Brachistochrone Problem PhL 10.29.16

Here I just finish the solution started by Goldstein page 36. To avoid confusion, I am going to assume the particle starts at t = 0 and y(0) = 0 and goes down to negative y (G likes that to be positive y which just makes a mess for me). We then have

$$f = \sqrt{(1+\dot{y}^2)/(-2gy)} \qquad (1) \qquad y \le 0$$

Here $\dot{y}$ *means* dy/dx.

Have Maple compute the required stuff. // I did this, and I end up with this ODE = 0 (see brachisto.mws)

$$eq1 := 1 + \left(\frac{\partial}{\partial x} y(x)\right)^2 + 2\left(\frac{\partial^2}{\partial x^2} y(x)\right) y(x)$$

or

$$2\, y(x)\, y''(x) + y'(x)^2 + 1 = 0 \qquad (2)$$

Wolfram claims this is equivalent to

$$[\,1 + y'(x)^2]\, y(x) = \text{constant} \qquad (3)$$

Proof: If I differentiate the latter, I get

$$[\,1 + y'(x)^2]\, y'(x) + [2y'(x)\, y''(x)]\, y(x) = 0$$

or

$$[\,1 + y'(x)^2] + [2\, y''(x)]\, y(x) = 0 \qquad (4)$$

and sure enough, this is exactly my equation (2). I have just shown that

(3) true $\Rightarrow$ (2) true

Can we go the other direction? This is what I just showed above:

$$d\{\,[\,1 + y'(x)^2]\, y(x)\,\}/dx = [\,1 + y'(x)^2] + [2\, y''(x)]\, y(x)$$

or

$$d\{\,[\,1 + y'(x)^2]\, y(x)\,\} = \{\,[\,1 + y'(x)^2] + [2\, y''(x)]\, y(x)\}dx$$

This is a perfect differential, so integrate this from $x = x_0$ to some t to get

$$[\,1 + y'(x)^2]\, y(x) - [\,1 + y'(x_0)^2]\, y(x_0) = \int_0^{x_0} \{\,[\,1 + y'(x)^2] + [2\, y''(x)]\, y(x)\}dx$$

Now if I know that $\{ [1 + y'(x)^2] + [2 y''(x)] y(x)\} = 0$, then I know that

$$[1 + y'(x)^2] y(x) = [1 + y'(x_0)^2] y(x_0)$$

and this then has a *specific* constant. So what I know is

(2) true $\Rightarrow$ (3) true with the specific constant shown.

Comment: Be careful, if you set $x_0 = 0$ and $y(x_0) = 0$ you get infinite slope times 0 here. This part of the curve only applies if your final point is directly underneath your starting point.

To summarize:

Equation 2 $\quad 2 y(x) y''(x) + y'(x)^2 + 1 = 0$

Equation 3 $\quad [1 + y'(x)^2] y(x) = K$

Theorem 1: Eq 3 true $\Rightarrow$ Eq 2 true for any value of K in Eq 3

Theorem 2: Eq 2 true $\Rightarrow$ Eq 3 but only for $K = [1 + y'(x_0)^2] y(x_0)$

Now Eq 2 is what really comes out of the Euler-Lagrange analysis.

Resume: Now I only have to solve a first order ODE

$$[1 + y'(x)^2] y(x) = C/2 \qquad (3)$$

We are then supposed to try this "**cycloid**" solution which is the ONLY solution in fact,

$$x = C(\theta - \sin\theta)$$
$$y = C(1 - \cos\theta)$$

$$dx/d\theta = C(1 - \cos\theta)$$
$$dy/d\theta = +C\sin(\theta)$$

$$\equiv dy/dx = \sin\theta/(1-\cos\theta)$$

$$[1 + y'(x)^2] y(x) = [1 + \sin^2\theta/(1-\cos\theta)^2]C (1-\cos\theta) = K \; ??$$

or

$$[(1-\cos\theta)^2 + \sin^2\theta] C(1-\cos\theta) = K (1-\cos\theta)^2 \; ??$$

or

$$[(1-\cos\theta)^2 + \sin^2\theta]C = K (1-\cos\theta) \; ??$$

or

$$[1+\cos^2\theta - 2\cos\theta + \sin^2\theta] C = K (1-\cos\theta) \; ??$$

or

$$[2 - 2\cos\theta\ ]C = K\ (1-\cos\theta) \quad ??$$

or

$$2C = K \quad ??$$

This is true if K = C/2. So then I claim that for the above cycloid solution this equation is true,

