Detail for Lai p 325: transverse isotropic tensor PhL 5.21.12

In these notes I take a quick shot (2 hours) at this problem and then give up for now. This needs more advanced methods than I am using right now and it is not in the main path really.

Now suppose you have a case of what I might call **cylindrical symmetry**. The axisym axis is taken to be the z axis. Recall from above

$$C'_{abcd} = F_{aa'} F_{bb'} F_{cc'} F_{dd'} C_{a'b'c'd'}$$

where this time F_{ab} rather than being a reflection in the x,y,or z plane, is a reflection in a plane that is rotated by angle β from the x=0 plan, and we have this symmetry for any β. This probably means the material is isotropic in the x-y direction.

How do you reflect in a plane at angle β? If **x** is a point in space, then let $\mathbf{x}_t$ = (x,y). Then we have

$$[R_z(\beta)\, \mathbf{x}_t] \rightarrow -[R_z(\beta)\, \mathbf{x}_t] \qquad \text{// This is 2D parity.}$$
$$z \rightarrow z \qquad \text{// no change in z}$$

Therefore I would say that $F_{ab} = R_z(\beta)_{ab}$ where this is the 3D matrix. So the condition on C then is

$$C_{abcd}(\beta) = R_z(\beta)_{aa'} R_z(\beta)_{bb'} R_z(\beta)_{cc'} R_z(\beta)_{dd'} C_{a'b'c'd'}$$

We know this C matrix will be a subset of the one shown above which is 5.49.3 p 325, since the one above applies for just the two reflections (x and y) which occur for β = 0 and β = π/2. But how on earth do we make sense out of the general case above? Since there are only 9 independent C elements here, we could put 9 distinct elements on the LHS and then we have 9 specific equations. For example

$$C_{1111}(\beta) = R_z(\beta)_{1a'} R_z(\beta)_{1b'} R_z(\beta)_{1c'} R_z(\beta)_{1d'} C_{a'b'c'd'}$$
$$C_{1122}(\beta) = R_z(\beta)_{1a'} R_z(\beta)_{1b'} R_z(\beta)_{2c'} R_z(\beta)_{2d'} C_{a'b'c'd'}$$
etc.

Is this not then 9 equations in 9 non-vanishing unknowns, and that for each value of β ? That seems to be too many conditions somehow, but they must not all be unique.

Let's try this directly with the C_{ab} matrix instead? But it is not a tensor so don't know what to do.

Two tricks might help here: (1) try specific angles like β = π/4 (2) anytime an index is 3, one R factor goes away. The 2D matrix for π/4 can be taken to be $(1/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Derivative attempt. (small β)

$C_{abcd}(\beta) = R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, R_z(\beta)_{cc'}\, R_z(\beta)_{dd'}\, C_{a'b'c'd'}$

$C_{3333}(\beta) = R_z(\beta)_{3a'}\, R_z(\beta)_{3b'}\, R_z(\beta)_{3c'}\, R_z(\beta)_{3d'}\, C_{a'b'c'd'} \;= C_{3333}$ // learn nothing

Next try

$C_{3331}(\beta) = R_z(\beta)_{3a'}\, R_z(\beta)_{3b'}\, R_z(\beta)_{3c'}\, R_z(\beta)_{1d'}\, C_{a'b'c'd'} \;= C_{3331}$

$= R_z(\beta)_{1d'}\, C_{333d'} \;= \cos\beta\, C_{3331} + \sin\beta\, C_{3332} \;= C_{3331}$

But how can this be true?

$C_{3331} = \cos\beta\, C_{3331} + \sin\beta\, C_{3332}$ for any β

The only way is if all three vanish, and we know that is the case from our theorem above. So let's only consider the non-zero entries in 5.49.3 and not waste time on the others. Try things with two 3's:

$C_{ab33}(\beta) = R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, R_z(\beta)_{3c'}\, R_z(\beta)_{3d'}\, C_{a'b'c'd'} = R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, C_{a'b'33}$

where a and b are both not 3. But even this is a painful problem.

Basically this is some kind of "ill proposed problem". I just don't grok it in terms of why there should be a unique solution, etc etc. My method is too weak.

