Most general rigid 2D displacement in polars PhL 4.14.12

Question 1: Consider only **rotations** for the moment, and do Cartesian only. Consider then $\mathbf{x} = R(\psi)\mathbf{X}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\psi-1 & -\sin\psi \\ \sin\psi & \cos\psi-1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \approx \begin{pmatrix} 0 & -\sin\psi \\ \sin\psi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where we assume ψ is small because all displacements are supposed to be small.

Question 2: How would you express the above result in polar coordinates? I think this is correct:

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = R_z(-\theta) \begin{pmatrix} u_x \\ u_y \end{pmatrix} \qquad \text{since } \hat{\mathbf{r}} = R_z(+\theta)\hat{\mathbf{x}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Then we get

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -\sin\psi \\ \sin\psi & 0 \end{pmatrix} \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = r\sin\psi \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= r\sin\psi \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This says that u_r is unchanged by this rotation but u_θ is and we get

$$u_r = 0 \qquad u_\theta = rk \qquad\qquad k = \sin\psi$$

This then seems to account for the "Fr" term in u_θ in 5.29.16.

Question 3: What does a translation do in polar coordinates

In Cartesians it does this

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} X+a_x \\ Y+a_y \end{pmatrix}$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Then in polars we get

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$u_r = a_x \cos\theta + a_y \sin\theta = G\cos\theta + H\sin\theta \qquad G = a_x \qquad H = a_y$$
$$u_\theta = -a_x \sin\theta + a_y \cos\theta = -G\sin\theta + H\cos\theta$$

and this accounts for the other terms in 5.29.15,16,

Question 4: How do you handle rotation and translation at the same time?

Let's try first translate the rotate. The translate part results in

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Now rotate to get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \left[\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix} \right]$$

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \left[\begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix} \right] - \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi - 1 & -\sin\psi \\ \sin\psi & \cos\psi - 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$\approx \begin{pmatrix} 0 & -\sin\psi \\ \sin\psi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} 0 & -\sin\psi \\ \sin\psi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

where the first term is the same as in Question 1, and in the second we drop order ε^2 terms.

Now as before we have

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

and since this is linear, we get the sum of the two results we got in the separate questions:

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -\sin\psi \\ \sin\psi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

$$= r\sin\psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

And so

$$u_r = G\cos\theta + H\sin\theta$$

$u_\theta = - G\sin\theta + H\cos\theta + Fr$