The Spin Tensor and Related Topics PhL 6.23.12

Every time this topic recurs in Lai, I am confused by it.

1. First, what notes have I already written? 1
2. Where does Lai talk about the spin tensor. Scan the pdf to find out: 1
3. Rotation of a vector: ds or $\dot{s}$ 1
4. An Antisymmetric Matrix associated with rotating a vector 3
5. What is the connection with tensor (∇u) ? 4
6. What is the connection with tensor (∇v) ? 7
7. Lai's page 379 discussion of what happens if n is a principle axis unit vector of D 10
8. Question: Can the ∇u and ∇v worlds be related somehow? 11
Appendix A. An example of simple shear and why it goes away in the "red frame". 13

1. First, what notes have I already written?

Lai folder:
- Ch 5 notes have nothing really, raw or meta. (searching on spin tensor)
- Ch3 has one tiny paragraph on spin tensor.

Goldstein folder: nothing on "spin tensor" but of course there is rotation stuff there.
Angular momentum in QM: nothing there, cross products was a red herring

Wizend doc all of "my interests" for spin tensor: (this includes all math and all physics of course)
I know I wrote at least something, maybe in Goldstein notes. I will start over here.

2. Where does Lai talk about the spin tensor. Scan the pdf to find out:

p 96, p 98, then p 370+, ignoring the problems.

3. Rotation of a vector: ds or $\dot{s}$

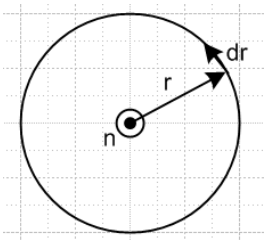
If you actively rotate a position vector $\mathbf{r}$ by amount $d\theta$ around axis $\hat{\mathbf{n}}$, then I claim that (and I will prove this in a moment)

$$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r} \qquad \text{where } d\boldsymbol{\theta} = d\theta\, \hat{\mathbf{n}} \ .$$

Dividing by dt then gives

$$\mathbf{v} = d\mathbf{r}/dt = D\mathbf{r}/Dt = \boldsymbol{\omega} \times r \quad \text{where} \quad \boldsymbol{\omega} = d\boldsymbol{\theta}/dt = \omega\, \hat{\mathbf{n}} \ \text{ where } \omega = d\theta/dt.$$

An assumed fact in the above scenario is that the axis $\hat{\mathbf{n}}$ has its tail end on the Origin of the coordinate system. So if we look at the rotation from a camera angle which stares backwards right down the vector $\hat{\mathbf{n}}$ to the origin, this is what we see (regardless of where $\hat{\mathbf{n}}$ points):



In this picture, the origin is located behind the plane of paper and we are looking at a position vector **r** which comes from that origin to a point on the circle which lies in the plane of paper. When we talk about "rotating vector **r** about $\hat{\mathbf{n}}$", what we mean is that the tip of the **r** vector moves around the circle shown. If we rotate all around 2π, **r** describes a cone. It seems completely clear that d**r** is perpendicular to both **r** and **n** (as our cross product formula requires). Also, the right hand rule agrees with the directions of the three vectors. Of course **n** and r are not perpendicular, they have some angle ψ between them.

Since we have the directions right, our only task is to check the magnitude. we need to show that

$$dr = d\theta \; r \; \sin\psi$$

But $r\sin\psi$ is the radius of the circle shown above, and we know that $dr = (r\sin\psi)d\theta$ just from the usual polar coordinates (ρ,θ) where $\rho = r\sin\psi$. So I regard the above formula as now officially "proved".

Having done this, we can now look at the "fancy" proof which is this. As outlined in "confusion about rotation operators" in ang mom, my rotation operators always do "active" rotations of vectors in space. And we have for such an active rotation

$$R = \exp(-i\boldsymbol{\theta}\,\hat{\mathbf{n}} \bullet \mathbf{J}) \qquad \text{where} \qquad (J_i)_{jk} = -i\,\varepsilon_{ijk}$$

Now consider for a small rotation,

$$d\mathbf{r} = R\,\mathbf{r} - \mathbf{r} = \exp(-id\theta\,\hat{\mathbf{n}} \bullet \mathbf{J})\mathbf{r} - \mathbf{r} \approx (1 - id\theta\,\hat{\mathbf{n}} \bullet \mathbf{J})\mathbf{r} - \mathbf{r} = -id\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}\,\mathbf{r}$$

$$dr_j = -id\theta\, n_i (J_i)_{jk} r_k = -id\theta\, n_i [-i\,\varepsilon_{ijk}] r_k = -d\theta\,\varepsilon_{ijk}\, n_i\, r_k = -d\theta\,\varepsilon_{jki}\, r_k\, n_i$$

$$= +d\theta\,\varepsilon_{jik}\, n_i\, r_k = \varepsilon_{jik}\,[d\theta\, n_i]\, r_k = \varepsilon_{jik}\,[d\theta_i]\, r_k = [d\boldsymbol{\theta} \times \mathbf{r}]_j$$

and therefore $d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}$, so this is our second "proof".

Now, Goldstein likes to write (see his page 128 4-94) $d\mathbf{r} = -d\boldsymbol{\Omega} \times \mathbf{r}$. This minus sign can be traced to his page 126 picture where he rotates *axes* by $+d\varphi$ which conflicts with my active rotation of a *vector* by $+d\varphi$. Goldstein uses the passive view of rotation, I use the active view, and there is always that minus sign between them, so fine.

Lai seems to agree with my active view looking at page 98 where he always writes $d\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}$.

In the above **r** is a position vector, but you can presumably rotate a velocity vector **v** the same way

$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$ where $d\boldsymbol{\theta} = d\theta\, \hat{\mathbf{n}}$

$\mathbf{a} = d\mathbf{v}/dt = D\mathbf{v}/Dt = \boldsymbol{\omega} \times \mathbf{v}$

or

$D_t\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$

Note that this is true only if a particle having $d\mathbf{v}$ or $\mathbf{v}$ is (at some instant in time) rotating around in a circle. It is not true in general for any $\mathbf{v}$ situation.