$$[\ 1 + y'(x)^2]\ y(x) = C/2$$

Now in my non-Goldstein world the y axis points up as usual, so we must have C < 0 which will make y < 0 on our trajectory. For example, we then have

```
restart;
C := -2;
```

$$C := -2$$

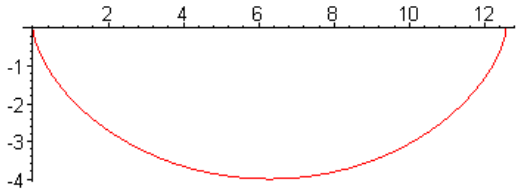
```
x := C*(theta - sin(theta));
```

$$x := -2\,\theta + 2\sin(\theta)$$

```
y := C*(1-cos(theta));
```

$$y := -2 + 2\cos(\theta)$$

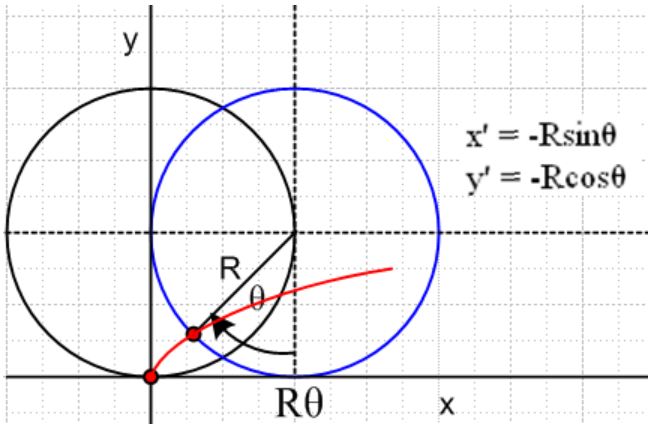
```
plot([x,y,theta=-2*Pi..0], scaling = constrained);
```



Remember that y axis goes up, x axis goes to the right, particle starts at origin and drops down the curve. It happens that the right side of the curve above has θ < 0 as Maple plotting shows.

Where is the bottom of this curve? Look for dy/dx = 0. But above dy/dx = sinθ/(1-cosθ) so the bottom occurs at θ = -π. And at that point y = 2C. And also x = -Cπ and y = 2C. Ratio is -y/x = 2/π ≈ 2/3. Interpret 2C = the lowest value that y attains on a hump. C = (1/2) y_{min} < 0.

Digression: The rolling wheel of radius R. Here is the rolling wheel model (it rolls to the right)



Relative to the center of the wheel the marked point has coordinates x' = -Rsinθ and y' = -Rcosθ. The center of the wheel is at $x_c = R\theta$ and $y_c = R$. Thus, the expressions for the point (x,y) are

$$x = x_c + x' = R\theta - R\sin\theta = R(\theta - \sin\theta)$$

$$y = y_c + y' = R - R\cos\theta = R(1 - \cos\theta)$$

So the dot painted on the wheel traces out the right side of the second red curve shown above, R > 0 here.

Back to the problem:

What then exactly is the curve? It starts at the origin, and you want it to go through some point (x,y) where x > 0 and y < 0. So consider (where we know C < 0)

$$x = C(\theta - \sin\theta)$$
$$y = C(1 - \cos\theta) .$$

What do we learn from the origin point (x,y) = (0,0)? Only that θ = 0.

But from the other point we

$$x_1 = C(\theta - \sin\theta)$$
$$-h_1 \equiv -|y_1| = C(1 - \cos\theta)$$

So this is two equations in two unknowns. Here h_i is the drop distance > 0. Take the ratio to get

$$x_1/h_1 = -(\theta - \sin\theta)/(1 - \cos\theta)$$

Use this to solve for θ, then use either of the above to solve for C which we know is C < 0:

```
restart;

h1 := 2;
```

$$h1 := 2$$

```
x1 := 1;
```

$$x1 := 1$$

```
eq1 := (x1/h1) = -(theta - sin(theta))/(1-cos(theta));
```

$$eq1 := \frac{1}{2} = -\frac{\theta - \sin(\theta)}{1 - \cos(\theta)}$$

```
S := fsolve(eq1,theta);
```

$$S := -1.401379456$$

```
print(S);
```

$$-1.401379456$$

```
(S - sin(S))/(1-cos(S));
```

$$-.5000000001$$

```
C := -h1/(1-cos(S));
```

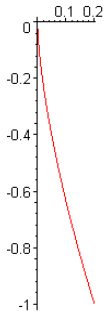
$$C := -2.405602930$$

So to make the brach path start at the origin and pass through the point (x,y) = (1,-2), we find that the solution angle parameter is θ = -1.40 radians (implies right side), and C = -2.41 (neg as expected).