Idea: what about using the rotation generators and just look near $\beta = 0$. Then

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad R_{\hat{n}}(\theta) = \exp(-i\theta \hat{\mathbf{n}} \bullet \mathbf{J})$$

$R_z(\beta) = 1 - i\beta J_3 \qquad (J_3)_{jk} = -i\,\varepsilon_{3jk}.$

$R_z(\beta)_{jk} = \delta_{jk} - i\beta(-i\varepsilon_{3jk}) = (\delta_{jk} - \beta\varepsilon_{3jk})$

At least this is something I can write down, all this last stuff is valid for small β only. Then

$C_{abcd}(\beta) = (\delta_{aa'} - \beta\varepsilon_{3aa'})(\delta_{bb'} - \beta\varepsilon_{3bb'})(\delta_{cc'} - \beta\varepsilon_{3cc'})(\delta_{dd'} - \beta\varepsilon_{3dd'})\, C_{a'b'c'd'}$

$= \beta^4 (\varepsilon_{3aa'}\, \varepsilon_{3bb'}\, \varepsilon_{3cc'}\, \varepsilon_{3dd'}\, C_{a'b'c'd'})$

$- \beta^3(\varepsilon_{3aa'}\, \varepsilon_{3bb'}\, \varepsilon_{3cc'}\, C_{a'b'c'd} + \varepsilon_{3aa'}\, \varepsilon_{3bb'}\, \varepsilon_{3dd'}\, C_{a'b'cd'} +$

$\varepsilon_{3aa'}\, \varepsilon_{3cc'}\, \varepsilon_{3dd'}\, C_{a'bc'd'} + \varepsilon_{3bb'}\, \varepsilon_{3cc'}\, \varepsilon_{3dd'}\, C_{ab'c'd'})$

$+ \beta^2$ (there are 4,2 = 4!/2!2! = 4!/4 = 3! = 6 terms here)

$+ \beta$ (there are 4,1 = 4!/3! = 4 terms here)

$+ C_{abcd}$

Now if the LHS is supposed to be independent of β, we should be able to say

$\partial^n C_{abcd}(\beta) = 0$

for any positive integer n. So lets try n = 4 and see what we get

$\varepsilon_{3aa'}\, \varepsilon_{3bb'}\, \varepsilon_{3cc'}\, \varepsilon_{3dd'}\, C_{a'b'c'd'} = 0$

This seems to be a set of $2^4 = 16$ conditions. For example, for abcd = 2222 we get

$\varepsilon_{321}\, \varepsilon_{321}\, \varepsilon_{321}\, \varepsilon_{321}\, C_{1111} = 0$

which says $C_{1111} = 0$ which is wrong! Something is wrong somewhere. Well, my approximation has omitted many β^4 terms so it is all garbage.

But what about the first derivative, since then I will have all those terms.

$C_{abcd}(\beta) = (\delta_{aa'} - \beta\varepsilon_{3aa'})(\delta_{bb'} - \beta\varepsilon_{3bb'})(\delta_{cc'} - \beta\varepsilon_{3cc'})(\delta_{dd'} - \beta\varepsilon_{3dd'})\, C_{a'b'c'd'}$

$\approx -\beta\,[\varepsilon_{3aa'}\, C_{a'bcd} + \varepsilon_{3bb'}\, C_{ab'cd} + \varepsilon_{3cc'}\, C_{abc'd} + \varepsilon_{3dd'}\, C_{abcd'}]$

So I think at least this is a valid set of conditions:

$\varepsilon_{3aa'}\, C_{a'bcd} + \varepsilon_{3bb'}\, C_{ab'cd} + \varepsilon_{3cc'}\, C_{abc'd} + \varepsilon_{3dd'}\, C_{abcd'}$

But this is 4! 4! = 24^2 equations! I am still no closer.

Question: why is there even a solution to this problem?

Why can you satisfy this equation for all β ???

$C_{abcd} = R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, R_z(\beta)_{cc'}\, R_z(\beta)_{dd'}\, C_{a'b'c'd'}$

If this equation is really true, then this must be true

$R_z(\beta)_{aa'}\, R_z(\beta)_{bb'}\, R_z(\beta)_{cc'}\, R_z(\beta)_{dd'} = \delta_{aa'}\, \delta_{bb'}\, \delta_{cc'}\, \delta_{dd'}$ // wrong

This is wrong because we know that only 21 of 81 elements are independent so there could be cancellations.

Try making symmetrized forms of the C matrix.

$C^{(L)}_{abcd} = (C_{abcd} + C_{bacd})/2$

$C^{(LR)}_{abcd} = (C^{(L)}_{abcd} + C^{(L)}_{abdc})/2$

$C^{(S)} = (C^{(LR)}_{abcd} + C^{(LR)}_{dcba})/2$

Looking at the web, I see that this is a much harder *general* problem than I want to do right now, but it is a good one.