The entire discussion above of course applies to rotations involving any vector $\mathbf{s}$, so I will restate the results here:

$d\mathbf{s} = d\boldsymbol{\theta} \times \mathbf{s}$ where $d\boldsymbol{\theta} = d\theta\, \hat{\mathbf{n}}$.

Dividing by dt then gives

$\dot{\mathbf{s}} = d\mathbf{s}/dt = D\mathbf{s}/Dt = \boldsymbol{\omega} \times \mathbf{s}$ where $\boldsymbol{\omega} = d\boldsymbol{\theta}/dt = \omega\, \hat{\mathbf{n}}$ where $\omega = d\theta/dt$.

4. An Antisymmetric Matrix associated with rotating a vector

I revert now from the general vector $\mathbf{s}$ to the prototype position vector $\mathbf{r}$. As presented here, this discussion applies only in 3 dimensions, since only there does the cross product of 2 vectors exist.

What does the above "stuff" have to do with some "antisymmetric matrix". Lai is always making some kind of association. Well, go back to

$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r} \quad \Rightarrow \quad dr_i = \varepsilon_{ijk} d\theta_j r_k = A_{ik} r_k$ where $A_{ik} = \varepsilon_{ijk} d\theta_j = -\varepsilon_{ikj} d\theta_j$

$d\mathbf{r} = A\, r$

So we can write $d\mathbf{r} = A\mathbf{r}$ in the usual linear algebra sense, and that is what A comes out being. You can see from the ε that indeed A is antisymmetric. We have

$A_{12} = -\varepsilon_{123}\, d\theta_3 = -d\theta_3$
$A_{13} = -\varepsilon_{132}\, d\theta_2 = +d\theta_2$
$A_{23} = -\varepsilon_{231}\, d\theta_1 = -d\theta_1$

$$A = \begin{pmatrix} 0 & -d\theta_3 & d\theta_2 \\ d\theta_3 & 0 & -d\theta_1 \\ -d\theta_2 & d\theta_1 & 0 \end{pmatrix}$$

Now let's turn the thing around. If you are given ANY 3x3 antisymmetric matrix, it can only have three distinct elements like those shown in the matrix above. Suppose we are *given* antisymmetric A_{ij}. Then we know that there is some vector $\mathbf{t}$ such that $A\mathbf{r}$ can be written this way.

$d\mathbf{r} = A\,\mathbf{r} = \mathbf{t} \times \mathbf{r}$ where $\mathbf{t} = (A_{32}, A_{13}, A_{21})$ (as shown right below)

In my Lai Ch1,2 notes page 8 I show that

$$t_i = -(1/2)\,\varepsilon_{ijk}A_{jk}$$

and Lai calls this **t** thing the "dual vector". This just says (which agrees with the above example)

$$t_1 = -A_{23} = A_{32}$$
$$t_2 = -A_{31} = A_{13}$$
$$t_3 = -A_{12} = A_{21}$$

In the above example, replacing **r** with a general vector **s**,

$$d\mathbf{s} = A\,\mathbf{s} = \mathbf{t} \times \mathbf{s} \qquad A = \begin{pmatrix} 0 & -d\theta_3 & d\theta_2 \\ d\theta_3 & 0 & -d\theta_1 \\ -d\theta_2 & d\theta_1 & 0 \end{pmatrix} \qquad \mathbf{t} = d\boldsymbol{\theta}$$

Here is another example, where the primes just indicate that we have a different matrix and vector,

$$\dot{\mathbf{s}} = A'\,\mathbf{s} = \mathbf{t}' \times \mathbf{s} \qquad A' = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad \mathbf{t}' = \boldsymbol{\omega} = d\boldsymbol{\theta}/dt$$

Thus, an antisymmetric 3x3 matrix can describe the differential change of a vector, *or* it can describe the rate of change of a vector.

Notice that the rotation matrix R is not an antisymmetric matrix like A. The connection is this:

$$R = \exp(-id\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}) = (1 - id\theta\,\hat{\mathbf{n}} \bullet \mathbf{J}) = 1 - i\,d\boldsymbol{\theta} \bullet \mathbf{J} = 1 + A \ .$$

To verify this claim, consider:

$$A_{ik} = -i\,d\boldsymbol{\theta} \bullet [\mathbf{J}]_{ik} = -i\,d\theta_j\,[J_j]_{ik} = -i\,d\theta_j\,[-i\,\varepsilon_{jik}] = -d\theta_j\varepsilon_{jik} = -\varepsilon_{ikj}\theta_j = A_{ik}$$ from above

Thus, the matrix A = R - 1 where we are only talking about small rotations by dθ. If R were a finite rotation matrix, then R-1 would have diagonal elements and would not be antisymmetric!

One more time: An antisymmetric matrix A of small values is associated with a rotation R = 1+A which moves points **s** to points **s** + d**s** and one can write d**s** = A**s** = **t** x **s** where **t** is the dual vector for A.

5. What is the connection with tensor (∇u) ?

(a) Recall how a differential vector d**x** moves in a flow,

$$d\mathbf{x}' = F\,d\mathbf{x} \qquad F \equiv (\nabla_{\mathbf{x}}\mathbf{x}') = \text{the deformation gradient} \ .$$

Lai always writes this in the following notation:

$d\mathbf{x} = F\, d\mathbf{X}$ $\qquad\qquad$ $F \equiv (\nabla_{\mathbf{X}}\mathbf{x})$ = the deformation gradient .

Right away, we are going to encounter the object $F^{T}F$ (= C) in elementary calculations such as

$d\mathbf{x}_1 \bullet d\mathbf{x}_2 = F\, d\mathbf{X}_1 \bullet F\, d\mathbf{X}_2 = d\mathbf{X}_1 \bullet F^{T}F\, d\mathbf{X}_2$.

