\mathbf{x}$	$\leftrightarrow$	$\mathbf{x}'$, $\mathbf{x}$		
$\mathbf{X} = \mathbf{X}(\mathbf{x},t)$	$\leftrightarrow$	$\mathbf{x}' = \boldsymbol{\mathcal{F}}(\mathbf{x})$		
$d\mathbf{X} = F^{-1}\ d\mathbf{x}$	$\leftrightarrow$	$d\mathbf{x}' = R\ d\mathbf{x}$	// as in Section 2	$\mathbf{x} = \boldsymbol{\mathcal{F}}(\mathbf{X})$
F^{-1}	$\leftrightarrow$	R		
F	$\leftrightarrow$	S	// $S = R^{-1}$	
$\mathbf{X}$ = Cartesian	$\leftrightarrow$	$\bar{g} = 1$		
$\mathbf{x}$ = Cartesian	$\leftrightarrow$	$\bar{g}' = 1$		
$C = F^TF$	$\leftrightarrow$	$\bar{\boldsymbol{g}}' = S^TS$	// as in Section 5 ($\boldsymbol{\ell}$)	// Lai p114 (3.23.2)

In this model, the x'-space of tensor doc matches Lai's X-space where we have an undistorted cubic particle, and the N-piped is the skewed particle at a later time in x-space in both notations. A key point being made here is that the skewed particle at the later time is spanned by the tangent base vectors of the inverse flow transformation $\mathbf{x}' = \boldsymbol{\mathcal{F}}(\mathbf{x})$. Therefore, we don't have to consider an "arbitrary area patch".

Comments:

1. Notice in the above discussion that the only contact with x'-space is through $d\mathbf{A}'^n\ = (\Pi_{i\neq n} dx'^i)\ (\mathbf{e}'_n)$ where the Cartesian-view is taken. We did not have to face the issue of g' versus $\boldsymbol{g}'$.

2. Secondly since $\mathbf{e}'_n = R\ \mathbf{e}_n \Rightarrow \mathbf{e}_n = S\ \mathbf{e}'_n$, we could have written

$d\mathbf{A}^n = \ \{(-1)^{k-1}\ S\mathbf{e}'_1 \times S\mathbf{e}'_2 \times \ldots\ldots \times S\mathbf{e}'_N\ \ //\ S\mathbf{e}'_k \text{ missing}\}\ (\Pi_{i\neq n} dx'_i)$

which has the special case

$$d\mathbf{A}^3 = S\mathbf{e}'_1 \times S\mathbf{e}'_2 \; dx'_1 dx'_2 \qquad // = [\det(S)\mathbf{E}_3] dx'_1 dx'_2$$

which is Lai's starting point 3.27.2. Lai quietly does what I did in Appendix A to bring out the det(S) factor, and I have to admit, it is quite concise.

2. What happens in the Alternate Derivation?

From Section 8 (c) item 9 tensor doc writes for an *arbitrary* area

$$d\mathbf{A}' = J\, R\, d\mathbf{A} \qquad \Rightarrow \qquad d\mathbf{A} = J^{-1}\, S\, d\mathbf{A}'$$

This seems a far cry from our result above which could be written

$$d\mathbf{A}^n = \det(S)\,(S^T)^{-1} d\mathbf{A}'^n \qquad \text{what we got above}$$
$$d\mathbf{A}^n = J^{-1}\; S\; d\mathbf{A}'^n \qquad \text{alternative approach } \textit{applied to } d\mathbf{A} = d\mathbf{A}^n$$

How do we reconcile these two results? Are they both valid, and if so, what is J ?

We need to examine the dev and std notations more carefully. Tensor doc shows that

$$(d\mathbf{A})_i = \varepsilon_{iabc..x}\,(dx^{[1]})^a (dx^{[2]})^b \ldots (dx^{[N-1]})^x$$

for an arbitrary area. The dx^a elements are contravariant, but the area is *covariant*! This fact is confused in the dev notation where the dA_i element would be regarded as contravariant! We really should write the above equation this way in dev notation:

$$(d\bar{\mathbf{A}})_i = \bar{\varepsilon}_{iabc..x}\,(dx^{[1]})_a (dx^{[2]})_b \ldots (dx^{[N-1]})_x$$

$$d\bar{\mathbf{A}} = \{(-1)^{k-1}\, \mathbf{e}_1 \times \mathbf{e}_2 \times \ldots\ldots \times \mathbf{e}_N \;\; // \; \mathbf{e}_k \text{ missing}\}$$

Then I think the result we really derived above was this one

$$d\bar{\mathbf{A}}^n = \det(S)\,(S^T)^{-1} d\bar{\mathbf{A}}'^n \qquad \text{what we got above}$$

