This is the Title PhL 3.26.05

I am going to try this problem before reading about it in Lai. The famous cantilever beam. Assume 2 is down and 1 is to the right. The Navier equation is local to a point as says ρg is the gravity force so

$0 = 0 + \partial_j T_{1j}$ // the 1 direction equation
$0 = \rho g + \partial_j T_{2j}$ // the 2 direction equation

We assume things stay in a vertical plane so $u_3 = 0$ and also $T_{3i} = 0$ for all i. Then the equations above are

$0 = \partial_1 T_{11} + \partial_2 T_{12}$
$0 = \rho g + \partial_1 T_{21} + \partial_2 T_{22}$

Assume T symmetric. And assume isoptropic elastic solid so that

$T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij}$ $E_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ $e = \text{div } u = \partial_1 u_1 + \partial_2 u_2$

So where does this lead us?

$T_{11} = \lambda e + 2\mu E_{11} = \lambda e + 2\mu \partial_1 u_1 = \lambda \partial_1 u_1 + \lambda \partial_2 u_2 + 2\mu \partial_1 u_1 = (\lambda+2\mu)\, \partial_1 u_1 + \lambda \partial_2 u_2$
$T_{22} = \lambda e + 2\mu E_{22} = \lambda e + 2\mu \partial_2 u_2 = \lambda \partial_1 u_1 + \lambda \partial_2 u_2 + 2\mu \partial_2 u_2 = \lambda\, \partial_1 u_1 + (\lambda+2\mu)\partial_2 u_2$
$T_{12} = 2\mu E_{12} = \mu(\partial_1 u_2 + \partial_2 u_1)$

Now we need derivatives:

$\partial_1 T_{11} = \partial_1[(\lambda+2\mu)\, \partial_1 u_1 + \lambda \partial_2 u_2] = (\lambda+2\mu)\, \partial_1^2 u_1 + \lambda\, \partial_1 \partial_2 u_2$

$\partial_2 T_{12} = \partial_2[\mu(\partial_1 u_2 + \partial_2 u_1)] = \mu \partial_1 \partial_2 u_2 + \mu \partial_2^2 u_1$

So insisting these add to zero requires that

$(\lambda+2\mu)\, \partial_1^2 u_1 + \lambda\, \partial_1 \partial_2 u_2 + \mu \partial_1 \partial_2 u_2 + \mu \partial_2^2 u_1 = 0$

$(\lambda+2\mu)\, \partial_1^2 u_1 + (\lambda+\mu)\, \partial_1 \partial_2 u_2 + \partial_2^2 u_1 = 0$ (1)

which is a strange 2D PDE. The other equation needs

$\partial_2 T_{22} = \lambda\, \partial_1 \partial_2 u_1 + (\lambda+2\mu)\partial_2^2 u_2$

and then it says

$-\rho g = \partial_1 T_{21} + \partial_2 T_{22} = [\mu(\partial_1^2 u_2 + \partial_1 \partial_2 u_1) + \lambda\, \partial_1 \partial_2 u_1 + (\lambda+2\mu)\partial_2^2 u_2$

$-\rho g = \mu \partial_1^2 u_2 + (\mu+\lambda)\, \partial_1 \partial_2 u_1 + (\lambda+2\mu)\partial_2^2 u_2$ (2)

So there are our two equations. Write again as

$$(\lambda+2\mu)\,\partial_1^2 u_1 + (\lambda+\mu)\,\partial_1\partial_2 u_2 + \partial_2^2 u_1 = 0 \qquad (1)$$
$$\mu\partial_1^2 u_2 + (\lambda+\mu)\,\partial_1\partial_2 u_1 + (\lambda+2\mu)\partial_2^2 u_2 = -\rho g \qquad (2)$$

In the small motion limit, I have done everything exactly. Now, suppose we assume that $u_1 << u_2$. As our cantilever bends down, the up down motion is larger than the left right motion of a particle. Then we could just set perhaps $u_1 = 0$ and then have

$$(\lambda+\mu)\,\partial_1\partial_2 u_2 = 0$$
$$\mu\partial_1^2 u_2 + (\lambda+2\mu)\partial_2^2 u_2 = -\rho g$$

This is all for a particle in our little beam, it is just self loading. Do I know the solution of this:

$$A\partial_x^2 f + B\partial_y^2 f = C$$

This is a 2nd order linear PDE, so I presume there are two solutions. Try

$$f = a + bx + cy + \alpha xy + \beta x^2 + \gamma y^2$$

$$\partial_x^2 f = 2\beta$$
$$\partial_y^2 f = 2\gamma$$

$$A2\beta + B2\gamma = C$$

So OK, we assume then that

$$u_2 = a + bx_1 + cx_2 + \alpha x_1 x_2 + \beta x_1^2 + \gamma x_2^2$$

Then our first two equations say

$$(\lambda+\mu)\,\partial_1[c + \alpha x_1 + 2\gamma x_2] = 0$$

$$(\lambda+\mu)\alpha = 0 \qquad \Rightarrow \alpha = 0 \text{ which I guess is OK by me}$$

The second equation says

$$\mu 2\beta + (\lambda+2\mu)2\gamma = -\rho g$$

and this just makes a connection between β and γ.

$$\beta + [\lambda+2\mu)/\mu]\gamma = -\rho g/2\mu \qquad \Rightarrow \qquad \beta = -\rho g/2\mu - [\lambda+2\mu)/\mu]\gamma$$

I would also require that $u_2(0,0) = 0$ and so then $a = 0$. Our solution might then be

$$u_2(x_1,y_1) = bx_1 + cx_2 + \beta x_1^2 + \gamma x_2^2$$

OK, stop. I am just goofing around, let's now see how it's done right then come back here and make repairs. I know I omitted the fact that the total Mg of the beam equals the total shear force at the connection. I think there are approximations I need to make beyond those above.