Time derivatives of differentials and Squirmy Integrals PhL 3.19.12

Comment: These are important notes. Something I did on my own way back in 3/12 was actually a derivation of both forms of the Reynold's Transport Theorem on page 418 which I encountered in 6/12 and which I am resuming with in 9/12. My notion of a "squirmy integral" in 1D, where the dx move and are written dx(t), becomes a "squirmy volume integral" in 3D, where the squirmy volume is the moving volume of fluid dynamics, as compared with the non-moving control volume. But in the end you can derive RTT with nothing squirmy, as shown in item 5.

I just can't seem to get comfortable with these things, not sure they are "real" or just fiddle-faddle.

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1. Dealing with $d_t(dx)$: the dumbbell model

Page 95. (Ch3) This talks about $d_t(d\mathbf{x})$. The physical meaning of this is that $d\mathbf{x}$ represents a little dumbbell having two endpoints, and as we flow, this dumbbell rotates and stretches. The endpoints of the dumbbell track two points in the flow. That seems totally clear and somewhat specific to continuum mechanics. Think of $d\mathbf{x}$ as a probe in the flow.

Now here seems one way to derive this. Consider our very basic starting point

$$d\mathbf{x} = F\, d\mathbf{X}$$

This tells how the dumbbell which starts at time t_0 ends up at time t. $d\mathbf{X}$ is a rock solid constant. So

$$d_t(d\mathbf{x}) = (d_t F)\, d\mathbf{X} \ .$$

This must be valid! Write out in components in dev notation, and below ∂_i means $\partial/\partial X^i$

$$d_t(dx_i) = (d_t F_{ij}) dX_j \qquad\qquad F_{ij} = (\partial x_i/\partial X_j) = \partial^{(\mathbf{X})}{}_j x_i = (\nabla_{\mathbf{X}}\mathbf{x})_{ij}$$

$$d_t F_{ij} = d_t(\partial x_i/\partial X_j) \ .$$

$F_{ij} = (\partial x_i/\partial X_j)$ is telling us how components of dumbbells change. More detail on the above

$$F_{ij} = (\partial x_i/\partial X_j) = (\partial x_i(\mathbf{X},t)/\partial X_j) = F_{ij}(\mathbf{X},t) = \hat{F}_{ij}(\mathbf{X},t) \ .$$

Consider now (X is a constant, so $d_t \to \partial_t$ here),

$$(d_t F_{ij}) = d_t(\partial x_i(\mathbf{X},t)/\partial X_j) = \partial_t \partial^{(\mathbf{X})}{}_j \, x_i(\mathbf{X},t) = \partial^{(\mathbf{X})}{}_j \, \partial_t \, x_i(\mathbf{X},t) = \partial^{(\mathbf{X})}{}_j \, \hat{v}_i(\mathbf{X},t)$$

and we have then shown that

$$d_t(dx_i) = (d_t F_{ij}) dX_j = [\partial^{(\mathbf{X})}{}_j \, \hat{v}_i(\mathbf{X},t)] \, dX_j = [\nabla^{(\mathbf{X})} \hat{\mathbf{v}}]_{ij} \, dX_j$$

or

$$d_t(d\mathbf{x}) = [\nabla^{(\mathbf{X})} \hat{\mathbf{v}}] \, d\mathbf{X}$$

and this agrees with p 95 (3.12.5) left equality. Both sides have dimensions L/T. This tells how the dumbbell is changing at time t, but you have to know $\hat{\mathbf{v}}(\mathbf{X},t)$ and d**X** to compute it (in this expression). Notice that you cannot take the limit d**X**→0 all the way, so this equation is different from an equation such as f '(x)dx = df where f '(x) retains its meaning in the limit. When we write $d_{\mathbf{x}}(f) = f'(x)$, neither side is a differential so the limit is a function. But in the equation above, both sides are differentials, not functions. I don't recall this notion coming up in Stak's distribution theory.

Now go back a few lines and write

$$(d_t F_{ij}) = \partial^{(\mathbf{X})}{}_j \, \hat{v}_i(\mathbf{X},t) = \partial^{(\mathbf{X})}{}_j \, \tilde{\mathbf{v}}_i(\mathbf{x}(\mathbf{X}),t) = \partial/\partial X_j \, \tilde{\mathbf{v}}_i = (\partial x_k/\partial X_j)(\partial/\partial x_k)\tilde{\mathbf{v}}_i(\mathbf{x},t)$$

$$= F_{kj} \, \partial^{(\mathbf{x})}{}_k \, \tilde{\mathbf{v}}_i(\mathbf{x},t) \; .$$