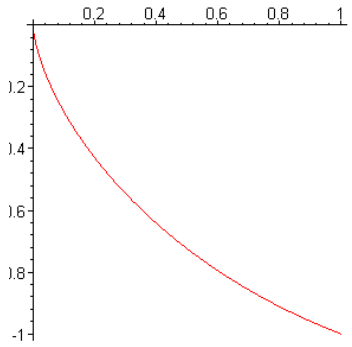
Selected Plots and Conclusions

The point we want to pass through is (x,y) = (x_1, $-h_1$) where x_1 and h_1 are positive.
Define a ratio ratio = x1/h1. If this ratio is small, you do a dive. If it is large, you dive and come back up again. Specific examples

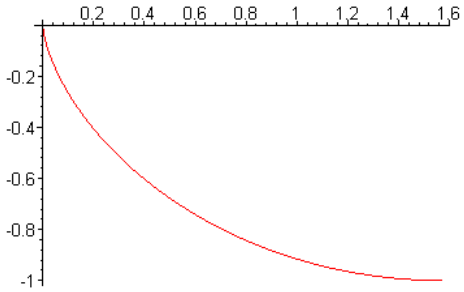
h1=1 x1=0.2 ratio = 0.2



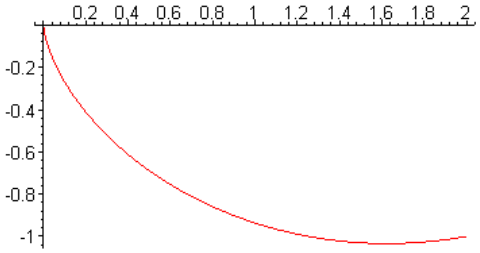
h1=1 x1=1 ratio = 1



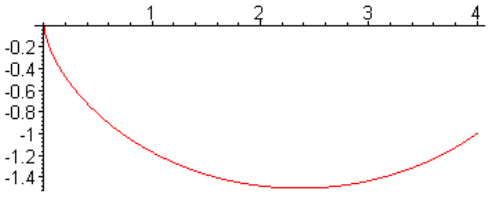
h1=1 x1=π/2 ratio = π/2 (zero slope at the end of the run)



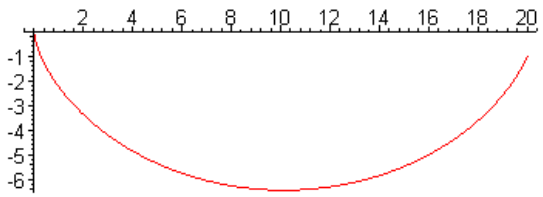
h1=1 x1=2 ratio = 2 (starting to come back up)



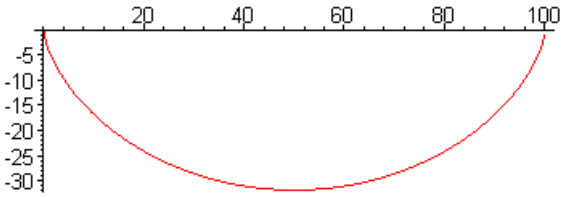
h1=1 x1=4 ratio = 4 (lots of coming back up)



h1=1 x1=20 ratio = 20 (coming WAY back up)



h1=1 x1=100 ratio = 100 (coming WAY WAYback up)



In this kind of limit, the ratio of x/2 at half way to ymax is $\pi/2 \approx 3/2$ as noted above. Another way to say it is that x/depth = 2 ($x_{1/2}$/depth) = 2 ($\pi/2$) = π. So the depth is $1/\pi$ times the distance for a long trip.

I found a note on Brachistochrone trains here:

https://brilliantuniverse.wordpress.com/2014/04/02/trains-and-tunnels-the-brachistochrone/

Question: What is the speed at any point on such a trip?
Question: What is position at any given time?

$x = C(\theta - \sin\theta)$
$y = C(1-\cos\theta)$.