In contrast, the object in our Section 8 (c) item 9 is the *contravariant* area which transforms with R

$$d\mathbf{A}^n = J^{-1}\; S\; d\mathbf{A}'^n \qquad (*)$$

Now in x-space we can say $d\bar{\mathbf{A}}^n = d\mathbf{A}^n$ but in x'-space the rule is this

$d\bar{\mathbf{A}}'^{\mathbf{n}} = \bar{\boldsymbol{g}}' \, d\mathbf{A}'^{\mathbf{n}}$ // $\bar{\mathbf{V}} = \bar{g}\,\mathbf{V}$ and $\bar{\mathbf{V}}' = \bar{\boldsymbol{g}}'\,\mathbf{V}'$ from Section 5 (g)

where we use script because this $\bar{\boldsymbol{g}}'$ is really a shorthand for $S^T S$ from Section 5 ($\boldsymbol{\ell}$) . Now we know that the product $\boldsymbol{g}'\,\bar{\boldsymbol{g}}' = 1$ so insert this into (*) to get

$$d\mathbf{A}^{\mathbf{n}} = J^{-1} \; S \; \boldsymbol{g}'\,\bar{\boldsymbol{g}}' \, d\mathbf{A}'^{\mathbf{n}}$$

and then we have

$$d\bar{\mathbf{A}}^{\mathbf{n}} = J^{-1} S \; \boldsymbol{g}' \, d\bar{\mathbf{A}}'^{\mathbf{n}}$$

$$d\bar{\mathbf{A}}^{\mathbf{n}} = J^{-1} S (RR^T) \, d\bar{\mathbf{A}}'^{\mathbf{n}} \qquad \boldsymbol{g}' = RR^T$$

$$= J^{-1} R^T \, d\bar{\mathbf{A}}'^{\mathbf{n}}$$

$$= J^{-1} (S^{-1})^T \, d\bar{\mathbf{A}}'^{\mathbf{n}} \qquad (**)$$

and now we are very close.

Now, so far, the object $d\bar{\mathbf{A}}'^{\mathbf{n}}$ is the covariant differential area in *curvilinear* x'-space, because that is what appears in our original formula involving the contravariant area $d\mathbf{A}^{\mathbf{n}'} = J\,R\,d\mathbf{A}^{\mathbf{n}}$. This area is, in Std notation for the moment,

$$(d\mathbf{A}')_{\mathbf{i}}{}^{\mathtt{curvilinear_view}} = \varepsilon'_{\mathbf{iabc..x}} (d\mathbf{x}'^{[1]})^{\mathbf{a}} (d\mathbf{x}'^{[2]})^{\mathbf{b}} \ldots (d\mathbf{x}'^{[N-1]})^{\mathbf{x}}$$

But we also know that

$$(d\mathbf{A}')_{\mathbf{i}}{}^{\mathtt{Cartesian_view}} = \varepsilon_{\mathbf{iabc..x}} (d\mathbf{x}'^{[1]})^{\mathbf{a}} (d\mathbf{x}'^{[2]})^{\mathbf{b}} \ldots (d\mathbf{x}'^{[N-1]})^{\mathbf{x}}$$

where there is no prime on the ε. And we also know that, when g = 1, $\varepsilon' = J^2 \varepsilon$ at the end of section (e) in Appendix. So the conclusion is that

$$(d\mathbf{A}')_{\mathbf{i}}{}^{\mathtt{curvilinear_view}} = J^2 \; (d\mathbf{A}')_{\mathbf{i}}{}^{\mathtt{Cartesian_view}}$$

and if we make this replacement in (**) above with implied labeling we get

$$d\bar{\mathbf{A}}^{\mathbf{n}} = J^{-1} (S^{-1})^T \; J^2 \; d\bar{\mathbf{A}}'^{\mathbf{n}} = J (S^{-1})^T \; d\bar{\mathbf{A}}'^{\mathbf{n}}$$

$$= \det(S) (S^{-1})^T \; d\bar{\mathbf{A}}'^{\mathbf{n}}$$

and this then agrees with "what we got above".

So there are many lessons here, and I think this implies some tensor doc updates. I also don't like how I have to keep hunting through Appendix D for the simple ε' type results. So I see two updates coming.

3. How does Lai handle having no contravariant and covariant indices?

(a) First, consider this equation for a weight -1 vector density,

$$\mathbf{V'} = J\, R\, \mathbf{V}$$

In standard notation, this equation is correct whichever way you think of it

$$V'^{a} = J\, R^{a}{}_{b}\, V^{b}$$
$$V'_{a} = J\, R_{a}{}^{b}\, V_{b}$$

The R matrix of course is different on these two lines, but both these equations match what is shown schematically above.

