Polar Decomposition PhL 1.31.12

The claim is that, if F is any matrix with detF ≠ 0, you can write it as

F = RU "the right polar decomposition"

where R is a rotation matrix and U is symmetric and positive definite, and this decomposition is unique. In **Section 1** I find an R and U that works for F = RU and then in **Section 2** I show that R and U with their properties as stated are *unique*. This uniqueness proof requires the use of a theorem which states that the square root of a pos def sym matrix which is itself pos def sym is unique. I prove this theorem in another doc called Crystal...doc, but I comment on it here in **Section 3**. Then in **Section 4** I do a similar treatment of the left polar decomposition which I write as V = VR' and again I get expressions for V and R' and again they are unique. Finally in **Section 5** I show that the R of the right decomposition is the same as the R' of the left decomposition. I have not dwelled much on this subject at all beyond just getting these things proved. The proofs were non-trivial and took me a day or two!

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1. First, find *a* solution for U and R in F = RU (Right Polar Decomposition)

Assume such orthogonal R and symmetric positive definite U exist, so that F = RU. [If we can compute R and U and show they have these properties, then this assumption is justified.]. Then

$F^TF = (RU)^T(RU) = U^RR^TRU = U^TU = UU = U^2$ $S \equiv F^TF = U^2$

Then the problem of determining U is merely one of finding the square root of the matrix $S \equiv F^TF$, which is manifestly symmetric. We know something else about such an S: the eigenvalues are positive. Here is why:

$\lambda|\varphi_\lambda|^2 = \langle\varphi_\lambda|\lambda\varphi_\lambda\rangle = \langle\varphi_\lambda|S\varphi_\lambda\rangle = \langle\varphi_\lambda| F^TF\ \varphi_\lambda\rangle = \langle F\varphi_\lambda|F\varphi_\lambda\rangle = |F\varphi_\lambda|^2$ QED

Therefore S is symmetric and positive definite, and we seek a square root of S, called U, which also has these properties.

We know this theorem:

A symmetric matrix S ($S = S^T$) can be diagonalized by a real orthogonal matrix ($Q^T = Q^{-1}$). This is an application of the more general theorem that a Hermitian matrix ($H = H^{T*}$) can be diagonalized by a unitary matrix ($Q^{T*} = Q^{-1}$).

Since S is symmetric, we therefore know that an orthogonal Q exists such that

$$Q^{-1}SQ = \Lambda \qquad \text{// a diagonal matrix} \qquad \Rightarrow \qquad S = Q\Lambda Q^{-1}$$

where the diagonal elements of Λ are the eigenvalues of S which we showed are positive.

So here is our *candidate* for U:

$$U = Q\Lambda^{1/2}Q^{-1}$$

where the meaning of $\Lambda^{1/2}$ is that we take the positive square root of all the diagonal elements of Λ, so then $\Lambda^{1/2}$ has all positive diagonal elements.

First, let's show that in fact $U^2 = S$:

$$U^2 = Q\Lambda^{1/2}Q^{-1}\, Q\Lambda^{1/2}Q^{-1} = Q\Lambda^{1/2}\Lambda^{1/2}Q^{-1} \quad = Q\Lambda\, Q^{-1} = \; S$$

Next, since $Q^{-1} = Q^T$ we find that

$$U^T = (Q\Lambda^{1/2}Q^T)^T = Q\Lambda^{1/2}Q^T = U$$

so we find that indeed U is symmetric. Looking at our form for U, we have

$$UQ = Q\Lambda^{1/2}$$

We know that the columns of orthogonal Q must be the normalized eigenvectors of the problem $U\varphi_n = \lambda_n^{1/2}\varphi_n$ and these columns are in the right order to match Λ and $\Lambda^{1/2}$. Thus, the above line really says

$$U\,\{.... \varphi_n ....\} \; = \{.... \varphi_n ....\}\, \Lambda^{1/2} \; = \; \{.... \lambda_n^{1/2}\varphi_n ....\}$$

$$\Rightarrow \; U\,\varphi_n = \lambda_n^{1/2}\varphi_n \qquad\qquad \lambda_n^{1/2} > 0$$

Therefore our candidate matrix U is both symmetric and positive definite, which was part of our starting requirement. U is also invertible because detU = product of the $\lambda_n^{1/2} \; > 0$.

Although we don't use this fact, we can apply U to the above equation to show that U and S have exactly the same eigenvectors, and that while S has eigenvalues $\lambda_n > 0$, U has eigenvalues $\lambda_n^{1/2} > 0$.

Meanwhile, our candidate for R is this.

$$R = F\,U^{-1}$$

How do we know this R is orthogonal?

$$R^T = (F\,U^{-1})^T \; = U^{-1,T}\,F^T \; = U^{T,-1}\,F^T \; = U^{-1}\; F^T$$

Then

$$R^T R = U^{-1} F^T F U^{-1} = U^{-1} U^2 U^{-1} = 1$$

Thus R is indeed orthogonal.

Thus, we have found explicit expressions for U and R which have the required properties. We must first find the normalized eigenvectors and the eigenvalues of $F^T F$. We know these eigenvalues are all positive, and we know those eigenvectors comprise the columns of orthogonal matrix Q. So our first step then is to obtain Λ (in some order, perhaps monotonic descending) and a corresponding Q, where

$$Q^{-1} (F^T F) Q = \Lambda$$

Our solution for U and R is then this:

$U = Q\Lambda^{1/2}Q^{-1}$ // U is symmetric and positive definite

$R = F U^{-1}$ // R is a rotation matrix.