Here is a little review of the associated theory:

$\mathbf{u} = \mathbf{x} - \mathbf{X}$ = the displacement field

$\mathbf{x} = \mathbf{X} + \mathbf{u}$

$(\nabla_{\mathbf{X}}\mathbf{x}) = 1 + (\nabla_{\mathbf{X}}\mathbf{u})$

$F = 1 + (\nabla\mathbf{u})$ $\qquad\qquad$ // now I drop the X subscript on ∇

$d\mathbf{x} = F\, d\mathbf{X} = [1 + (\nabla\mathbf{u})]d\mathbf{X} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$

$d\mathbf{u} = (\nabla\mathbf{u})d\mathbf{X}$

Interpretation of the above: a small dumbbell $d\mathbf{X}$ maps into some $d\mathbf{x}$ when stress is applied. The change in this dumbbell is $d\mathbf{u} = d\mathbf{x} - d\mathbf{X}$. That is to say, each end of the dumbbell displaces some **u**, and the difference of these two displacements is $d\mathbf{u}$. You can compute $d\mathbf{u}$ from $d\mathbf{u} = (\nabla\mathbf{u})d\mathbf{X}$ as shown.

Next we have

$F^{T}F = [1 + (\nabla\mathbf{u})^{T}][\, 1 + (\nabla\mathbf{u})] \approx 1 + 2\,(\nabla\mathbf{u})^{S}$ $\qquad$ // dropping higher terms !!!!

Now define

$E = (\nabla\mathbf{u})^{S}$ $\qquad$ => $\qquad$ $F^{T}F = 1 + 2E$

Then we get

$d\mathbf{x}_1 \bullet d\mathbf{x}_2 = d\mathbf{X}_1 \bullet F^{T}F\, d\mathbf{X}_2 = d\mathbf{X}_1 \bullet (1 + 2E)\, d\mathbf{X}_2$ $\qquad$ (*)

and from this we find the "interpretation" of the E_{ij}, as follows. First, suppose 1 = 2 = (some axis direction) and $|d\mathbf{x}| = ds$ and $d\mathbf{X} = dS\,\mathbf{n}$ where **n** = unit vector: (here ds^2 means always $(ds)^2$ etc)

$ds^2 = dS\,\mathbf{n} \bullet (1 + 2E)\, dS\,\mathbf{n} = dS^2 + dS^2\mathbf{n} \bullet 2E\,\mathbf{n} = dS^2 + 2dS^2E_{nn}$

$ds^2 - dS^2 = 2dS^2 E_{nn} \approx (ds - dS)(2dS)$

$\Rightarrow E_{nn} = (ds-dS)/dS$ = fractional stretch in the n direction

Meanwhile, if $1 \neq 2$ we get from (*)

$ds_1\, ds_2 \cos\theta = d\mathbf{X}_1 \bullet (1 + 2E)\, d\mathbf{X}_2 = d\mathbf{X}_1 \bullet 2E\, d\mathbf{X}_2 = dS_1 dS_2\, \mathbf{n}_1 \bullet 2E\, \mathbf{n}_2$

$\cos\theta \approx \mathbf{n}_1 \bullet 2E\, \mathbf{n}_2 = 2E_{12} = \cos(\pi/2-\psi) = \sin\psi$ ψ = amount of decrease in angle

$\Rightarrow \psi \approx 2E_{12}$

(b) Now given all the above, consider

$(\nabla \mathbf{u}) = (\nabla \mathbf{u})^S + (\nabla \mathbf{u})^A = E + \Omega$ = infinitesimal strain tensor + infinitesimal rotation tensor

[this Ω is NOT the spin tensor, we have to wait till we get to the next section]

The antisymmetric matrix Ω has some associated dual vector $d\boldsymbol{\theta}$ and we know how to compute it, Then we know that for any vector **s**,

$d\mathbf{s} = \Omega\mathbf{s} = d\boldsymbol{\theta} \times \mathbf{s}$ $R = 1 + \Omega$ $d\theta_i = -(1/2)\, \varepsilon_{ijk}\Omega_{jk}$

describes a rotation of vector **s** by small vector amount $d\boldsymbol{\theta}$. Of special interest is vector **X** and **x**. I am not sure how to write this, since we usually think of $d\mathbf{X}$ as a fixed thing and $d\mathbf{x}$ the variable,

$d\mathbf{x} = \Omega\mathbf{x} = d\boldsymbol{\theta} \times \mathbf{x}$ $R = 1 + \Omega$ $d\theta_i = -(1/2)\, \varepsilon_{ijk}\Omega_{jk}$

where earlier we wrote **x** as **r**. So the point is that the Ω portion of $(\nabla \mathbf{u})$ causes a small rotation of a particle of continuous matter about the origin. In the case of a rigid body rotating about axis $\hat{\boldsymbol{\theta}}$, we have that $(\nabla \mathbf{u}) = \Omega$ and that is all there is. We can sequence a set of our small rotations and we find that the point **x** just goes around in some circle. In this case, E = 0 because a particle does not scale its edges, nor does it undergo shear.

In the detailed example of Appendix A below, we consider a certain very simple "shear" displacement field with its $(\nabla \mathbf{u})$ and we think of this as $E + \Omega$. We go to a special viewing frame S' where $(\nabla \mathbf{u})' = E' + \Omega'$ where E' is diagonal. We show that in this frame, a sugar cube particle undergoes no shear whatsoever, though it does undergo scaling due to the diagonal elements of E' being different. The particle also undergoes a rotation $\Omega' = \Omega$ in our example, and this is all demonstrated graphically. The example seems to be in 2D, but it is really in 3D where the 3rd axis is out of the plane of paper and plays no real role.

Comment: In the above result, we see that if we write $d\mathbf{x} = ds\, \mathbf{n}$ where **n** starts out as a unit vector, we will end up with (in general) some $d\mathbf{x}' = ds'\, \mathbf{n}'$ where $ds' \neq ds$, $\mathbf{n}' \neq \mathbf{n}$. In other words, we have