$v_x(t) = C(\partial_t\theta - \cos\theta\partial_t\theta) = C(\theta-\cos\theta)(\partial_t\theta)$
$v_y(t) = C(1-\cos\theta) = C\sin\theta(\partial_t\theta)$

How do you do stuff like this when you only have a parametric form??

At some instant you know it is sliding at this angle

$dy/dx = \sin\theta/(1-\cos\theta) = \tan(\varphi)$

We know

$ds = \sqrt{1 + (dy/dx)^2}\ dx$

But this just tells you ratio of speeds.

Acceleration along the curve might be this

$a_s = g\sin(\varphi) = d^2s/dt^2$ // could use above to get some $a_s(\theta)$

Then work back to velocity and then position.

Day Two 10.30.16

Idea for finding θ(t): use the two equations obtained by setting $a_x = 0$ and $a_y = -g$ by twice diffing the expressions for x and y, all in Maple. Here are the results

x = C(θ - sinθ)
y = C(1-cosθ) .

v_x = C(1 - cosθ)θ'
v_y = C(sinθ θ') // since C<0, θ<0,v_y<0 must have θ' < 0

a_x = C[(1 - cosθ)θ" + sinθ θ'²]
a_y = C(cosθ θ'² + sinθ θ")

Now I should be able to inject "the physics" into the problem using Newton's law:

a_x = 0
a_y = - g // because y is up in my world, gravity pulls down

STOP!!! I am using **F** = m**a** but I am neglecting the constraint force in these equations !!!!
So ignore everything in blue below this point in this doc!! Resume after

The two equations are then

0 = C[(1 - cosθ)θ" + sinθ θ'²]
-g = C[cosθ θ'² + sinθ θ"] wrong

The first equation tells us that

(1 - cosθ)θ" + sinθ θ'² = 0

which relates θ" to θ' which seems helpful. Try now to eliminate θ'² :

θ'² = - [(1 - cosθ)/sinθ]θ" or θ" = - sinθ/(1-cosθ) * θ'² (*)

Note: for small negative θ, θ" > 0.

Note: Examine a_y for small negative θ:

a_y = C(cosθ θ'² + sinθ θ")
- + - + // cannot determine the sign of a_y in this way

Then the second equation says

-g/C = cosθ θ'² + sinθ θ"

$= - \cos\theta[(1 - \cos\theta)/\sin\theta]\theta'' + \sin\theta\ \theta''$

$= [- \cos\theta(1 - \cos\theta)/\sin\theta + \sin\theta] \theta''$

$= [- \cos\theta(1 - \cos\theta) + \sin^2\theta] \theta'' / \sin\theta$

$= [- \cos\theta + \cos^2\theta + \sin^2\theta] \theta'' / \sin\theta$

$= [- \cos\theta + 1] \theta'' / \sin\theta$

So we finally have an ODE for θ

$$(1- \cos\theta)\ \theta'' + (g/C) \sin\theta = 0$$

Now use half angles

$$2\sin^2(\theta/2)\ \theta'' + (g/C)\ 2\sin(\theta/2)\cos(\theta/2) = 0$$

or

$$\sin(\theta/2)\ \theta'' + (g/C)\ \cos(\theta/2) = 0$$

or

$$\theta'' + (g/C)\ \cot(\theta/2) = 0$$

How can I verify that I did this correctly? Here is an attempt:

```
restart;
x := C*(theta(t) - sin(theta(t)));
```

$$x := C\,(\theta(t) - \sin(\theta(t)))$$

```
y := C*(1 - cos(theta(t)));
```

$$y := C\,(1 - \cos(\theta(t)))$$

```
vx := diff(x,t);
```

$$vx := C\left(\left(\frac{\partial}{\partial t}\theta(t)\right) - \cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\right)$$

```
vy := diff(y,t);
```

$$vy := C\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$

```
ax := diff(vx,t);
```

$$ax := C\left(\left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + \sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 - \cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\right)$$

```
ay := diff(vy,t);
```

$$ay := C\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 + C\sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

```
ax := subs({diff(theta(t),t,t)= tpp, diff(theta(t),t)=tp},ax);
```

$$ax := C\,(tpp + \sin(\theta(t))\,tp^2 - \cos(\theta(t))\,tpp)$$

```
ay := subs({diff(theta(t),t,t)= tpp, diff(theta(t),t)=tp},ay);
```