2. Show the solution is unique

Suppose the decomposition is not unique so we have in fact two distinct solutions:

$$F = R_1 U_1 = R_2 U_2$$

Then write $F^T F$ two ways to get

$$F^T F = F^T F$$

$$(R_1 U_1)^T (R_1 U_1) = (R_2 U_2)^T (R_2 U_2)$$

$$U_1^T U_1 = U_2^T U_2$$

$\Rightarrow U_1^2 = U_2^2$ where U_i are each symmetric and positive definite.

or

$S = S$ where $S = U_1^2$ and $S = U_2^2$

Now we need this theorem:

Theorem: Consider $U^2 = S$. If S is a given positive definite matrix, we claim that *there is only one* positive definite symmetric U which solves this equation. Call this unique solution U_0.

Assume we have proven this theorem. The problem $S = U_1^2$ then has only the solution $U_1 = U_0$, and the problem $S = U_2^2$ has only the solution $U_1 = U_0$. Therefore we end up with $U_1 = U_2$ so our U candidate given above is in fact unique. Now from the start we have

$R_1 = F\ U_1^{-1}$
$R_2 = F\ U_2^{-1}$

If the U's are the same, then the R's must also be the same.

3. Comments on the Required Theorem

Theorem: Consider $U^2 = S$. If S is a given positive definite matrix, we claim that *there is only one* positive definite symmetric U which solves this equation. Call this unique solution U_0.

This theorem is "the hard part" of the uniqueness proof. After various attempts, I was unable to prove this theorem on my own, so I went hunting out on the web and found this slightly more general theorem proved by a master's student named Crystal (see Crystal kth root of a matrix.doc) :

> **Theorem 4.3:** Let $A \in M_n$ be positive semidefinite and let $k \geq 1$ be a given integer. Then there exists a unique positive semidefinite Hermitian matrix B such that $B^k = A$. We also have
>
> (a) $BA = AB$ and there is a polynomial $p(t)$ such that $B = p(A)$;
>
> (b) rank B = rank A, so B is a positive definite if A is; and
>
> (c) B is real if A is real.

First of all,

A is positive semidefinite means $\lambda_i \geq 0$
A is positive definite means $\lambda_i > 0$

so the case I need is a subset of her general theorem.

Secondly, I only need k = 2, but it is shown for any integer k, so again my need is a subset.
Crystal goes beyond the basic theorem I need and shows that:

If $A = B^k$ and B is the unique solution of interest, then [A,B] = 0 and B = poly(A).
In my positive definite case, A and B both have full rank N, and (c) is obvious.

It is this polynomial business that allows the uniqueness proof to be done, something I would never have thought of.

4. Same polar decomposition with V on the other side (Left Polar Decomposition F = VR' .

You can also decompose this way (where R is the same as above), it is claimed:

$$F = VR'$$

Let's try imitating some of the steps above, not knowing that it is the same R. I won't repeat all the words, just the basic equations:

$$FF^T = VR'(VR')^T = VR'R'^{-1}V^T = VV^T = V^2 \qquad S' = FF^T = V^2$$

S' is positive definite by the same argument as above. Diagonalize S' as above

$$Q'^{-1}S'Q' = \Lambda'$$

Candidate is

$$V = Q'\Lambda'^{1/2}Q'^{-1}$$

Since S ≠S', everything is (can be) different now, the eigenvectors and values are different, Λ' is different, Q' is different. The candidate for R' is

$$R' = V^{-1}F$$

and we show this is orthogonal as before,

$$R'^T = F^TV$$

$$R'R'^T = V^{-1}F\ F^TV = V^{-1}\ V^2V = 1$$

where again V is symmetric and positive definite, and R' is a rotation matrix.

5. Show that R = R'

This is completely non-obvious to me! We do know this much:

$$F = VR'$$
$$F = RU$$

$$\Rightarrow RU = VR' \qquad \Rightarrow R' = V^{-1}RU \qquad \Rightarrow \qquad V = RUR'^{-1}$$

Horn verifies that it is <u>true</u> that R = R', but it is given as an exercise to verify it. I tried and flailed and then found the answer in Google books Dover *Continuum Mechanics* by Peter Chadwick, 1999, 192 pages, $7 Amazon. Here it is:

$$F = RU = (R'R'^T)\ VR' = R'\ (R'^T\ VR') \equiv R'\ T$$

Consider the matrix $T = R'^{T} VR'$. Since V is symmetric, we have $T^{T} = T$ so T is symmetric. Written as $T = R'^{-1} VR'$, it is trivial to show that T has the same characteristic equation as V and therefore T has the same eigenvalues as V. Therefore, since V is positive definite, so is T. So we have just shown that

$$F = RU = R'T$$

This represents two right polar decompositions of F! Since we know that the two pieces with the usual properties are *unique*, we must conclude that

$$R = R'$$

$$T = U \quad \Rightarrow \quad R'^{T} VR' = U \quad \Rightarrow \quad R^{T}VR = U$$

Thus, in fact U and V both have the same eigenvalues since they are related as shown on the right. Thus in fact $\Lambda = \Lambda'$ as used above.