$d\mathbf{x}' = ds'\, \mathbf{n}' = (ds + \Delta ds)\, (\mathbf{n} + \Delta \mathbf{n})$

$$= ds\,\mathbf{n} + ds\,\Delta\mathbf{n} + \Delta(ds)\,\mathbf{n} + \Delta(ds)\,\Delta\mathbf{n}$$
$$\quad \varepsilon \qquad \varepsilon \quad \varepsilon \qquad \varepsilon^2 \qquad\quad \varepsilon^2 \quad \varepsilon$$

$$\approx ds\,\mathbf{n} + ds\,\Delta\mathbf{n} + \Delta(ds)\,\mathbf{n}$$

$$= ds'\,\mathbf{n} + ds\,\Delta\mathbf{n}$$

$$= (\nabla\mathbf{u})d\mathbf{x} = E d\mathbf{x} + \Omega d\mathbf{x}$$

The first term ds' **n** (scaling only) we would associate with Ed**x**. The second term ds Δ**n** would be a pure rotation if it happened that Δ**n** was perp to **n**, and would then be covered by the Ωd**x** term *if* this were true for all three Δ**n** **'s** in the three directions. But if the three Δ**n**'s were different, then the three axes rotate differently and we have "shear", and that goes in the Ed**x** term. On the other hand, if Δ**n** were along **n**, then this would be a scaling and ds Δ**n** would then be part of Edx as well. So I think all we can say is

ds' **n**	is part of the Ed**x** term (since simple scaling in each direction)
ds Δ**n**	can be distributed into both the Ed**x** term and the Ωd**x** term

6. What is the connection with tensor (∇v) ?

This tensor is called just "the velocity gradient" (notice usual order reversal)

$$(\nabla\mathbf{v})_{ij} = \partial_j v_i$$

Where does it show up in Lai's book?

(a) <u>First</u> of all, if you take some general material derivative of a vector $\tilde{\mathbf{p}}$ you get this

$$d_t\tilde{p}_j(\mathbf{x(t)}, t) = \partial_t\tilde{p}_j(\mathbf{x(t)}, t) + (\nabla\tilde{p}_j) \bullet \mathbf{v}$$

$$d_t\tilde{\mathbf{p}}(\mathbf{x(t)}, t) == \partial_t\tilde{\mathbf{p}} + (\nabla\tilde{\mathbf{p}})\mathbf{v}$$

and notice that (∇**v**) does NOT appear. But if you have the special case $\tilde{\mathbf{p}} = \mathbf{v}$, then you get

$$a_j = d_t v_j(\mathbf{x(t)}, t) = \partial_t v_j(\mathbf{x(t)}, t) + (\nabla v_j) \bullet \mathbf{v}$$

$$\mathbf{a} = d_t\mathbf{v}(\mathbf{x(t)}, t) == \partial_t\mathbf{v} + (\nabla\mathbf{v})\mathbf{v}$$

so (∇**v**) shows up in the acceleration of a particle. We shall have nothing more to say about this particular appearance of (∇**v**) in this document.

(b) Second of all, we know that

$$D_t(d\mathbf{x}) = (\nabla \mathbf{v})\, d\mathbf{x}$$

and this is derived quite easily on Lai p 95. This describes how a "dumbbell" of continuous matter tumbles as it flows.

(c) Now we make a decomposition of $(\nabla \mathbf{v})$. D (rate of deformation tensor) and W (the spin tensor) are defined in this manner

$$(\nabla \mathbf{v}) = D + W \qquad \text{where } D = (\nabla \mathbf{v})^{S} \text{ and } W = (\nabla \mathbf{v})^{A}$$

so at once we have

$$D_t(d\mathbf{x}) = (\nabla \mathbf{v})d\mathbf{x} = Dd\mathbf{x} + Wd\mathbf{x}$$

Since $W = (\nabla \mathbf{v})^{A}$ is an antisymmetric 3x3 matrix, we can associate with it some kind of dual vector $\boldsymbol{\omega}$. We know from Section * that $Wd\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x}$ so we can expand the second term this way to write

$$D_t(d\mathbf{x}) = Dd\mathbf{x} + \boldsymbol{\omega} \times d\mathbf{x} \qquad \text{// think } \dot{\mathbf{s}} = \boldsymbol{\omega} \times \mathbf{s} \text{ where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt = \omega\, \hat{\mathbf{n}} \text{ where } \omega = d\theta/dt.$$

Now recall from Section * the idea that $\dot{\mathbf{s}} = d\mathbf{s}/dt = D\mathbf{s}/Dt = \boldsymbol{\omega} \times \mathbf{s}$ where $\boldsymbol{\omega}$ was $d\boldsymbol{\theta}/dt$ which we might call an "angular rate of rotation vector". We might compare the above equations to something in the previous section:

$$d\mathbf{u} = (\nabla \mathbf{u})d\mathbf{X} = E\, d\mathbf{X} + \Omega\, d\mathbf{X}$$

$$d\mathbf{u} = E\, d\mathbf{X} + d\boldsymbol{\theta} \times d\mathbf{X} \qquad \text{// think } d\mathbf{s} = d\boldsymbol{\theta} \times \mathbf{s} \Rightarrow d(d\mathbf{X}) = d\mathbf{u} = d\boldsymbol{\theta} \times d\mathbf{X}$$

Things are very similar. In our current context, D says *the rate* at which a sugar cube is shearing and is stretching on its three axes, while W says *the rate* at which the cube is rotating (spinning).

What is the vector $\boldsymbol{\omega}$? We know that it is a dual vector to W and in fact $\omega_i = -(1/2)\, \varepsilon_{ijk} W_{jk}$. We can write this as

$$\omega_i = -(1/2)\, \varepsilon_{ijk}\, (\partial_k v_j - \partial_j v_k)/2 = (1/2)\, \varepsilon_{ijk}\, (\partial_j v_k) = 1/2\, [\text{curl } \mathbf{v}]_i$$

which says

$$2\boldsymbol{\omega} = \text{curl } \mathbf{v}$$

which was mentioned in 2.30.1 page 52. After chapter 2, you don't see "curl" mentioned in Lai until page 380! The big deal then is that if you velocity flow has no curl, then your sugar cubes have no rotation!

(d) Based on the discussion of Section 3 above, we know that if we *just* have

$$D_t(d\mathbf{x}) = W d\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x} ,$$

then this tells us that our little dumbbell d**x** is instantaneously rotating about axis $\hat{\boldsymbol{\omega}}$ at angular rate ω. So think of one end of the dumbbell d**x** at an Origin, and **ω** also points directly away from this Origin. The **ω** is the same for all d**x** (three axes of a sugar cube): rigid rotation of the continuous particle. But of course **ω** can be a function of position, so this does not mean global rigid body motion. Of course such motion if it occurred would be accounted for in this term.