$$ay := C\cos(\theta(t))\,tp^2 + C\sin(\theta(t))\,tpp$$

```
eq1 := ax=0;
```

$$eq1 := C\,(tpp + \sin(\theta(t))\,tp^2 - \cos(\theta(t))\,tpp) = 0$$

```
eq2 := ay = -g;
```

$$eq2 := C\cos(\theta(t))\,tp^2 + C\sin(\theta(t))\,tpp = -g$$

```
solve({eq1,eq1}, {tpp, tp});
```

$$\{tp = tp,\ tpp = -\frac{tp^2\,(\cos(\theta(t)) + 1)}{\sin(\theta(t))}\}$$

Last line says $\theta'' = -\theta'^2\,(1+\cos\theta)/\sin\theta$ which disagrees with my (*) above which was

$\theta'' = -\sin\theta/(1-\cos\theta) * \theta'^2$

Is it possible that

$(1+\cos\theta)/\sin\theta = \sin\theta/(1-\cos\theta)$?

$(1+\cos\theta) = \sin^2\theta/(1-\cos\theta)$?

$(1-\cos\theta)(1+\cos\theta) = \sin^2\theta$?

$1 - \cos^2\theta = \sin^2\theta$? yes!

So we agree on that result. I then do more Maple and it seems to say

$a_y = -C\,\theta'^2$

But something is wrong here because I know C< 0 but want $a_y = -g$, sides then have different signs.

Can I get that result by hand? Back up to this point,

$a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\,\theta'^2]$
$a_y = C(\cos\theta\,\theta'^2 + \sin\theta\,\theta'')$

$\theta'^2 = -[(1 - \cos\theta)/\sin\theta]\theta''$ or $\theta'' = -\sin\theta/(1-\cos\theta) * \theta'^2$ (*)

Use the second in a_y to get

$a_y = C(\cos\theta\,\theta'^2 + \sin\theta[-\sin\theta/(1-\cos\theta)\theta'^2])$

$= C\theta'^2(\cos\theta + \sin\theta[-\sin\theta/(1-\cos\theta)])$

$= C\theta'^2(\cos\theta - \sin^2\theta/(1-\cos\theta))$

$= C\theta'^2(\cos\theta(1-\cos\theta) - \sin^2\theta)/(1-\cos\theta)$

$= C\theta'^2(\cos\theta-\cos^2\theta - \sin^2\theta)/(1-\cos\theta)$

$= C\theta'^2(\cos\theta-1)/(1-\cos\theta)$

$= -C\theta'^2$

OK, it is verified. But I still have the sign of sides problem!

We now have this fairly simple ODE to deal with

$-C\theta'^2 = a_y = -g$

$C\theta'^2 = g$

$\theta'^2 = (g/C)$

$\theta' = -\sqrt{g/C}$

$\theta(t) = -\sqrt{g/C}\,t + \theta(0)$

Now $\theta(0)$ = left end of curve = 0, so we have

$\theta(t) = -\sqrt{g/C}\,t$

as the connection between the two "parameters" .

There must be some trivial way I could know this. Energy conservation says

$mg(-y) = (1/2)m(v_x^2+v_y^2)$

Above I had

$v_x = C(1 - \cos\theta)\theta'$
$v_y = C(\sin\theta\ \theta')$

$y = C(1-\cos\theta)$

Then energy says

$-mg\ C(1-\cos\theta) = (m/2)C^2\ [\ (1-\cos\theta)^2 + \sin^2\theta\ ]\ \theta'^2$

$-g\ (1-\cos\theta) = (1/2)C\ [\ (1-\cos\theta)^2 + \sin^2\theta\ ]\ \theta'^2$

$-g\ (1-\cos\theta) = (1/2)C\ [\ 1 + \cos^2\theta - 2\cos\theta + \sin^2\theta\ ]\ \theta'^2$

$-g\ (1-\cos\theta) = (1/2)C\ [\ 2 - 2\cos\theta\ ]\ \theta'^2$

$-g\ (1-\cos\theta) = C\ [\ 1 - \cos\theta\ ]\ \theta'^2$

$-g = C\ \theta'^2$

Paradox: In the equation $a_y = -C\theta'^2$ we have a sign disagreement on the two sides.

1. Why is C negative? Go back to $y = C(1-\cos\theta)$. We know $1-\cos\theta > 0$ and $y < 0$, therefore we know that C is negative.