Then we have

$$d_t(dx_i) = (d_t F_{ij}) dX_j = F_{kj} [\partial^{(\mathbf{x})}{}_k \, \tilde{\mathbf{v}}_i(\mathbf{x},t)] \, dX_j = [\partial^{(\mathbf{x})}{}_k \, \tilde{\mathbf{v}}_i(\mathbf{x},t)] [F_{kj} \, dX_j] = [\partial^{(\mathbf{x})}{}_k \, \tilde{\mathbf{v}}_i(\mathbf{x},t)] \, dx_k$$

$$= [\nabla^{(\mathbf{x})} \tilde{\mathbf{v}}]_{ik} \, dx_k$$

which says

$$d_t(d\mathbf{x}) = [\nabla^{(\mathbf{x})} \tilde{\mathbf{v}}] \, d\mathbf{x}$$

and this then is the other form shown in (3.12.5). Lai in 3.12.4 gets to these results much more quickly than I did, and I agree with his stuff. In any event, this second form is much more practical. We look at a dumbbell d**x** and ask how it changes in time. All we need is the Eulerian velocity field $\tilde{\mathbf{v}}(\mathbf{x},t)$ since we already know d**x .**

Since our dumbbell d**x** has a non-zero derivative, we can consider it as d**x**(t). That is to say, this thing is "squirmy".

2. Dealing with $d_t(dV)$

What does this one mean: $d_t(dV)$. The volume of our constant-mass flow volume dV is in fact changing, and it's shape is also changing. I know that $dV = detF\ dV_0$, our famous p 130 result. This is the change after a long finite flow. But certainly then we can say

$$d_t(dV) = d_t[det(F)]dV_0$$

This looks like it will be "the hard way" to do things. I will just try brute force as usual

$$d_t[det(F)] = d_t\varepsilon_{ijk}F_{1i}F_{2j}F_{3k}$$

$$= \varepsilon_{ijk}\,(d_tF_{1i})\,F_{2j}F_{3k} + \text{cyclic on } 1,2,3$$

From above we had

$$(d_tF_{ij}) = F_{aj}\,\partial^{(x)}{}_a\,\tilde{v}_i(\mathbf{x},t)$$

so

$$(d_tF_{Ij}) = F_{aJ}\,\partial^{(x)}{}_a\,\tilde{v}_I(\mathbf{x},t)$$

so

$$(d_tF_{1i}) = F_{ai}\,\partial^{(x)}{}_a\,\tilde{v}_1(\mathbf{x},t)$$

The first term above is then

$$\varepsilon_{ijk}\,F_{ai}\,\{\partial^{(x)}{}_a\,\tilde{v}_1(\mathbf{x},t)\}\,F_{2j}F_{3k} = \{\partial^{(x)}{}_a\,\tilde{v}_1(\mathbf{x},t)\}\,\varepsilon_{ijk}\,F_{ai}\,F_{2j}F_{3k}$$

Now I claim that

$$\varepsilon_{ijk}\,F_{ai}\,F_{2j}F_{3k} = \delta_{a,1}\,\varepsilon_{ijk}\,F_{ai}\,F_{2j}F_{3k} = \delta_{a,1}\,det(F)$$

so our first term is then

$$= \{\partial^{(x)}{}_a\,\tilde{v}_1(\mathbf{x},t)\}\,\delta_{a,1}\,det(F) = \{\partial^{(x)}{}_1\,\tilde{v}_1(\mathbf{x},t)\}\,det(F)$$

The sum of the three terms will then be

$$d_t[det(F)] = div^{(x)}[\,\tilde{v}_1(\mathbf{x},t)]\,det(F)$$

and then we get

$$d_t(dV) = d_t[det(F)]dV_0 = div^{(x)}[\,\tilde{\mathbf{v}}(\mathbf{x},t)]\,det(F)\,dV_0 = div^{(x)}[\,\tilde{\mathbf{v}}(\mathbf{x},t)]\,dV$$

$$= div(\tilde{\mathbf{v}})dV$$

and hurray, this agrees with p 97 (3.13.14). Once again, if we want to know how dV is changing right now at time t, first we need to know dV, which of course we DO know, and then we need that same Eulerian velocity field that we needed before. So here are our two results side by side:

$$d_t(d\mathbf{x}) = [\nabla\tilde{\mathbf{v}}]\, d\mathbf{x} \qquad \Rightarrow\ d\mathbf{x}(t)$$

$$d_t(dV) = \mathrm{div}(\tilde{\mathbf{v}})dV \qquad \Rightarrow dV(t)$$

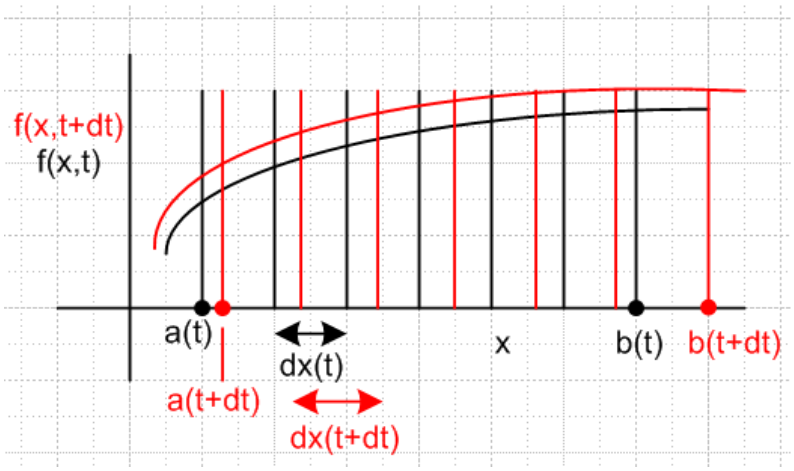
The volume element is thus also "squirmy".