In developmental notation, this is not true. In this notation the vector equation

$$\mathbf{V'} = J\, R\, \mathbf{V}$$

is only valid for the contravariant V, and one would write

$$V'_{a} = J\, R_{ab}\, V_{b}$$

To get a covariant vector equation, we have to do this

$$\bar{\boldsymbol{g}}'\, \mathbf{V'} = J\, \bar{\boldsymbol{g}}'\, R\, g\, \bar{g}\mathbf{V} \qquad \Rightarrow \qquad \bar{\mathbf{V}}' = J\, \bar{\boldsymbol{g}}'\, R\, g\, \bar{\mathbf{V}} = J\, S^{T}\, \bar{\mathbf{V}}$$

So notice then the difference:

$$\mathbf{V'} = J\, R\, \mathbf{V}$$
$$\bar{\mathbf{V}}' = J\, S^{T}\, \bar{\mathbf{V}}$$

The point is that in std notation, these two vector equations look different! Looking again at our alternate derivation above

$$d\mathbf{A}' = J\, R\, d\mathbf{A}$$
$$d\bar{\mathbf{A}}' = J\, S^{T}\, d\bar{\mathbf{A}} \qquad \Rightarrow \qquad d\bar{\mathbf{A}} = J^{-1}\, (S^{T})^{-1} d\bar{\mathbf{A}}'$$

which is almost the final result, but then we have to say $d\bar{\mathbf{A}}' \rightarrow J^{2}\, d\bar{\mathbf{A}}'$ to get the final result.

(b) So, what does Lai mean when he writes this:

$$d\mathbf{A}_{0} = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)}$$

$$d\mathbf{A} = Sd\mathbf{X}^{(1)} \times Sd\mathbf{X}^{(2)}$$

I think this is the answer: $d\mathbf{A}_0$, $d\mathbf{A}$, $\mathbf{n}$ and $\mathbf{n}_0$ are all *covariant* vectors and in std notation they would all have bars over them.

Question: Suppose we assume g = 1 in x-space, then in $\bar{\boldsymbol{g}}' = S^T S$ in x'-space (X-space) and g' = 1. Which of these g' objects do we use to convert a contravariant vector to covariant one in x'-space (X-space)? I see two possible answers:

Answer #1: If you use $\bar{\boldsymbol{g}}'$ to do it, then both these equations will be true (first implies the second)

$$\mathbf{V}' = J\,R\,\mathbf{V}$$
$$\bar{\mathbf{V}}' = J\,S^T\,\bar{\mathbf{V}}$$

Answer #2: If you use g' = 1 to do it, then the second equation above will not be true. Furthermore, in either space will have in this case $\bar{\mathbf{V}} = \mathbf{V}$ and $\bar{\mathbf{V}}' = \mathbf{V}'$ so the above pair is replaced by this pair

$$\mathbf{V}' = J\,R\,\mathbf{V}$$
$$\bar{\mathbf{V}}' = J\,R\,\bar{\mathbf{V}}$$

which is to say, the two equations are the same. I think this is the situation of the Lai World. But of course this is not compatible with my dev notation!

4. The general connection between Lai World and Tensor Doc

Tensor doc is jammed with facts of many types. Some apply to Lai World, and some do not. Here we want to partition the tensor doc facts into these two bins.

- The transformation of contravariant vectors as described in tensor doc applies in Lai world. An example of course is the fact that $d\mathbf{x}' = R\,d\mathbf{x}$. This is a "vector space" concept and there is no implication here regarding norm or distance (metric) except that in x-space we regard g = 1.

- Anything that involves g' in x'-space is going to require examination. Basically, in Lai world, the x'-space described in tensor doc is discarded and is replaced with a new x'-space which has g' = 1. This space still contains contravariant vectors like $d\mathbf{x}'$ and they are still obtained as $d\mathbf{x}' = R\,d\mathbf{x}$. But the length of such vectors is given now by $|dx'|^2 = g'_{ij}dx'^i dx'^j$ where g' = 1.

- Certainly if we use g' = 1 to define dot products and norms in x'-space, then the magnitude of a vector is no longer a scalar under the transformation, so $|d\mathbf{x}'|^{g'=1} \neq |d\mathbf{x}|$. So any parts of tensor doc (there are many) that assume invariance of vector magnitudes do not apply in Lai world.

