Surface Tractions PhL 5.13.12

1. Consider an arbitrarily oriented differential cube in R^3 at some location $\mathbf{x}$, and consider some face on this cube which has unit normal $\mathbf{n}$. This cube is embedded in a solid object. The rest of the solid object applies a stress vector $\mathbf{t}$ to this face in the amount $\mathbf{t} = T\mathbf{n}$ where $T = T(\mathbf{x})$ at this point.

2. If this face of the cube lies on a boundary of a solid object, so there is nothing outside, then there is nothing there to apply a stress to that face, so we must have $\mathbf{t} = 0$ on such a boundary face. In this case one says that we have "no surface traction" at such a point. This is a boundary condition:

$$T\mathbf{n} = 0$$

that the tensor T must meet.

3. On the other hand, one might have some external object outside our test object which external object applies a stress $\mathbf{t}_a$ to our face of interest. Then our boundary condition at this point is

$$T\mathbf{n} = \mathbf{t}_a \quad .$$

4. One interesting possible external object is a "frictionless wall" which can apply a normal stress to our face, but cannot apply a shear stress. In this special case of the above, we would have $\mathbf{t}_a = t_a\mathbf{n}$ as the only possible applied external stress. In this face the surface traction is "normal only" and there is no shear surface traction.

5. There are various ways to writing things. For example

$$\mathbf{t} = T\mathbf{n} = \mathbf{T}\,(n_1\hat{\mathbf{x}}_1 + n_2\hat{\mathbf{x}}_2 + n_3\hat{\mathbf{x}}_3) = n_1\,\mathbf{T}\hat{\mathbf{x}}_1 + n_2\,\mathbf{T}\hat{\mathbf{x}}_2 + n_3\,\mathbf{T}\hat{\mathbf{x}}_3$$

A component of this equation would be

$$t_i = n_1\,(\mathbf{T}\hat{\mathbf{x}}_1)_i + n_2\,(\mathbf{T}\hat{\mathbf{x}}_2)_i + n_3\,(\mathbf{T}\hat{\mathbf{x}}_3)_i$$

But

$$(\mathbf{T}\hat{\mathbf{x}}_1)_i = T_{ij}(\hat{\mathbf{x}}_1)_j = T_{ij}\delta_{1,j} = T_{i1}$$

Therefore we can write

$$t_i = n_1\,T_{i1} + n_2\,T_{i2} + n_3\,T_{i3}$$
$$\mathbf{t} = n_1\,\mathbf{T}\hat{\mathbf{x}}_1 + n_2\,\mathbf{T}\hat{\mathbf{x}}_2 + n_3\,\mathbf{T}\hat{\mathbf{x}}_3$$

In Lai's notation we would write

$$t_i = n_1\,T_{i1} + n_2\,T_{i2} + n_3\,T_{i3}$$

$$\mathbf{t} = n_1 \mathbf{Te}_1 + n_2 \mathbf{Te}_2 + n_3 \mathbf{Te}_3$$

6. Example. Consider a prismatic bar in the x_1 direction and let **n** be a normal on the lateral surface. If nothing is touching the lateral surface, then one must have $T\mathbf{n} = 0$ at all points on the lateral surface. So any candidate T must meet this boundary condition.