**Essay on Continuum Mechanics** PhL 2.4.12

1. What is often called classical Lagrangian mechanics concerns the motion of point particles, as presented for example in Goldstein's book Classical Mechanics. The "kinematics" of a system of particles could be stated in this way: for each particle i there is some position function $\mathbf{x}_i(t)$. Kinematics is a part of the subject which can be separated from the rest of the subject which involves forces which cause the particles to *have* those trajectories $\mathbf{x}_i(t)$. Basically this rest of the subject is Newton's $\mathbf{F} = m\mathbf{a}$. Lagrangian mechanics has what we might call "the movie" or "the flow" where the particles might have some positions $\mathbf{x}_i(t_0) = \mathbf{X}_i$ at time $t_0$, and then at some later time t they all have some new positions $\mathbf{x}_i(t)$. You imagine that the film runs from $t=t_0$ to t=t and you just watch all the particles move along their trajectories. So kinematics is just whatever set of parameters you need to "describe" the particles positions. Each particle might also have a mass $m_i$ which we might regard as part of the kinematics. We can derive things like velocity and acceleration $\mathbf{v}_i(t)$ and $\mathbf{a}_i(t)$ by applying $\partial_t$ to $\mathbf{x}_i(t)$, so these latter items are derived items.

2. Continuum mechanics can be regarded as some kind of limit of Lagrangian mechanics in which the number of particles becomes infinite and the particles form "continuous matter", solid, liquid, gas, plasma, whatever. The "particles" of this continuous matter are regarded as infinitely small in a differential sense, but in practice such a particle of continuous matter is always large enough to contain very many "atoms" or "molecules" or whatever the "underlying constituents" are. A particle of continuous matter has a differential volume dV, a differential mass $dm = \rho dV$ where $\rho$ is the mass density per $cm^3$. The particle might be a tiny orthogonal 3-piped whose edges are spanned by differential vectors $d\mathbf{x}_1 = ds_1\hat{\mathbf{u}}_1$ and similarly for the other two. Because this particle contains many constituents, it can have thermodynamic properties such as temperature, something that does not exist in the Lagrangian picture stated above. This is a new feature which appears in continuum mechanics (an emergent property).

3. In Lagrangian mechanics the basic kinematics was $\mathbf{x}_i(t)$ with $\mathbf{x}_i(t_0) = \mathbf{X}_i$. In continuum mechanics this index i becomes a continuous index. We in fact can label each "particle" by the location it had at time $t=t_0$, which position we just call **X**. Instead of writing $\mathbf{x}_{\mathbf{X}}(t)$ we write $\mathbf{x}(\mathbf{X},t)$ as a function which describes the position of the particle (which started at point **X** at $t=t_0$) at any later time t. Now the "movie" which describes the flow of a continuous matter blob from $t=t_0$ to t=t is fully described by $\mathbf{x}(\mathbf{X},t)$. You pick any particle at **X** in the initial blob at the movie start time $t_0$, and then $\mathbf{x}(\mathbf{X},t)$ tells you where that particle ends up at time t. Sometimes instead of using **x**, one uses $\mathbf{u} = \mathbf{x}-\mathbf{X}$ which is called "the displacement field". It is a "field" because it is a function of points in space **X**. The displacement of course just says where each particle goes *relative to* its starting position.

4. Lagrangian mechanics, describing separated particles like planets going round the sun, does not have any notion of the differential distance between two particles, but in continuum mechanics, this differential distance is almost the whole name of the game! If two closely spaced particles at time $t_0$ are spaced by d**X**, then at time t they will still be closely spaced, but the separation will be some d**x**. The elapsed time of the movie $t-t_0$ does not have to be a differential time dt here, it can be a finite time. As long as the flow of the blob is "smooth", a tiny d**X** will map into a tiny d**x** even after a finite duration $t-t_0$. It is fairly easy to

see how d**x** is related to d**X** : [ since both ends of these each of these vectors are at the same time, there is no dt time contribution to the following differentials. Also, implied summation is used for j. ]

$$d\mathbf{x} = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t)$$

$$dx_i = x_i(\mathbf{X}+d\mathbf{X},t) - x_i(\mathbf{X},t) = (\partial x_i/\partial X_j)\, dX_j = F_{ij}dX_j \qquad \text{where } F_{ij} = (\partial x_i/\partial X_j)$$

so matrix F is a set of derivatives of the function **x**(**X**,t). In general, $F_{ij} = F_{ij}(\mathbf{X})$ since **x**(**X**,t) is in general a non-linear transformation from **X** to **x**. If we define $\partial_j$ to mean $\partial/\partial X_j$ then the above says

$$dx_i = \partial_j x_i \; dX_j = F_{ij}dX_j$$

It is traditional to define the following matrix ( I call it a reverse dyadic, but it is just a definition of a certain matrix and nothing more)

$$(\nabla \mathbf{w})_{ij} \equiv \partial_j w_i \qquad \mathbf{w}(\mathbf{X}) \text{ is any vector function of } \mathbf{X}$$