2. Why do I think $a_y = -g$? The mass is being pulled down by gravity. Down is the minus y direction because that is how I have set things up.

3. Is x really positive? I have $x = C(\theta - \sin\theta) = C(-|\theta| + \sin|\theta|)$ for small neg θ on the right, so

$x = -C(|\theta| - \sin|\theta|)$ here $|\theta| - \sin|\theta| > 0$ and $C < 0$ to $x > 0$, correct.

4. Again from $y = C(1-\cos\theta)$ for small θ we have $y < 0$ if $C<0$ and this is correct.

Resume. The ramp has a normal force N and if we draw a little picture, we see that $\mathbf{N} + \mathbf{g}$ must point along the ramp [**wrong**]. Total force is $\mathbf{F} = \mathbf{N} + \mathbf{g}$ and $\mathbf{g} = -g\hat{\mathbf{y}}$ with $g>0$. So write $\mathbf{F} = \mathbf{N} - g\hat{\mathbf{y}}$ and then we have $\mathbf{N} = \mathbf{F} + g\hat{\mathbf{y}}$ and $N_x = F_x$ and $N_y = F_y + g$

$F_x = N_x + g_x = N_x$ **ok**
$F_y = N_y + g_y = N_y - g$ **ok**

$F_y/F_x = (N_y - g)/N_x$ **ok**

We then need **F** to point down along the curve [**wrong**]. Then

wrong

$F_y/F_x = dy/dx = \sin\theta/(1-\cos\theta)$ // from far above

\- + - + - +

We then have

wrong

$(N_y - g)/N_x = \sin\theta/(1-\cos\theta)$

Now we can use **F** = m**a** so // **F** = m**a** = **N** + **g**

$F_x = ma_x$ $F_y = ma_y$ **ok**

ok **wrong** **ok**

$a_y/a_x = F_y/F_x = dy/dx = \sin\theta/(1-\cos\theta)$ **ok**

$a_y = [\sin\theta/(1-\cos\theta)]\, a_x$ **wrong**

\- - + +

At least we now have a relationship between the two acceleration components [**wrong**] . Maple tells us that

$a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\, \theta'^2]$

$a_y = C(\cos\theta\, \theta'^2 + \sin\theta\, \theta'')$ // checked this with new Maple code

Divide to get

wrong

$a_y/a_x = [\cos\theta\, \theta'^2 + \sin\theta\, \theta''] / [(1 - \cos\theta)\theta'' + \sin\theta\, \theta'^2] = [\sin\theta/(1-\cos\theta)]$

Don't know if this will help, but at least it is a new fact of interest:

$[\cos\theta\, \theta'^2 + \sin\theta\, \theta''] / [(1 - \cos\theta)\theta'' + \sin\theta\, \theta'^2] = [\sin\theta/(1-\cos\theta)]$

or

$(1-\cos\theta)[\cos\theta\, \theta'^2 + \sin\theta\, \theta''] = [\sin\theta]\, [(1 - \cos\theta)\theta'' + \sin\theta\, \theta'^2]$

or

$(1-\cos\theta)[\cos\theta\, \theta'^2 + \sin\theta\, \theta''] - [\sin\theta]\, [(1 - \cos\theta)\theta'' + \sin\theta\, \theta'^2] = 0$

or

$\theta'^2\{(1-\cos\theta)\cos\theta - \sin^2\theta\} + \theta''\{(1-\cos\theta)\sin\theta - [\sin\theta]\, [(1 - \cos\theta)]\} = 0$

or

$\theta'^2\{(1-\cos\theta)\cos\theta - \sin^2\theta\} = 0$

or

$\theta'^2\{\cos\theta - \cos^2\theta - \sin^2\theta\} = 0$

or

$\theta'^2\{\cos\theta - 1\} = 0$

$\Rightarrow \theta' = 0$

which of course is a wrong result. [and now I know why it is wrong]

Paradox of 8:15AM

This says that θ'(t) = 0 which says θ(t) = constant = θ(0) = 0 so θ(t) = 0 which is total garbage. I checked the algebra in red. There is then a mental logic error somewhere in the above foot of file. Here are the possibilities of what is wrong:

1. **F** = m**a** is wrong where **F** is the total force on m (seems unlikely that this is wrong).

