1. Cauchy Equation of Motion. The general framework of Lai's Chapter 5 Part A is that of *infinitesimal* motions in the solid. The quantities of interest are the displacement u_i, the strain E_{ij} and the stress T_{ij}. The fundamental equation for motion which replaces **F** = m**a** is $\rho\mathbf{B} + \mathbf{divT} = \rho\mathbf{a}$, where **divT** plays the role of "force". This is called Cauchy's equation of motion, p 213. In components it reads $\rho B_i + \partial_j T_{ij} = \rho a_i$. For a statics problem with no body forces, the equation is simply $\partial_j T_{ij} = 0$. Of course if you have a problem in which the BC's are specified in terms of u_i, then solving $\partial_j T_{ij} = 0$ requires extra work, it is not just a "problem in T_{ij}". [the Cauchy equation applies to any CM situation, not just infinitesimal.]

In curvilinear coordinates, the Cauchy equation when written in components looks quite different from the Cartesian form. One must compute **divT** in the curvilinear system of interest. Lai shows how to do this on page 54+ for polar, cylindrical and spherical coordinates. In my tensor doc, I show how to write **divT** in any curvilinear system.

2. Relation between E and u. In Cartesian coordinates the relation between E and u is given by $E_{ij} = (\partial_i u_j + \partial_j u_i)/2 = (\nabla\mathbf{u})^S{}_{ij}$. Under the transformation F which defines some curvilinear coordinates, neither $\partial_i u_j$ nor $\partial_i u_j + \partial_j u_i$ are tensors, so to relate the E_{ij} in some curvilinear system to the u_i in that system, one must do some work. Back in Chap 2 (p 54+) Lai gives expressions for a generic tensor $(\nabla\mathbf{v})$ in 3 basic curvilinear systems, and so it is then an easy matter to first change **v** to **u** to get $(\nabla\mathbf{u})$ and then from that to compute $(\nabla\mathbf{u})^S$. In polar coordinates, one finds for example that

$E_{rr} = (\nabla\mathbf{u})^S{}_{rr} = \partial_r u_r$
$E_{\theta\theta} = (\nabla\mathbf{u})^S{}_{\theta\theta} = (1/r)(\partial_\theta u_\theta + u_r)$
$E_{r\theta} = (\nabla\mathbf{u})^S{}_{r\theta} = [\,(1/r)(\partial_\theta u_r - u_\theta) + \partial_r u_\theta]/2 = (\nabla\mathbf{u})^S{}_{\theta r} = E_{\theta r}$. // agrees tensor doc

For cylindrical coordinates we would add to the above list (see page 60)

$E_{rz} = (\partial_z u_r + \partial_r u_z)/2 = E_{zr}$
$E_{\theta z} = (\partial_z u_\theta + (1/r)\partial_\theta u_z)/2 = E_{z\theta}$
$E_{zz} = \partial_z u_z$

and for sphericals see page 64.

In all of Lai's work, tensor components like $E_{r\theta}$ are the tensor components obtained when the tensor E is expanded on the orthogonal unit vectors of the curvilinear coordinate system. One can write

$E = \Sigma_{ij} [E^{(\hat{e})}]^{ij} (\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j)$ example: $[E^{(\hat{e})}]^{12} = E_{r\theta}$

and here is how these components are related to the Cartesian components

$[E^{(\hat{e})}]^{ij} = (h'_i R^{i}{}_{i'})(h'_j R^{j}{}_{j'}) E^{i'j'} = M^{i}{}_{i'} M^{j}{}_{j'} E^{i'j'}$ $M^{a}{}_{b} \equiv h'_a R^{a}{}_{b}$

where R is the usual R matrix for the curvilinear transformation F, and h'_i are the usual scale factors. Matrix M is always a rotation matrix. The above can be written in this manner in developmental notation,

$$[E^{(\hat{e})}]_{ij} = M_{ii'} M_{jj'} E_{i'j'} = (MEM^T)_{ij}$$

and when written in matrix form we then have $E^{(\hat{e})} = MEM^T$. For polar coordinates, one has then

$$\begin{pmatrix} E_{rr} & E_{r\theta} \\ E_{\theta r} & E_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which is just a way to write things.

Similarly one has

$$\mathbf{u} = \Sigma_i [u^{(\hat{e})}]^i \hat{\mathbf{e}}_i \qquad \text{example: } [u^{(\hat{e})}]^1 = u_r$$

and then components like u_r are related to the Cartesian components u^i in this manner

$$[u^{(\hat{e})}]^i = (h'_i R^i{}_{i'}) u^{i'}.$$

Given **u**, it is pretty easy to find E_{ij} since we have $E_{ij} = (\nabla \mathbf{u})^S{}_{ij}$ and we know how to compute this in any coordinate system. However, given E, it is less easy to compute **u** because you have to do integrations in a PDE world. Lai does this in various places as illustration. The "integration constants" type terms you get when you do these integrations can be associated with a $\mathbf{u}_a$ field which represents a rigid motion of the solid and which is usually not so interesting to the problem solver.