(e) So, this second W term ONLY describes dumbbell rotation (in fact, about the **ω** axis), no stretching or shearing. In contrast, the D term can describe stretching, shearing, *and* rotation. This requires a little explanation. If you think of just one d**x**, then that can only be stretched and rotated. But if you take the three d**x** which span a tiny sugar cube, then each of those d**x** can stretch and rotate at different rates, and the result is that the sugar cube can be "shearing" as well as stretching differently in the three directions as well as it can be rotated. [I change my mind on this: if all the d**x** rotate in the same manner so the sugar cube really is rotating, this would be picked up in the second term. So I don't think one should associate the D term with any rotation of the sugar cube, even though D rotates the axes to make shearing. This is clarified again below. I need a rewrite here.]

Now just to show that an individual d**x** can be rotated by D, consider the frame of reference in which D is diagonal with eigenvalues on the diagonal. Then D d**x** is a small vector in which each component of d**x** has been scaled *differently*. If you take a vector and scale each component differently, you end up with a vector that in general has a different direction and a different length, so certainly D is capable of "rotating" d**x**.

If it happens that d**x** lies along a principle axis of D, then in that special case we have D d**x** = λ d**x** (where λ is the EV for that axis), and in this case the vector d**x** is only scaled and is not rotated.

(f) Now we have said that the D term can cause a rotation of d**x**. Can it cause a rotation of the entire sugar cube? In order to do so, it would have to rotate all three edge d**x** the same way. But since these edges form a basis, that means D would have to rotate *all* vectors d**x** the same way (ie, same **ω**). But we showed above that a matrix which rotates all d**x** the same way must be antisymmetric, so I conclude that no, the D matrix cannot rotate the entire sugar cube. It can only shear it by rotating the three vectors differently. So the rigid body rotation of the cube (if there is any) is entirely described by the W term.

Note:
- for d**x** you can use any of the edges of a particle cube.
- The spin tensor tells you how this cube is spinning!

(g) Question: Go back to $D_t(d\mathbf{x}) = (\nabla \mathbf{v})(d\mathbf{x})$ which applies for any material d**x**. What happens if you write this out as d**x** = ds **n** where **n** is a unit vector. Then we have

$$D_t(d\mathbf{x}) = (\nabla \mathbf{v})(d\mathbf{x})$$

$$D_t(ds\ \mathbf{n}) = (\nabla \mathbf{v})(\ ds\ \mathbf{n})$$

$D_t(ds)\ \mathbf{n} + dx\ D_t(\mathbf{n}) = ds\ (\nabla \mathbf{v})\ \mathbf{n}$

$[D_t(ds)/ds]\ \mathbf{n} + D_t(\mathbf{n}) = (\nabla \mathbf{v})\ \mathbf{n}$

$D_t(\mathbf{n}) = (\nabla \mathbf{v})\ \mathbf{n} - [D_t(ds)/ds]\ \mathbf{n}$

$D_t(\mathbf{n}) = W\mathbf{n} + D\mathbf{n} - [D_t(ds)/ds]\ \mathbf{n}$ (*)

In this equation, the first term can only describe a rotation of vector **n**, the last term can only describe a scaling of **n**, while the D**n** term can describe both a scaling and rotation of **n**. In the case that **n** is an eigenvalue of D, then only the first term can describe a rotation of **n**, while the other two are only scalings of **n**. Obviously if there are any "scaling" changes of **n**, it is moving away from being a unit vector.

7. Lai's page 379 discussion of what happens if n is a principle axis unit vector of D

The topic here is this equation

$D_t\mathbf{n} = W\mathbf{n} = \boldsymbol{\omega} \times \mathbf{n}$

The second equality is true for any vector **n**. But what about the left equality? It would be true only if the vector **n** is rotating according to **ω**. You cannot claim that an arbitrary vector in continuum mechanics just happens to be rotating according to **ω**. In fact, in (*) above we found that if **n** is a unit vector for d**x**,

$D_t\mathbf{n} = W\mathbf{n} + D\mathbf{n} - [D_t(ds)/ds]\ \mathbf{n}$ (*)

so in general there are extra terms.

Now back to

$D_t\mathbf{n} = W\mathbf{n} = \boldsymbol{\omega} \times \mathbf{n}$

Lai claims that the left equality is true for any vector **n** which is an eigenvector of D! Recall that D is symmetric and so has eigenvectors. These eigenvectors are the principle axes of D.

Look at Lai's work on page 379. At first, **n** is just a unit vector in the direction of d**x** which is arbitrary. He shows that (it was shown on page 97 that) $[D_t(ds)/ds] = \mathbf{n} \bullet D\mathbf{n} = \mathbf{n}^T D\mathbf{n} = \langle \mathbf{n} \mid D \mid \mathbf{n} \rangle$. Then our (*) becomes

$D_t\mathbf{n} = W\mathbf{n} + D\mathbf{n} - (\mathbf{n} \bullet D\mathbf{n})\ \mathbf{n}$. (*)

If **n** is an eigenvector of D, the last two terms cancel!

$D\mathbf{n} - (\mathbf{n} \bullet D\mathbf{n})\ \mathbf{n} = \lambda\mathbf{n} - (\lambda)\mathbf{n} = 0$

and we are left just with $D_t\mathbf{n} = W\mathbf{n}$. We could at this point normalize each eigenvector **n** and write this equation the same way with the understanding that the three **n** are unit vectors.

Conclusion: the three unit normalized eigenvectors $\mathbf{n}_i$ of D all rotate according to the same $\boldsymbol{\omega}$ (W) rotation. So the axes all rotate together (locked hands George Shearing) causing the cube to rotate. Meanwhile, we have

$$[D_t(ds)/ds] = \mathbf{n}_i \bullet D\mathbf{n}_i = \lambda_i$$

so each edge of the cube is changing length. This is what Lai says in A on page 380. What the cube does not do is shear!! Remember that the three principal axes as EV's are orthogonal.

So if you look specifically at a sugar cube whose edges are principle axes, it rotates and scales edges, but it does not shear.

If it happens that curl $\mathbf{v} = 0$ for a fluid flow, then $\boldsymbol{\omega} = 0$ and this says that a principle axis sugar cube does not rotate at all (nor does it shear), it just keeps its orientation and scales its edges.