3. The Squirmy 1D Integral

(a) The General Idea. Consider the following "squirmy integral"

$$I(t) \equiv \int_{\mathbf{a(t)}}^{\mathbf{b(t)}} f(x,t)\, dx(t)$$

where we now plot things at time t in black, and at time t+dt in red,



Remember that an integral is just the limit of a sum, and we can choose to do the sum in this funny way where the dx(t) vary with time. The only requirement is that the dx's "fill out" the interval. There is no rule that they have to be independent of t. Think then of dx(t) as 1D dumbbells in a flow. This sum is shown after the next paragraph. [all the dx(t) at a given time t are the same size, we don't have dx(x,t)]

At time t we arbitrarily set all the dx intervals equal, but as time progresses, they become unequal in size, and they move a little bit. This is some kind of "moving domain integral" of the type encountered in continuum mechanics. The tiny wormlike $dx_i(t)$ intervals "squirm" in time, they move and change in length (like dumbbells changing in a flow over time). Of course the picture should show 1000 of these intervals, each changing only a tiny amount. The integral at either time t or time d+dt is completely well defined as the limit of the sum of the intervals times the function value. Notice that the function also is allowed to move a small amount (red versus black curve above). We might define the black time t

situation as the "control" integration. The "control volume" is from a(t) to b(t), and the intervals for this control volume (the interval at time t) happen to be equally spaced.

The question is this: how might we compute $d_t I(t)$ where d_t is shorthand for d/dt? I propose we think of the integral at each time as a limit of sums as suggested by the above picture. Then

$$I(t) \equiv \int_{a(t)}^{b(t)} f(x(t),t)\, dx(t) = \Sigma_i\, f(x_i(t),t)\, dx_i(t)$$

$$I(t+dt) \equiv \int_{a(t+dt)}^{b(t+dt)} f(x(t+dt),t+dt)\, dx(t+dt) = \Sigma_i\, f(x_i(t+dt),t+dt)\, dx_i(t+dt)$$

Notice that $\Sigma_i = \Sigma_{i=1}^{N}$ where N is the fixed number of intervals, and N = 6 in the above example.

Now subtract

$$I(t+dt) - I(t) = \Sigma_i\, f(x_i(t+dt),t+dt)\, dx_i(t+dt) - \Sigma_i\, f(x_i(t),t)\, dx_i(t)$$

$$= \Sigma_i\, [f(x_i(t+dt),t+dt)\, dx_i(t+dt) - f(x_i(t),t)\, dx_i(t)\,]$$

Divide by dt to conclude that

$$d_t I(t) = \Sigma_i\; d_t[f(x(t),t)dx(t)] \ .$$

Now in this sum we can use the black equally spaced intervals since we are doing the sum at time t. In other words, the sum here is done in the "control volume" sense just mentioned. We then take the limit to obtain

$$d_t I(t) = \Sigma_i\; d_t[f(x(t),t)dx(t)] = \int_{a(t)}^{b(t)} d_t[f(x(t),t)dx(t)] \ .$$

Thus, we obtain our first conclusion:

$$I(t) \equiv \int_{a(t)}^{b(t)} f(x(t),t)\, dx(t) \quad \Rightarrow \quad d_t I(t) = \int_{a(t)}^{b(t)} d_t[f(x(t),t)dx(t)] \ . \qquad (*)$$

We move the d_t through the integral, but it acts not just on f(x,t), but also on dx(t). This is the unique feature of "the squirmy integral" (which must have some official name but I don't know what it is). This conclusion is "the first step" that Lai takes at the top of page 188 where he uses D/Dt for our d_t.

Of course we can now expand in the usual manner,

$$d_t[f(x(t),t)dx(t)] = d_t[f(x(t),t)]\, dx(t) + f(x(t),t)\, d_t[dx(t)] \ .$$

(b) Comment: The integral I(t) in (*) is a "moving integral" inasmuch as the integration domain changes in time. This is expressed not only in the fact that the endpoints can move, but also that the dx(t) move. In

fact, the endpoint motion is just a result of the motion of the dx(t), and the motion of the dx(t) is the driving force here. You might imagine a situation where the dx(t) all squirm, but the end points in fact don't change at all.