3. Relation between T and E. To close the infinitesimal equation of motion, one must have a relationship between force and displacement. In the general (infinitesimal) case one has $T_{ij} = C_{ijab} E_{ab}$ where C is the rank 4 elasticity tensor. But for isotropic solids, this simplifies and only 2 constants remain and we have

$$T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij} \qquad e = \mathrm{tr}(E) = E_{kk}$$

This equation is covariant under rotation M above (which M is the "R matrix for non-linear transformation F_M") and for that reason has the same form in any orthogonal coordinate system. For example,

$$T_{rr} = \lambda [T_{rr} + T_{\theta\theta}] + 2\mu E_{rr}$$
$$T_{r\theta} = 2\mu E_{r\theta} \quad .$$

Notice that there are no derivatives involved in the E/T relation, so it is just as easy to get T from E as it is to get E from T (for our isotropic solid). The inverse equation is given bottom page 209,

$$E_{ij} = (1/2\mu)[T_{ij} - \lambda(3\lambda+2\mu)^{-1} T_{kk} \delta_{ij}] \qquad \text{and} \qquad e = E_{kk} = (3\lambda+2\mu)^{-1} T_{kk}$$

and again this has the same appearance in other coordinates systems as per above.

4. The Navier Equation. From above the Cauchy equation says $\rho B_i + \partial_j T_{ij} = \rho a_i$ and we can then write this as

$$\rho B_i + \partial_j[\lambda E_{kk} \delta_{ij} + 2\mu E_{ij}] = \rho a_i \qquad a_i = \partial_t^2 u_i$$

or

$$\rho B_i + [\lambda (\partial_i E_{kk}) + 2\mu(\partial_j E_{ij})] = \rho a_i$$

Then finally we can replace $E_{ij} = (\nabla \mathbf{u})^S{}_{ij}$ so this becomes

$$\rho B_i + [\lambda (\partial_i \partial_k u_k) + \mu(\partial_j(\partial_i u_j + \partial_j u_i)] = \rho a_i$$

or

$$\rho B_i + [\lambda (\partial_i \partial_k u_k) + \mu (\partial_i \partial_j u_j + \partial_j \partial_j u_i)] = \rho a_i$$

or

$$\rho B_i + [\lambda (\partial_i e) + \mu (\partial_i e + \partial_j \partial_j u_i)] = \rho a_i$$

or

$$\rho B_i + (\lambda+\mu) (\partial_i e) + \mu \nabla^2 u_i = \rho a_i$$

or

$$\rho \mathbf{B} + (\lambda+\mu) (\nabla e) + \mu \nabla^2 \mathbf{u} = \rho \mathbf{a} \qquad e = \text{div } \mathbf{u}$$

which is the Navier or Navier/Cauchy equation. This equation can be viewed as a PDE system for displacement $\mathbf{u}(x)$. This then is the equation of motion for the displacement inside an isotropic solid with infinitesimal displacements and $\rho = \rho_0$ in Lai. On the last line it is written in vector form and we know how to write div u in any coordinate system, and from that we can write $\nabla(\text{div}\mathbf{u})$ in any system. We also know how to write the vector Laplacian shown, so we ultimately know how to write this Navier equation in any coordinate system we want. For a static situation with no body force we have

$$(\lambda+\mu) \nabla(\nabla \bullet \mathbf{u}) + \mu \nabla^2 \mathbf{u} = 0 \ .$$

On page 216+ Lai writes out both the component Navier equations and the T/E relationship in his three basic curvilinear systems and I have checked it all.

5. Solving Basic Problems in Statics: boundary conditions. Solving a problem usually means assuming some reasonable solution form for T_{ij} or E_{ij} or u_i and then showing that this form satisfies the Navier/Cauchy equation and meets the required boundary conditions for the problem at hand. The first solved problems concern waves of various types and reflection of such waves, and each of these problems has it's BC's to worry about. A very typical BC is the statement that some free part of the solid's bounding surface has "no surface tractions". This means that $T\mathbf{n}=0$ at all points on the free bounding surface, where $\mathbf{n}$ is the normal. This of course is a boundary condition on the object T_{ij}. Another type of boundary condition on T might be that $\mathbf{t} = T\mathbf{n} = \mathbf{t}_s$ where t_s is some driving (prescribed) surface traction. A completely different type of BC might say that $\mathbf{u} = 0$ on some part of the boundary. A frictionless wall boundary condition says that $\mathbf{t} = T\mathbf{n} = t\,\mathbf{n}$ which says that the shear surface tractions only are zero.

6. Planar Static Problems. Lai spends quite a bit of time on problems which have a certain simple form for either the T or E matrix, the plane-strain and plane-stress class problems. Within this class of problem, he shows how the Airy function φ which is related to the planar T_{ij} elements in a certain way allows for a certain solution method which involves the biharmonic equation. This Airy function is "potential like", and only one such function is needed, the cost being $\nabla^4\varphi$ to deal with. Especially in non-Cartesian coordinate systems, there is lot of "Airy technology" available.

7. For more general static problems, one uses the vector and scalar potentials. The general idea here is that we introduce a whole new set of fields, these potentials, which are related to **u** as follows:

$$\mathbf{u} = \boldsymbol{\Psi} - [4(\nu-1)]^{-1} \nabla (\mathbf{x} \bullet \boldsymbol{\Psi} + \Phi)$$

Lai shows on page 279 that you can satisfy the static Navier equation quoted above if you can find the vector and scalar potentials $\boldsymbol{\Psi}$ and Φ such that

$$\nabla^2 \boldsymbol{\Psi} = - \mathbf{B}/\mu \qquad \text{and} \qquad \nabla^2\Phi = \mathbf{x}\bullet\mathbf{B}/\mu$$

and of course if there is no body force, you have just

$$\nabla^2 \boldsymbol{\Psi} = 0 \qquad \text{and} \qquad \nabla^2\Phi = \mathbf{0}$$

vector Laplacian scalar Laplacian

with an alternate form $\boldsymbol{\psi}$ and φ given on page 280. Of course now you have to worry about the boundary conditions on these potentials and how that might relate to physical BC's for example on **u**.