8. Question: Can the ∇u and ∇v worlds be related somehow?

Although the $\nabla\mathbf{u}$ world as presented by Lai seems to deal with elastic solids and the $\nabla\mathbf{v}$ world with fluids, you would think that the $\nabla\mathbf{u}$ stuff would also apply in the fluid world in a differential dt sense. Maybe start by comparing the most basic equations in the two worlds,

$$d\mathbf{x} = F\, d\mathbf{X} = [1 + (\nabla\mathbf{u})]d\mathbf{X} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$$

$$d\mathbf{u} = (\nabla\mathbf{u})d\mathbf{X} \qquad (\nabla\mathbf{u}) = (\nabla\mathbf{u})^{S} + (\nabla\mathbf{u})^{A} = E + \Omega \qquad d\mathbf{x} = \Omega\mathbf{x} = d\boldsymbol{\theta} \times \mathbf{x}$$

$$D_t(d\mathbf{x}) = (\nabla\mathbf{v})d\mathbf{x} = Dd\mathbf{x} + Wd\mathbf{x} \qquad Wd\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x}$$

$$(\nabla\mathbf{v}) = D + W = \mathcal{L}\,(\text{web sources}) \qquad \text{where } D = (\nabla\mathbf{v})^{S} \text{ and } W = (\nabla\mathbf{v})^{A}$$

Somehow we want to divide the first world by dt. Start with

$$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X} \qquad d\mathbf{X} = \text{dumbbell} \qquad d\mathbf{x} = \text{dumbbell after stress applied}$$

Then have

$$D_t(d\mathbf{x}) = 0 + D_t\{(\nabla\mathbf{u})d\mathbf{X}\} = D_t\{(\nabla\mathbf{u})\}\, d\mathbf{X}$$

$$D_t(dx_i) = = [\, D_t\{(\nabla\mathbf{u})\}]_{ij}\, dX_j$$

$$[\, D_t\{(\nabla\mathbf{u})\}]_{ij} = D_t\,[(\nabla\mathbf{u})_{ij}] = D_t\,[\partial_j u_i\,] = \partial_j\, d_t u_i = \partial_j\, d_t x_i = \partial_j\, v_i = (\nabla\mathbf{v})_{ij}$$

It this is correct, we have then shown that

$$d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$$

$\Rightarrow \quad D_t(d\mathbf{x}_i) = (\nabla \mathbf{v})_{ij}\, d\mathbf{X}_j$

$\Rightarrow \quad D_t(d\mathbf{x}) = (\nabla \mathbf{v})\, d\mathbf{X}$

Question: does d**X** move in time? It is a starting dumbbell at $t = t_0$ in a flow. It is the value of d**x**(t) at time t_0, and as a value, I regard it as a constant. So no, dX is not time dependent, and so the D_t derivative above has been done correctly. Now consider a small dt, so then as usual $x \approx X$ to small order, then the above says

$D_t(d\mathbf{x}) = (\nabla \mathbf{v})\, d\mathbf{x}$

and this agrees with the $\nabla \mathbf{v}$ world equation as quoted above.

So let's start again and repeat the above, but replaces ∇u by $E + \Omega$:

$d\mathbf{x} = d\mathbf{X} + E d\mathbf{X} + \Omega\, d\mathbf{X}$ d**X** = dumbbell d**x** = dumbbell after stress

$D_t(d\mathbf{x}) = 0 + D_t\{E d\mathbf{X}\} + D_t\{\Omega d\mathbf{X}\}$

$= (D_t E) d\mathbf{X} + (D_t \Omega) d\mathbf{X}$

$\approx (D_t E) d\mathbf{x} + (D_t \Omega) d\mathbf{x}$

$= D d\mathbf{x} + W d\mathbf{x}$ // from the ∇v world

Therefore, it certainly seems that

$D = D_t E \qquad W = D_t \Omega$

Since for example $E = E(\mathbf{x},t)$, we cannot replace these with ∂_t. As reinforcement, start again with

$F = [1 + E + \Omega]$ F = "deformation gradient"
E = "infinitesimal strain tensor"
Ω = "infinitesimal rotation tensor"

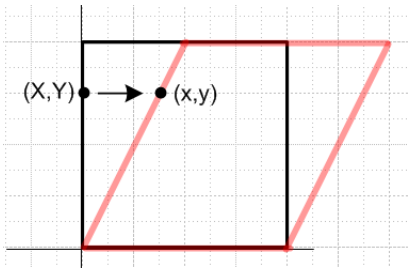
$D_t F = D_t E + D_t \Omega = D + W = (\nabla \mathbf{v})$ $(\nabla \mathbf{v}) = D_t F$ = "velocity gradient"
D = "rate of deformation tensor"
W = "spin tensor"

I cannot find these relations on line in a quick search. In general people don't see a need to relate the two worlds. Certainly the notion that $D = D_t E$ agrees with the interpretation of the two tensors! So I think this is correct

Appendix A. An example of simple shear and why it goes away in the "red frame".

Imagine that we have some displacement field **u** which indicates a shear perhaps with other things. For infinitesimal displacements, this deformation is described by symmetric tensor E which plays such a major role in elastostatics. Since we know there is a frame of reference in which E' will be diagonal, which means no shear, we have a little mystery question. How can you have shear in one camera view and no shear in another camera view. I plan to answer that right here.

Let's do a simple example. My pictures at the moment are inside Ng's covariance1.vsd. Start with



where black is starting situation and red is after deformation. The displacement field going with this picture is

$$x = X + kY$$
$$y = Y$$

so

$$u_x = kY$$
$$u_y = 0$$

Then we get

$$\partial_1 u_1 = \partial_x u_x = 0 \qquad \partial_2 u_1 = \partial_Y u_x = k$$
$$\partial_1 u_2 = \partial_x u_y = 0 \qquad \partial_2 u_2 = \partial_Y u_Y = 0$$

$$(\nabla u) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \quad => \quad E = (\nabla u)^S = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix} \qquad \Omega = (\nabla u)^A = \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}$$

where of course k is supposed to be "small", k << 1 I guess.