AREA

How does Lai deal with area in x'-space, and how does this compare with how tensor doc deals with area in x'-space? First, here is the tensor doc notion of area in x'-space (in dev notation)

$$d\bar{\mathbf{A}}'_{\mathbf{c}}\,|^{\mathbf{td}} = \bar{\boldsymbol{\varepsilon}}'_{\mathbf{cab}} dx'_{\mathbf{a}} dx'_{\mathbf{b}} = (\boldsymbol{g}'/g)\,\bar{\boldsymbol{\varepsilon}}_{\mathbf{cab}} dx'_{\mathbf{a}} dx'_{\mathbf{b}} \qquad \boldsymbol{g}' = S^{\mathbf{T}}\,\bar{g}\,S \qquad (\boldsymbol{g}'/g) = J^{\mathbf{2}}$$

and we note that $d\bar{\mathbf{A}}'_{\mathbf{c}}$ are components of a *covariant* vector. If g=1 we can write

$$d\bar{\mathbf{A}}'_{\mathbf{c}}\,|^{\mathbf{td}} = \bar{\boldsymbol{\varepsilon}}'_{\mathbf{cab}} dx'_{\mathbf{a}} dx'_{\mathbf{b}} = \boldsymbol{g}'\,\varepsilon_{\mathbf{cab}} dx'_{\mathbf{a}} dx'_{\mathbf{b}} \qquad \boldsymbol{g}' = S^{\mathbf{T}}S \text{ matrix} \qquad \boldsymbol{g}' = J^{\mathbf{2}} \text{ number}$$

where ε is the permutation tensor and we can put its indices however we like. But even when g = 1, the area $d\bar{\mathbf{A}}'_{\mathbf{c}}\,|^{\mathbf{td}}$ is the component of a *covariant* vector in x'-space.

The transformation of this tensor doc area is given by the following general rule for a covariant vector density which has weight -1 (assume g = 1)

$$d\bar{\mathbf{A}}'|^{\mathbf{td}} = J\,S^{\mathbf{T}}\,d\bar{\mathbf{A}} \qquad \text{where} \quad \bar{\mathbf{V}}' = \boldsymbol{g}'\,\mathbf{V}' \quad \text{and} \quad \bar{\mathbf{V}} = \mathbf{V} \quad \text{are the covariant vectors.}$$

In Lai World, where g = 1 always, a different x'-space area definition is used, namely

$$dA'_{\mathbf{c}}|^{\mathbf{Lai}} \equiv \varepsilon_{\mathbf{cab}} dx'_{\mathbf{a}} dx'_{\mathbf{b}} \qquad \varepsilon_{\mathbf{cab}} = \text{permutation tensor}$$

so this gives a different area! The connection between these two areas is:

$$d\bar{\mathbf{A}}'\,|^{\mathbf{td}} = J^{\mathbf{2}}\,d\mathbf{A}'|^{\mathbf{Lai}} \qquad \boldsymbol{g}' = S^{\mathbf{T}}S \qquad \boldsymbol{g}' = \det(S^{\mathbf{T}}S) \qquad \boldsymbol{g}' = J^{\mathbf{2}} \qquad J = \det(S)$$

Since Lai defines x'-space area by fiat in this way, we don't really have a "transformation rule" for the Lai area going between the two spaces. Since we have no such rule, we have no way to say whether the Lai area is contravariant, covariant, or neither. But knowing how it is related to the td area, perhaps we can reach a conclusion as to how it transforms:

$$d\bar{\mathbf{A}}'|^{\mathbf{td}} = J\,S^{\mathbf{T}}\,d\bar{\mathbf{A}} = J^{\mathbf{2}}\,d\mathbf{A}'|^{\mathbf{Lai}}$$

$$\Rightarrow\; d\mathbf{A}'|^{\mathbf{Lai}} = J^{\mathbf{-1}}\,S^{\mathbf{T}}\,d\bar{\mathbf{A}} \qquad \Rightarrow \qquad d\bar{\mathbf{A}} = J\,(S^{\mathbf{T}})^{\mathbf{-1}}\,d\mathbf{A}'|^{\mathbf{Lai}} \qquad \text{// roughly 3.27.10}$$

$$d\bar{\mathbf{A}}'|^{\mathbf{td}} = J\;\; S^{\mathbf{T}}\,d\bar{\mathbf{A}} \quad \text{for comparison}$$

The Lai area "seems" more covariant than contravariant, since $S^{\mathbf{T}}$ appears, but the J power is wrong. Perhaps you could say the Lai area is a covariant vector density of weight +1. Yes, this is exactly right, since we are basically saying $d\mathbf{A}'|^{\mathbf{Lai}} = \boldsymbol{g}'^{\mathbf{-1}} d\bar{\mathbf{A}}'\,|^{\mathbf{td}}$ and we then add weights.