Then what the above says is

$$F_{ij} = \partial_j x_i = (\nabla \mathbf{x})_{ij}$$

so we can then identify the matrix F with the matrix (**∇x**), and one they has in vector/matrix notation

$$d\mathbf{x} = F\, d\mathbf{X} = (\nabla \mathbf{x})\, d\mathbf{X}$$

From now on, we just use the matrix F and work with

$$d\mathbf{x} = F(\mathbf{X},t)\, d\mathbf{X}$$

where we emphasize that matrix F is in some function of the vector **X** and of time t. F is a "matrix field" or a "rank-2 tensor field".

5. Having now "done the math", we must comment on a major new feature of continuum mechanics which is not present in Lagrangian particle mechanics: the fact that particles of continuum matter have a "shape" and that the shape *changes* during the flow of a blob of such matter. At time $t=t_0$ one can consider somewhere in the blob at location **X** a particular tiny rectangular solid orthogonal box spanned by three differential vectors (a "particle"), and later at time t=t this same particle of matter (the one that started at location **X**) has some different shape. Those three differential bounding vectors, call them $d\mathbf{X}^{(i)} = dS_i\, \hat{\mathbf{u}}_i$ for i=1,2,3 ( the u things are unit vectors) have changed into three non-orthogonal vectors $d\mathbf{x}^{(i)} = ds_i \hat{\mathbf{n}}_i$, which vectors now span a 3D parallelepiped which I like to call a 3-piped. The matter inside this tiny skewed 3-piped is exactly the same matter that started in the little box. Since the box might be larger or smaller, the mass density ρ could be different for our particle at **x**, compared to when that same particle was at point **X**. The temperature and pressure might be different. Lots of things might be different.

The way the tiny matter particles change shape is completely described by the matrix F(**X**,t) discussed above. The information is all there. After all, F tells how any tiny spatial vector changes, so it can tell you how the areas of the box surfaces change and how the volume of the box changes and how the edges of the box have moved. The box is very likely to be skewed in some manner, and this change away from the starting orthogonal shape is called "**deformation**" or "**shear**" or "**strain**".

To review, then, in Lagrangian mechanics, the only real kinematic quantities of interest are $\mathbf{x}_i(t)$and $m_i$, the position and mass of each particle. In continuum mechanics, the position becomes **x**(**X**,t) and the mass of a particle would be ρ(**x**)dV. Moreover, the shape information for a continuum particle which began at time $t_0$ and position **X** is at time t and position **x** contained in the matrix F(**X,**t) .

It turns out that it is more convenient to use a matrix which is derived from F, namely $C = F^T F$ where $F^T$ is just the transpose of matrix F. It turns out that the diagonal elements of matrix C tell us how much each little box side vector stretched (or shrank), while the off-diagonal elements of matrix C tell us about how these same little side arrows of the box are skewed away from 90 degrees from each other. Matrix C has a fancy name: C is called the Right Cauchy-Green Deformation Tensor.

A fact of the matrix world is that a general matrix like F can be written as $F = \mathcal{R}U$ where $\mathcal{R}$ is a rotation matrix and U is a stretching matrix. It is easy to show that $C = U^2$ since a rotation has $\mathcal{R}^T\mathcal{R} = 1$. It is possible in a flow for some particle (a tiny cube say) that the particle simply rotates and does not get deformed at all. In this case, for such a particle which started at location X, we have U = 1 and $F = \mathcal{R}$, so the little cube edge vectors just get rotated, and the cube acts as a rigid body. In a true macroscopic rigid body, *all* the particles of the body would get rotated only, not stretched (they might get translated as well in some direction).

Various other matrices can be derived from F and each of them is convenient to use in some particular area of application.