Now there is supposed to exist a rotated frame in which E' is diagonal (I will use my Ng notation)

$$E' = R^{-1} E R \qquad \hat{u}^{(i)} = R\, \hat{e}^{(i)} \qquad R = [\, \hat{u}^{(1)}, \hat{u}^{(2)} \,]$$

We know that $\hat{u}^{(i)}$ are the normalized eigenvalues of E. Maple is happy to tell us the eigenvalues and eigenvectors of any matrix:

```
E := matrix(2,2,[0,k/2,k/2,0]);
```

$$E := \begin{bmatrix} 0 & \frac{1}{2}k \\ \frac{1}{2}k & 0 \end{bmatrix}$$

```
eigenvectors(E);
```

$$\left[-\frac{1}{2}k,\ 1,\ \{[-1,\ 1]\}\right],\left[\frac{1}{2}k,\ 1,\ \{[1,\ 1]\}\right]$$

So here is the situation: (above gives eigenvalue, multiplicity, eigenvector within [...])

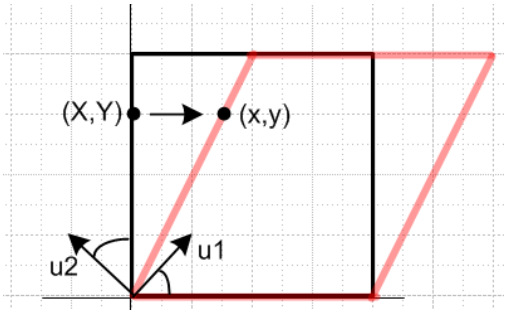
$E\,\hat{\mathbf{u}}^{(1)} = (k/2)\,\hat{\mathbf{u}}^{(1)}$ $\qquad \hat{\mathbf{u}}^{(1)} = (1/\sqrt{2}\ ,\ 1/\sqrt{2}\ )$

$E\,\hat{\mathbf{u}}^{(2)} = -(k/2)\,\hat{\mathbf{u}}^{(1)}$ $\qquad \hat{\mathbf{u}}^{(2)} = (-1/\sqrt{2}\ ,\ 1/\sqrt{2}\ )$

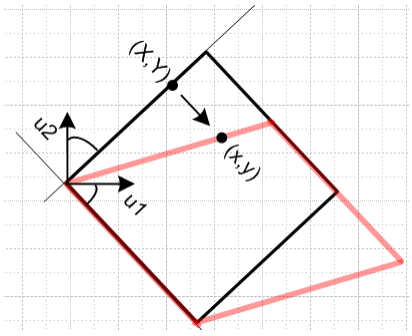
where I list them in order of decreasing eigenvalue. So we get

$$E' = \begin{pmatrix} k/2 & 0 \\ 0 & -k/2 \end{pmatrix} \qquad R = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = R_z(\pi/4) \qquad \text{// rot mat doc}$$

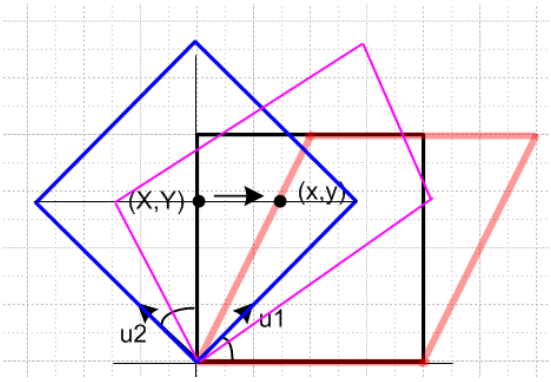
and we have that $\hat{\mathbf{u}}^{(2)} = R_z(\pi/4)\,\hat{\mathbf{e}}^{(1)}$ and $\hat{\mathbf{u}}^{(2)} = R_z(\pi/4)\,\hat{\mathbf{e}}^{(2)}$ and then we have



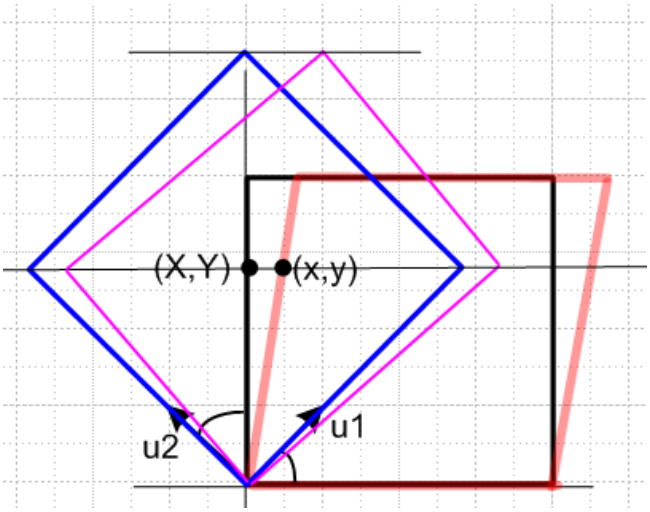
If we now go to frame S' and observe things, we get this picture



and of course the square is still showing shear. But what happens in this S' frame if we instead start with an axis-aligned square. It is most convenient to draw this new blue square in the original picture. I can then manually compute the position of the boundaries of this diamond as shown in violet



The violet thing still shows shear, but that is because I did not do this infinitesimally ! If we now make k somewhat smaller, this picture becomes



When k is quite small, the four corners of the purple thing will all be 90 degrees. This means that under deformation, the corner angles are not changing, and that means $E'_{12} = 0$, and that agrees with the known fact that E' has no diagonal elements. On the other hand, we can see that the u_1 edge is longer after deformation than it was in blue, and the u_2 edge is shorter. This agrees with

$$E' = \begin{pmatrix} k/2 & 0 \\ 0 & -k/2 \end{pmatrix}$$

because now k/2 is the fractional stretch of the u1 edge, and -k/2 that of the u2 edge.

In the above picture, the orange deformed boundary still shows quite a bit of shear. but the purple boundary is very close to being a rectangle! I suspect that the deviation from 90 degrees will be a second order effect. Let's try to compute these angles.