But then we throw in the final zinger as part of the x'-space being Cartesian. Instead of talking about a "covariant vector" being defined by the way it transforms (and in tensor doc, contravariant and covariant vectors transform very differently!) , we are going to say that in x'-space, any vector is both contravariant and covariant based on the different definition of making partners with the metric tensor g' = 1. Once this

is done, one concludes that the Lai area vector is both contravariant and covariant in x'-space. But so doing does NOT cause this to be true,

$$d\mathbf{A}'|^{Lai} = J^{-W} R\, d\mathbf{A}$$

whatever weight W you want to use. This is NOT true because the following IS true

$$d\mathbf{A}'|^{Lai} = J^{-1} S^{T}\, d\mathbf{A}$$

and they cannot both be true!

So my conclusion is that (1) the Lai area transforms as a covariant vector density of weight +1 as indicated by the last line above. It really is "covariant in terms of transformation". (2) The Lai area is both covariant and contravariant in terms of using g' = 1 in x'-space to convert back and forth. (3) So the safest way to thing of it is as "covariant" and they everybody is happy.

5. A fast derivation of 3.27.10 in light of the above.

I showed in 4 above that the Lai area is related to the td area in this way (I use overbar on Lai now since it is "mostly" covariant as per the above discussion)

$$d\bar{\mathbf{A}}'|^{Lai} = \boldsymbol{g}'^{-1} d\bar{\mathbf{A}}'\,|^{td}$$

and that the Lai area in fact transforms as *covariant* vector density of weight +1 so that, all in developmental notation, and valid for an *arbitrary* area patch,

$$d\mathbf{A}'|^{Lai} = J^{-1} S^{T}\, d\bar{\mathbf{A}} \qquad = J^{-1} S^{T}\, d\mathbf{A}$$

This can be inverted to get

$$d\mathbf{A} = J\,(S^{T})^{-1}\, d\mathbf{A}'|^{Lai} \qquad (*)$$

and THIS is what equation 3.27.10 says if I just write

$$d\mathbf{A}'|^{Lai} = d\mathbf{x}'^{(1)} \times d\mathbf{x}'^{(2)} = dA'|^{Lai}\, \mathbf{e}'_3$$

and then we get since J = det(S)

$$d\mathbf{A} = dA'|^{Lai} \det(S)\,(S^{T})^{-1}\, \mathbf{e}'_3$$

which then translates to

$$d\mathbf{A} = dA_0 \det(F)\,(S^{T})^{-1}\, \mathbf{e}'_3$$

and this really is 3.27.10. Alternatively, one could write

$d\mathbf{A}'|^{Lai} = (dA'|^{Lai})\ \hat{\mathbf{n}}'$

$d\mathbf{A} = (dA)\ \hat{\mathbf{n}}$

and then (*) above becomes

$(dA)\ \hat{\mathbf{n}} = (dA'|^{Lai})J\ (S^T)^{-1}\ \hat{\mathbf{n}}'$

Finally, using the inverse flow model, where x' = X at $t = t_0$ and where S = F, we can write

$dA\ \hat{\mathbf{n}} = dA_0 J\ (F^T)^{-1}\ \hat{\mathbf{n}}_0$ J = det(F)'

and this agrees with Lai p 175 4.10.9.

6. New material just added to tensor doc

I made several additions which I will outline here:

- In Section 8 I added another level of TOC because so much is going on.
- In Section 8 (c) item 9, I broke off the little section on "covariant d**A** and dV"
- Same place, I added a section showing how d**A** and dV look like in developmental notation, and a key item added here is that $d\bar{\mathbf{A}}' = J\ S^T\ d\bar{\mathbf{A}}$ whch will relate to Lai.
- I then repaginated this Section 8

- In Section 5 (o) I greatly enlarged the footnote reatring the area ratio. I managed to work into this little section the two key Lai equations concerning area transformation. Everything done above in this doc is now shown in this little tensor doc section, namely

$dA\ \mathbf{n} = d\mathcal{A}_0\ \text{def}(F)\ (F^T)^{-1}\ \mathbf{n}_0$ // Lai p 175 (4.10.8)

$dA^{(k)}\ \mathbf{n} = d\mathcal{A}^{(k)}{}_0\ \det(F)\ (F^T)^{-1}\ \mathbf{u}_k$ // Lai p 129 (3.27.10) with k=3

These are the two equations that have been annoying me for a long time.
- Below that, I comment on Lai's use of partial developmental notation.
- I then repaginated the tail of Section 5.

The upshot is that all the material covered in THIS document