Now the integral $d_t I(t)$ shown in (*) is also a "moving integral" and we would have to treat it as such IF we were doing to compute something like $d_t\{d_t I(t)\}$. But at least for now, we have no intention of computing this second time derivative, and therefore we can imagine that this second integral is a fixed integral, not a moving one, and we integrate over the control volume. It is then only in this sense and for this application range that we can write

$$I(t) \equiv \int_{Im} f(x(t),t)\, dx(t) \qquad \Rightarrow \qquad d_t I(t) = \int_{Ic} d_t[f(x(t),t)dx(t)] \ ,$$

where I_m means we are integrating over the moving (squirmy) interval, while I_c means we are integrating over the control interval. As an example, if $f = \rho$, the linear mass density, then $I(t) = M$ (total mass in our interval or volume) because it is a moving integral! Our scenario is that the matter in the interval moves (squirms) but none leaves the interval, so M never changes, and that is precisely what we mean by the moving interval. It tracks the matter. It would seem then that $d_t I(t) = 0$ in this case.
We can then apply this same tracking integral to functions other than ρ if we want.

(c) Expanding the integrand. At this point, we need some kind of model for the function dx(t). We did this above in 3D, but let's do it again here in 1D. I think we cannot avoid at least temporarily involving the dX reference "dumbbell", so

$$d_t[dx(t)] = d_t[\, x(X+dX,t) - x(X,t)] = \hat{v}(X+dX,t) - \hat{v}(X,t) \ .$$
dumbbell at t=t

We can then of course express each velocity in its Eulerian sense and write

$$d_t[dx(t)] = \tilde{v}(x+dx,t) - \tilde{v}(x,t) = \partial_x \tilde{v}(x,t)\, dx(t) \ .$$

So we have now added some "physics" into the picture of a squirmy integral. Given this model for the interval function dx(t), which model really was just dx(t) = F(t)dX, where F is our linearized transformation normally called S(t), we can now continue on to say (Leibniz),

$$d_t[f(x(t),t)dx(t)] = \{\, d_t[\tilde{f}(x,t)] + \tilde{f}(x,t)\, \partial_x \tilde{v}(x,t) \,\}\, dx(t)$$

where now we might as well show the modality of both functions f and v -- they are both Eulerian.

We have now arrived at this further conclusion:

$$I(t) \equiv \int_{Im} \tilde{f}(x(t),t)\, dx(t)$$

$$\Rightarrow\ d_t I(t) = \int_{Ic} d_t[\tilde{f}(x(t),t)dx(t)] = \int_{Ic} \{\, d_t[\tilde{f}(x,t)] + \tilde{f}(x,t)\, \partial_x \tilde{v}(x,t) \,\}\, dx(t)$$

Now consider that

$$d_t[\widetilde{f}(x,t)] = \partial_t[\widetilde{f}(x,t)] + \partial_x[\widetilde{f}(x,t)]\,(\partial x/\partial t) = \partial_t[\widetilde{f}(x,t)] + \partial_x[\widetilde{f}(x,t)]\,\widetilde{v}(x,t) \ .$$

Then our integrand above is in fact

$$\{ d_t[\widetilde{f}(x,t)] + \widetilde{f}(x,t)\,\partial_x\widetilde{v}(x,t) \} = \partial_t[\widetilde{f}(x,t)] + \widetilde{f}(x,t)\,\partial_x\widetilde{v}(x,t) + \partial_x[\widetilde{f}(x,t)]\,\widetilde{v}(x,t)$$

$$= \partial_t[\widetilde{f}(x,t)] + \partial_x[\,\widetilde{f}(x,t)\,\widetilde{v}(x,t)]$$

so we now have shown the following:

$$I(t) \equiv \int_{Im} \widetilde{f}(x(t),t)\,dx(t)$$

$$\Rightarrow d_tI(t) = \int_{Ic} d_t[\widetilde{f}(x(t),t)dx(t)] = \int_{Ic} \{ \partial_t[\widetilde{f}(x,t)] + \partial_x[\widetilde{f}(x,t)\,\widetilde{v}(x,t)] \}\,dx(t)$$

As Lai notes on page 189, the twiddle is omitted when it is obvious that we are dealing with Eulerian functions, but I will maintain it for a while anyway.

Application of the 1D Squirmy Integral. Suppose we have a distribution of mass dm = ρ(x)dx along a line. Suppose we select some segment of the line which contains some of this mass distribution. Suppose the blob of mass is moving in some smooth manner. Let a(t) be the left edge of our selected blob, and let b(t) be the right edge. As the blob smoothly evolves, a(t) continues to be the left edge and b(t) continues to be the right edge. This in fact is how a(t) and b(t) are defined for this application. That means that if we just look at the integral from a(t) to b(t) of dm = ρdx, we always get the same total M, and that means the time derivative of that particular integral $d_tI(t)$ would be zero. This is the type of application we are interested in! The interval which moves and is always between a(t) and b(t) is the "moving volume" or the "material volume". We could pick a "control volume" at some instant in time (like t), but that is not the application we are interested in. We are interested in the physics of the matter in a moving blob. So the integral to which we apply d_t will always be a "moving volume" in our applications in fluid mechanics. The integrals on the RHS can be either moving or fixed volume integrals, they are the same at time t. You only need to use a moving volume when you are going to apply d_t to an integral.