Assume the squares (blue and black) have edge e. Their diagonals are then $e\sqrt{2}$ and their half diagonals are $e/\sqrt{2}$. The distance up to the black dot pair is $y = e/\sqrt{2}$ so the horizontal distance between the black dots is then $k\, e/\sqrt{2}$. The lower right vector in purpose is then

$$-\mathbf{v}_{LR} = (e/\sqrt{2} + k\, e/\sqrt{2}\ ,\ e/\sqrt{2}\ ) - (0,0) = (e/\sqrt{2}\ )\ (1+k,1)$$

The upper right vector is then

$$\mathbf{v}_{UR} = (2\, k\, e/\sqrt{2}\ ,\ e\sqrt{2}\ ) \ - \ (e/\sqrt{2} + k\, e/\sqrt{2}\ ,\ e/\sqrt{2}\ ) = (k\, e/\sqrt{2} \ - e/\sqrt{2}\ ,\ e/\sqrt{2}\ ) \ = (e/\sqrt{2}\ )\ (k-1,1)$$

Note that

$$|\, \mathbf{v}_{LR}| = (e/\sqrt{2}\ )\ \sqrt{2+2k+k^2}$$

$$|\, \mathbf{v}_{UR}| = (e/\sqrt{2}\ )\ \sqrt{2-2k+k^2}$$

$$-\mathbf{v}_{LR} \bullet \mathbf{v}_{UR} = (e^2/2)\ [\ (1+k)(k-1) + 1*1\ ]$$
$$= (e^2/2)\ [\ (k+1)(k-1) + 1\ ] = (e^2/2)\ [\ (k^2 - 1) + 1\ ] = (e^2/2)\ k^2$$

Then we find that

$$\cos\theta = -(e^2/2)\ k^2\ /\ \{[(e/\sqrt{2}\ )\ \sqrt{2+2k+k^2}\ ][\ (e/\sqrt{2}\ )\ \sqrt{2-2k+k^2}\ ]\ \}$$

$$= -k^2\ /\ \{\ \sqrt{2+2k+k^2}\ \ \sqrt{2-2k+k^2}\ \ \} \ \approx\ -k^2/2$$

Now write $\theta = \pi/2 + \psi$ so that

$$\cos\theta = \cos(\pi/2 + \psi) = -\sin(\psi)$$

So we get that the deviation angle from 90 is ψ where

$$\sin(\psi) \approx k^2/2 \qquad \text{or} \qquad \psi \approx k^2/2$$

Thus, as predicted, the angle deviation (in radians) is quadratic in k.

For example, if $k = 1/5.6 = 0.18$ as shown in the picture, then

$$\text{shear angle} = \tan^{-1}(0.18) \sim 10 \text{ degrees}$$

$$\text{right purple corner deviation from 90} = \psi \approx (0.18)^2/2 = .016 \text{ radians}\ ,\ * 180/\pi \ \sim 1 \text{ degree}$$

Now go back to

$$(\nabla u) = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} = (\nabla u)^{S} + (\nabla u)^{A} = \begin{pmatrix} 0 & k/2 \\ k/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix} = E + \Omega$$

On scratch paper I show by direct hand calculation that

$$\Omega' = R^{-1}\Omega R = \Omega = \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}$$

Well, this is just because in this example, R and Ω are both z rotations, so everything commutes, fine. So in the rotated frame we have

$$(\nabla \mathbf{u})' = E' + \Omega' = \begin{pmatrix} k/2 & 0 \\ 0 & -k/2 \end{pmatrix} + \begin{pmatrix} 0 & k/2 \\ -k/2 & 0 \end{pmatrix}$$

This is a combination of an aligned rectangle stretch (the first term) plus a rotation (1 + the second term) by a small angle about the origin (backwards by $\sin\theta = k/2$, in agreement with the picture!)

Summary:

(1) We know from Lai page 86 that

$$d\mathbf{x}_1 \bullet d\mathbf{x}_2 = d\mathbf{X}_1 \bullet F^{T}F\, d\mathbf{X}_2 = d\mathbf{X}_1 \bullet (1+\nabla \mathbf{u}^{T})(1+\nabla \mathbf{u})\, d\mathbf{X}_2$$

$$\approx d\mathbf{X}_1 \bullet (1+2E)\, d\mathbf{X}_2 \qquad E = (\nabla \mathbf{u})^{S}$$

If the vectors are the same, so that $d\mathbf{X}_1 = d\mathbf{X}_2 = dX\,\mathbf{n}$, then

$$ds^2 = dS^2 + d\mathbf{X}_1 \bullet 2E d\mathbf{X}_1 = dS^2 + dS^2\,\mathbf{n} \bullet 2E\,\mathbf{n} = dS^2(1 + \mathbf{n} \bullet 2E\,\mathbf{n})$$

$$= dS^2(1 + 2E_{nn}) \quad \Rightarrow \quad ds^2 - dS^2 = 2E_{nn} \quad \Rightarrow (ds - dS)\,2dS = 2E_{nn}$$

$$\Rightarrow E_{nn} = (ds-dS)/dS \qquad \text{// as we know}$$

If the vectors are different (then orthogonal), then

$$d\mathbf{x}_1 \bullet d\mathbf{x}_2 \approx d\mathbf{X}_1 \bullet (1+2E)\, d\mathbf{X}_2 = d\mathbf{X}_1 \bullet 2E)\, d\mathbf{X}_2 = dS^2\,\mathbf{n}_1 \bullet 2E\,\mathbf{n}_2 = dS^2\, 2E_{12}$$

But $d\mathbf{x}_1 \bullet d\mathbf{x}_2 \approx dS^2\,\mathbf{n}_1 \bullet \mathbf{n}_2$ $\Rightarrow \mathbf{n}_1 \bullet \mathbf{n}_2 \approx 2E_{12}$ // as we know

So the interpretation of the elements of E_{ij} is only valid in the infinitesimal limit.

(2) Staying then reasonably close to this limit, so the theory is valid, we take a situation in frame S which has some very obvious shear. We wonder how it is that if observed from a special rotated frame S', we won't see any shear. The catch is that in the rotated frame, we have to take a *different* sugar cube -- one that aligns with the new frame axes. This is like the blue square in the picture above. We find that in this

frame, the blue square is deformed into a purple rectangle and this is in exact agreement with our calculation of the diagonal E' in this frame. This deformation involves stretching of axes, a rotation, but no shear! In the original frame we had only shear.