6. A continuous material's particles are fully described for our purposes by the position of the particle, and by the particle's orientation and amount of deformation (= strain) it has (at any time t), which is to say, by the shape of the particle. Kinematics does not say HOW a particle gets flowed into some strained condition and some new position, it just provides handles by which we can *talk about* the shape and the position. Similarly, in Lagrangian mechanics, the fact that a particle has a path $\mathbf{x}_i(t)$ does not tell us HOW a particle gets to some position at time t. One has to add into the mix Newton's law **F** = m**a** to learn what Lagrangian particles are doing, where **F** means "forces". One is then involved not just with kinematics, but also with the dynamics of the motion of the set of particles.

In the realm of continuous matter, a particle of matter of some tiny volume dV is just a mathematical cube inside the matter. The forces in this case are the forces of the rest of the matter (the blob minus this particular cube) which act somehow on the 6 faces of the cube. In this context, these forces have a special name: they are called **stresses**. When one applies a huge stress to a tiny cube of steel, it will bend a bit so it will deform and it is then in a condition of "strain". But this is caused by the stress applied by the material to a tiny cube of its interior. At the other end of the spectrum, a tiny cube of air which is flowing in some manner gets into a strained or deformed shape with very small stresses.

7. Connector to my tensor document. In that document, I talk about a transformation **x**' = **G**(**x**) and how for a small region near **x** (which corresponds to a small region near **x**') , the transformation can be treated as a linear transformation and we then have

$$d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x} \qquad R_{ik}(\mathbf{x}) \equiv (\partial x'_i/\partial x_k) \qquad R = S^{-1} \qquad // \; dx'_i = R_{ij}\, dx_j$$

$d\mathbf{x} = S(\mathbf{x}')\, d\mathbf{x}'$ $\qquad S_{ik}(\mathbf{x}') \equiv (\partial x_i/\partial x'_k)$ $\qquad S = R^{-1}$ $\qquad // \; dx_i = S_{ij}\, dx'_j$

Only in the very near neighborhood of **x** in x-space (and the corresponding point **x'** in x'-space) are we allowed to use these linear forms. I go on to show that vectors like d**x** are called contravariant vectors with respect to the transformation **G**. I show how in each of the two spaces there is a metric tensor called g. In fact, if x-space is a Cartesian space, then the metric tensors are simply related to the matrices R and S : $g = RR^T$ and $\bar{g} = S^T S$, where $\bar{g}$ is called the covariant metric tensor and g the contravariant one. I also talk about the notion of tangent base vectors which transform according to $\mathbf{e'}_n = R(\mathbf{x})\, \mathbf{e}_n$ where the non-unit vectors $\mathbf{e}_n$ live in x-space and are tangent to the coordinate lines there, while the $\mathbf{e'}_n$ are axis aligned unit vectors in x'-space.

The above is a mathematical structure which applies to any transformation $\mathbf{x}' = \mathbf{G}(\mathbf{x})$. I will now change the names of the variables such that we replace **x** by **X**, and replace **x'** by **x**. Then we have a transformation from variables **X** to variables **x** :

$\mathbf{x} = \mathbf{G}(\mathbf{X})$

$d\mathbf{x} = R(\mathbf{X})\, d\mathbf{X}$ $\qquad R_{ik}(\mathbf{X}) \equiv (\partial x_i/\partial \mathbf{X}_k)$ $\qquad R = S^{-1}$ $\qquad // \; dx_i = R_{ij}\, d\mathbf{X}_j$

$d\mathbf{X} = S(\mathbf{x})\, d\mathbf{x}$ $\qquad S_{ik}(\mathbf{x}) \equiv (\partial \mathbf{X}_i/\partial x_k)$ $\qquad S = R^{-1}$ $\qquad // \; dx_i = S_{ij}\, dx_j$

$g = RR^T$ $\qquad \bar{g} = S^T S$ $\qquad \mathbf{e'}_n = R(\mathbf{x})\, \mathbf{e}_n$

Comparing with the discussion above where we had $d\mathbf{x} = F(\mathbf{X},t)\, d\mathbf{X}$, we immediately identify the matrix called R in tensor doc with the matrix called F in continuum mechanics. The R used here does not mean a rotation matrix by the way, it is a general matrix R = F. Also we $\mathbf{x} = \mathbf{G}(\mathbf{X}) = \mathbf{x}(\mathbf{X},t)$ .

I thought that I was going to find that $C = F^T F$ would come out being one of the metric tensors, but I see now that does not work. The metric tensor is $g = FF^T$ which is a different animal. I think we can identify g with the *Left* Cauchy tensor B. I will have to put a hold on this subject for now. I just want to point out that there is a connection between the continuous matter flow and my tensor doc's general transformation.