4. The Squirmy 3D Integral and contact with Lai page 188

We know that $d_t(dV) = \mathrm{div}(\widetilde{\mathbf{v}})dV$ from section 2 above, so we can see what the 3D version of the above equations will look like: [just as we had a(t) and b(t) above, we write $V_m(t)$ as the moving region)

$$I(t) \equiv \int_{Vm(t)} \widetilde{f}(\mathbf{x}(t),t)\,dV(t)$$

$$\Rightarrow\ d_t I(t) = \int_{V_c} d_t[\tilde{f}(\mathbf{x}(t),t)dV(t)] \quad = \int_{V_c} \{\, d_t[\tilde{f}(x,t)] \ +\ \tilde{f}(x,t)\ \mathrm{div}(\tilde{\mathbf{v}})\,\}\ dV(t)\ .$$

Again, I(t) is a moving volume integral so we write V_m as the integration domain. As long as no further time derivatives are planned, we can regard the integrals for $d_t I(t)$ as V_c, over the control volume. Now doing again what was done above

$$d_t[\tilde{f}(\mathbf{x},t)] \ = \partial_t[\tilde{f}(\mathbf{x},t)] \ + \partial_i\ \tilde{f}(\mathbf{x},t)\ (\partial x_i/\partial t) = \ \partial_t[\tilde{f}(\mathbf{x},t)] \ + \partial_i \tilde{f}(\mathbf{x},t)\ \tilde{\mathbf{v}}_i(\mathbf{x},t)$$

$$= \ \partial_t[\tilde{f}(\mathbf{x},t)] \ + \nabla \tilde{f}(\mathbf{x},t) \bullet \tilde{\mathbf{v}}(\mathbf{x},t)$$

Then the integrand is

$$\{\, d_t[\tilde{f}(x,t)] \ +\ \tilde{f}(x,t)\ \mathrm{div}(\tilde{\mathbf{v}})\,\} \ = \ \partial_t[\tilde{f}(\mathbf{x},t)] \ + \nabla \tilde{f}(\mathbf{x},t) \bullet \tilde{\mathbf{v}}(\mathbf{x},t) \ +\ \tilde{f}(x,t)\ \mathrm{div}(\tilde{\mathbf{v}})$$

Now write the following vector identity,

$$\nabla\bullet(\varphi \mathbf{A}) = \mathbf{A} \bullet \nabla\varphi + \varphi\ (\nabla\bullet\mathbf{A})$$

$$\nabla\bullet(\tilde{f}\tilde{\mathbf{v}}) = \tilde{\mathbf{v}} \bullet \nabla\tilde{f} + \tilde{f}\ (\nabla\bullet\tilde{\mathbf{v}})$$

to conclude that

$$\{\, d_t[\tilde{f}(x,t)] \ +\ \tilde{f}(x,t)\ \mathrm{div}(\tilde{\mathbf{v}})\,\} \ = \ \partial_t[\tilde{f}(\mathbf{x},t)] \ + \mathrm{div}\ (\tilde{f}\ \tilde{\mathbf{v}})$$

and we end up with this situation:

$$I(t) \equiv \ \int_{V_m(t)}\ \tilde{f}(\mathbf{x},t)\ dV(t)$$

$$\Rightarrow\ d_t I(t) = \int_{V_c} d_t[\tilde{f}(\mathbf{x},t)dV(t)] \quad = \int_{V_c} \{\, \partial_t[\tilde{f}(\mathbf{x},t)] \ + \mathrm{div}\ (\tilde{f}\ \tilde{\mathbf{v}})\} \ dV(t)$$

Example: Suppose f = ρ, the 3D mass density. Then we know that I(t) = 0 as a moving integral. We conclude then that

$$\int_{V_c} \{\, \partial_t[\tilde{\rho}(\mathbf{x},t)] \ + \mathrm{div}\ (\tilde{\rho}\ \tilde{\mathbf{v}})\} \ dV(t) \ = 0$$

Since this must be valid for any control volume V_c, if we shrink that down, we conclude that in fact it must be true that

$\partial_t \widetilde{\rho} + \text{div}\,(\widetilde{\rho}\,\widetilde{\mathbf{v}}) = 0$ as a local differential statement

Now $\widetilde{\rho}\,\widetilde{\mathbf{v}} = \widetilde{\mathbf{J}}$, the mass current, so this says

$$\partial_t \widetilde{\rho} + \text{div}\,(\widetilde{\mathbf{J}}) = 0$$

which is "the usual" equation of continuity for any "flow" situation where the stuff that is flowing is neither being created nor destroyed.

Our general analysis above and this specific example appear in Lai p 188 top.

Notes added 6.30.12. Go back one step in the above to where we had,

$$I(t) \equiv \int_{Vm(t)} \widetilde{f}(\mathbf{x}(t),t)\, dV(t)$$

$$\Rightarrow\ d_t I(t) = \int_{Vc} d_t[\widetilde{f}(\mathbf{x}(t),t) dV(t)] \ = \int_{Vc} \{\, d_t[\widetilde{f}(x,t)] + \widetilde{f}(x,t)\, \text{div}(\widetilde{\mathbf{v}}) \,\}\, dV(t)\ .$$

Write this using the Lai derivative notation,

$$D/Dt\ \{ \int_{Vm(t)} \widetilde{f}(\mathbf{x}(t),t)\, dV(t) \} = \int_{Vc} [\ D/Dt\ [\widetilde{f}(x,t)] + \widetilde{f}(x,t)\, \text{div}(\widetilde{\mathbf{v}})\ ]\, dV(t)$$

Now, $\widetilde{f}(\mathbf{x},t)$ is an arbitrary scalar function in this whole analysis. It could be a component of some tensor of arbitrary rank. We would then have the above statement for each component of that tensor:

$$D/Dt\ \{ \int_{Vm(t)} \widetilde{T}_{ijk..}(\mathbf{x}(t),t)\, dV(t) \} = \int_{Vc} [\ D/Dt\ [\widetilde{T}_{ijk..}(\mathbf{x},t)] + \widetilde{T}_{ijk..}(\mathbf{x},t)\, \text{div}(\widetilde{\mathbf{v}})\ ]\, dV(t)$$

or in shorter notation

$$D/Dt\ \{ \int_{Vm(t)} \widetilde{T}_{ijk..}(\mathbf{x},t)\, dV \} = \int_{Vc} [\ D/Dt\ [\widetilde{T}_{ijk..}(\mathbf{x},t)] + \widetilde{T}_{ijk..}(\mathbf{x},t)\, \text{div}(\widetilde{\mathbf{v}})\ ]\, dV(t)$$

or in still tighter notation

$$D/Dt\ \{ \int_{Vm(t)} \widetilde{T}_{ijk..}\, dV \} = \int_{Vc} [\ D/Dt\ [\widetilde{T}_{ijk..}] + \widetilde{T}_{ijk..}\, \text{div}(\widetilde{\mathbf{v}})\ ]\, dV$$

and if we then leave off the tensor subscripts (same in each term), and remove the twiddle,

$$D/Dt\ \{ \int_{Vm(t)} T(\mathbf{x},t)\ dV \} = \int_{Vc} [\ DT/Dt + T\, \text{div}(\mathbf{v})\ ]\, dV$$

Much to my amazement, this is precisely the **Reynolds Transport Theorem** of 7.4.2 p 418, written in one of its two forms! So before I even started into Chapter 7, I had already derived this thing! I just didn't know what it was called.

I will now produce the other form 7.4.1. Go back to the above where we had

$$I(t) \equiv \int_{vm(t)} \tilde{f}(\mathbf{x},t)\, dV(t)$$

$$\Rightarrow d_t I(t) = \int_{vc} d_t[\tilde{f}(\mathbf{x},t) dV(t)] = \int_{vc} \{ \partial_t[\tilde{f}(\mathbf{x},t)] + \text{div}\,(\tilde{f}\,\tilde{\mathbf{v}})\}\, dV(t) \ .$$

Now use the divergence theorem to replace the last volume integral

$$d_t I(t) = \int_{vc} \{ \partial_t[\tilde{f}(\mathbf{x},t)]\, dV(t) + \int_{sc} d\mathbf{S} \bullet (\tilde{f}\,\tilde{\mathbf{v}})$$

$$= \int_{vc} \{ \partial_t[\tilde{f}(\mathbf{x},t)]\, dV(t) + \int_{sc} dS\, \mathbf{n} \bullet (\tilde{f}\,\tilde{\mathbf{v}})$$

$$= \int_{vc} \{ \partial_t[\tilde{f}(\mathbf{x},t)]\, dV(t) + \int_{sc} \tilde{f}\, (\tilde{\mathbf{v}} \bullet \mathbf{n})\, dS$$

Again replace $\tilde{f}$ with our generic tensor component to get

$$D/Dt \{ \int_{vm(t)} T(\mathbf{x},t)\ dV \} = \int_{vc} \partial_t T\, dV + \int_{sc} T\, (\tilde{\mathbf{v}} \bullet \mathbf{n})\, dS$$

and this then verifies the form 7.4.1 page 418.

5. Derivation of 1D then 3D RTT without need for squirmy anything.

Here is such an integral:

$$I(t) \equiv \int_{a(t)}^{b(t)} f(x,t)\, dx$$

$$I(t+dt) \equiv \int_{a(t+dt)}^{b(t+dt)} f(x,t+dt)\, dx =$$

$$= \int_{a(t)}^{b(t)} f(x,t+dt)\, dx + \int_{b(t)}^{b(t+dt)} f(x,t+dt)\, dx - \int_{a(t)}^{a(t+dt)} f(x,t+dt)\, dx$$

where I have assumed that a(t) and b(t) are both monotonic increasing functions. If they are not, I think the same equation is true, but the second two integrals could be negative. Now keep going :

$$\approx \int_{a(t)}^{b(t)} f(x,t+dt)\,dx + f(b,t)\int_{b(t)}^{b(t+dt)} dx - f(a,t)\int_{a(t)}^{a(t+dt)} dx$$

$$= \int_{a(t)}^{b(t)} f(x,t+dt)\,dx + f(b,t)\,[b(t+dt) - b(t)] - f(a,t)\,[a(t+dt) - a(t)]$$

$$= \int_{a(t)}^{b(t)} f(x,t+dt)\,dx + f(b,t)b'(t)dt - f(a,t)\,a'(t)dt$$

This is all I(t+dt). Rewrite the first term as

$$\int_{a(t)}^{b(t)} f(x,t+dt)\,dx = \int_{a(t)}^{b(t)} [f(x,t) + dt\,\partial_t f(x,t)]\,dx = I(t) + dt\int_{a(t)}^{b(t)} \partial_t f(x,t)dx$$

Therefore we have

$$I(t+dt) = \int_{a(t)}^{b(t)} f(x,t+dt)\,dx + f(b,t)b'(t)dt - f(a,t)\,a'(t)dt$$

$$= I(t) + dt\int_{a(t)}^{b(t)} \partial_t f(x,t)dx + f(b,t)b'(t)dt - f(a,t)\,a'(t)dt$$

It follows that

$$d_t I(t) = \int_{a(t)}^{b(t)} \partial_t f(x,t)dx + f(b,t)b'(t) - f(a,t)\,a'(t)$$

Now let $v_b = b'(t)$ be the velocity of the right endpoint, and think of it as $\mathbf{v_b} = v_b\,\hat{\mathbf{x}}$. At that end point the normal vector to the region is $\hat{\mathbf{n}} = +\hat{\mathbf{x}}$ so then $f(b,t)b'(t) = f(b,t)\,(\mathbf{n} \bullet \mathbf{v_b})$. At the other end we have the opposite sign for n, so we have $-f(a,t)\,a'(t) = f(b,t)\,(\mathbf{n} \bullet \mathbf{v_a})$ where $\mathbf{v_a} = v_a\,\hat{\mathbf{x}}$. Then we have

$$d_t I(t) = \int_{a(t)}^{b(t)} \partial_t f(x,t)dx + \mathrm{sum}_i\, f(x_i,t)\,(\mathbf{n} \bullet \mathbf{v_i})$$

and once again, where Σ_i represents a sum over points on the boundary of region (a(t),b(t))

$$D_t \int_{a(t)}^{b(t)} f(x,t)\,dx = \int_{a(t)}^{b(t)} \partial_t f(x,t)dx + \Sigma_i\, f(x_i,t)\,(\mathbf{n} \bullet \mathbf{v_i})$$

Now one more step. Let $t = \tau$ = some particular fixed time. Then the above says

$$[\,D_t\,\{\int_{a(t)}^{b(t)} f(x,t)\,dx\}\,]^{t=\tau} = \int_{a(\tau)}^{b(\tau)} [\partial_t f(x,t)]^{t=\tau}\,dx + \Sigma_i\, f(x_i,\tau)\,(\mathbf{n} \bullet \mathbf{v_i})$$

We can then extract the ∂_t to get

$$[\, D_t \{ \int_{a(t)}^{b(t)} f(x,t)\, dx \} \,]^{t=\tau} = \{\partial_t [\int_{a(\tau)}^{b(\tau)} f(x,t) dx \,]\}^{t=\tau} + \Sigma_i\, f(x_i,\tau)\, (\mathbf{n} \bullet \mathbf{v}_i)$$

We now regard $\int_{a(t)}^{b(t)} = \int_{V_m}$ and $\int_{a(\tau)}^{b(\tau)} = \int_{V_c}$ to get

$$[\, D_t \{ \int_{V_m} f(x,t)\, dx \} \,]^{t=\tau} = \int_{V_c} [\partial_t f(x,t)]^{t=\tau}\, dx + \Sigma_i\, f(x_i,\tau)\, (\mathbf{n} \bullet \mathbf{v}_i)$$

$$[\, D_t \{ \int_{V_m} f(x,t)\, dx \} \,]^{t=\tau} = \{\partial_t [\int_{V_c} f(x,t) dx \,]\}^{t=\tau} + \Sigma_i\, f(x_i,\tau)\, (\mathbf{n} \bullet \mathbf{v}_i)$$

The first of these two lines is fairly unambiguous if we then set $\tau = t$

$$D_t \{ \int_{V_m} f(x,t)\, dx \} = \int_{V_c} [\partial_t f(x,t)]\, dx + \Sigma_i\, f(x_i,t)\, (\mathbf{n} \bullet \mathbf{v}_i)$$

but the second line, which often appears in literature, is a little hazy since we have to write

$$D_t \{ \int_{V_m} f(x,t)\, dx \} = \partial_t [\int_{V_c} f(x,t) dx \,] + \Sigma_i\, f(x_i, t)\, (\mathbf{n} \bullet \mathbf{v}_i)$$

But in this second line, the reader just "thinks of" V_c as independent of time so there is then no confusion. But when you actually write out $\int_{V_c} = \int_{a(t)}^{b(t)}$, the confusion comes back a bit. One simply has to understand that the ∂_t passes through the integral. Lai only uses the first form.

Now, the last term is the "outflow of f through the boundary". There is no ambiguity concerning the sign of this last term, it is what we derived it to be. So we have

$D_t \{ \int_{V_m} f(x,t)\, dx \}$	=	$\partial_t [\int_{V_c} f(x,t) dx \,]$	+	$\Sigma_i\, f(x_i,t)\, (\mathbf{n} \bullet \mathbf{v}_i)$
change in amount of f in moving V_m		change in amount of f in fixed V_c		outflow of f thru boundary of V_c

If we now add the physical extra input and say that V_m is a boundary which tracks fluid velocity v, then we know that V_m contains the same particles all the time and can be tried as a "particle" in the sense of particle mechanics. In this case, if f(x,t) = mass density per cm $\rho(x,t)$, then LHS = 0 and we get that the change in the V_c amount of mass equals - ouflow = +inflow of mass, so signs are right.

The above line then becomes the RTT for a flow in 1D. In 3D things look that same, but the boundary is then not just two points but a 2D surface and we then have

$D_t \{ \int_{V_m} f(x,t)\, dx \}$	=	$\partial_t [\int_{V_c} f(x,t) dx \,]$	+	$\int_{S_c} f(x,t)\, (\mathbf{n} \bullet \mathbf{v})$
change in amount		change in amount		outflow of f thru

of f in moving V_m of f in fixed V_c boundary of V_c

Now again, if f = ρ, we find $\partial_t[\int_{V_c} \rho(x,t)dx\ ] = -\int_{S_c} \rho(x,t)\ (\mathbf{n} \bullet \mathbf{v})$ which says that the increase in mass in the fixed control volume equals the mass inflow.

Thus, we have derived the Reynolds Transport Theorem without the need for any squirmy dx segments. In this section the dx were just constants. That is one main point of this subsection.

The other point is that the sign of the last term is plus and is always an *out*flux of f.

Now, my derivation here did not use squirmy anything. But moving from the 1D to 2D or 3D is not immediately obvious. I could do it, but it takes some work. The extra pieces above in the 1D case,

$$\int_{b(t)}^{b(t+dt)} f(x,t+dt)\ dx - \int_{a(t)}^{a(t+dt)} f(x,t+dt)\ dx$$

must become

$$\int_{\Delta V} f(x,t+dt)\ dV$$

where $\Delta V = V_m - V_c$ = the difference volume. Of course the shape of ΔV is determined by the local velocity **v** at each point on the boundary V_c, and that is how **v** will get into the picture. In fact, a little dV piece of ΔV would be $dV = (\mathbf{n}\bullet\mathbf{v})\ dt\ dS$ so there it is :

$$I(t+dt) = \int_{V_c} f(x,t+dt)\ dV + \int_{\Delta V} f(x,t+dt)\ dV$$

$$= \int_{V_c} f(x,t)\ dV + dt\ \partial_t \int_{V_c} f(x,t)\ dV + \int_{S_c} f(x,t+dt)\ (\mathbf{n}\bullet\mathbf{v})\ dt\ dS$$

$$= I(t) + dt\ \partial_t \int_{V_c} f(x,t)\ dV + \int_{S_c} f(x,t+dt)\ (\mathbf{n}\bullet\mathbf{v})\ dt\ dS$$

And then we get our desired 3D result

$$D_t I(t) = \partial_t \int_{V_c} f(x,t)\ dV + \int_{S_c} f(x,t)\ (\mathbf{n}\bullet\mathbf{v})\ dS$$

and again we have an outflow term with a + sign on the right.

So I have now derived the 3D RTT without needing anything "squirmy".

You can always "check" the sign of the second term by treating f = ρ.