2. F_y/F_x = dy/dx is wrong. (drew careful picture, seems right) **wrong**

3. $a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\ \theta'^2]$ one or both is wrong
   $a_y = C(\cos\theta\ \theta'^2 + \sin\theta\ \theta'')$ (both lines copied directly from Maple and x,y correctly entered)

4. dy/dx = sinθ/(1-cosθ) is wrong (but this is verified below)

5. Algebra above is done wrong. (Maple says θ'(t) = 0 as well, see below)

Look at item 4:

x = C(θ - sinθ)
y = C(1-cosθ) .

dx = C(dθ - cosθdθ)
dy = C(+ sinθdθ)

dy/dx = sinθ/ (1-cosθ) agrees with 4 as quoted.

Let Maple do algebra of step 5. First we have this

```
ax := diff(vx,t):ax := collect(% ,diff(theta(t),t,t));
```

$$ax := C\,(1 - \cos(\theta(t)))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + C\,\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

```
ay := diff(vy,t);
```

$$ay := C\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 + C\,\sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

```
ax := subs({diff(theta(t),t,t)= tpp, diff(theta(t),t)=tp},ax);
```

$$ax := C\,(1 - \cos(\theta(t)))\,tpp + C\,\sin(\theta(t))\,tp^2$$

```
ay := subs({diff(theta(t),t,t)= tpp, diff(theta(t),t)=tp},ay);
```

$$ay := C\,\cos(\theta(t))\,tp^2 + C\,\sin(\theta(t))\,tpp$$

Then we do this

```
eq1 := ay/ax - sin(theta(t))/(1 - cos(theta(t))) = 0;
```

$$eq1 := \frac{C\cos(\theta(t))\,tp^2 + C\sin(\theta(t))\,tpp}{C\,(1-\cos(\theta(t)))\,tpp + C\sin(\theta(t))\,tp^2} - \frac{\sin(\theta(t))}{1-\cos(\theta(t))} = 0$$

```
simplify(%);
```

$$\frac{tp^2}{-tpp + tpp\cos(\theta(t)) - \sin(\theta(t))\,tp^2} = 0$$

The only way this does NOT say θ' = 0 is if the denominator vanishes. That would require

$-\theta'' + \cos\theta\,\theta'' - \sin\theta\,\theta'^2 = 0$

or

$\theta''(\cos\theta - 1) = \sin\theta\,\theta'^2$

$\theta'' = [\sin\theta/(\cos\theta - 1)]\,\theta'^2$

But the second line above says $\theta''(\cos\theta-1) - \sin\theta\,\theta'^2 = 0$ or $\theta''(1-\cos\theta) + \sin\theta\,\theta'^2 = 0$. But recall above

$a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\,\theta'^2]$

So this little option says $a_x = 0$, but I know that is wrong because object does accelerate to the right as if slides down the ramp. How else would it pick up $v_x > 0$ after the vertical start?

Status of paradox 8:40AM. I made a list of 5 possible error sources, and checked each one and none of these sources appear to be wrong. Paradox lives on!!!

Did a whiteboard session and I found the mistake. It is item 2. Although v must point along the track, in general **a** and therefore **F** do not point along the track. Think ball on string going around, or perhaps circular track with track surface pointing to the center. In that case **a** points to the center, it does not point along the track. It is **v** which points along the track.

Did I make this same mistake somewhere in Lagrange doc??? Well there I had a linear track as an example in Appendix D, so in that case the force F *was* along the track, whew!

OK, maybe this is what I want to look at : it is **v** which points along the track, not **a**.

$\sin\theta/(1-\cos\theta) = dy/dx = v_y/v_x = C(\sin\theta\,\theta')/C(1 - \cos\theta)\theta' = \sin\theta/(1-\cos\theta)$

OK, this is true, but adds no new information. So here is what I know

$x = C(\theta - \sin\theta)$
$y = C(1-\cos\theta)$.

$v_x = C(1 - \cos\theta)\theta'$
$v_y = C(\sin\theta\,\theta')$

$a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\ \theta'^2]$
$a_y = C(\cos\theta\ \theta'^2 + \sin\theta\ \theta'')$

$dy/dx = v_y/v_x = \sin\theta/(1-\cos\theta)$

How about using this extra fact from energy conservation

$(1/2)mv^2 = mg(-y)$

This says

$v^2 = -2gy$

or

$v_x^2 + v_y^2 = -2gy$

or

$C^2(1 - \cos\theta)^2\theta'^2 + C^2(\sin^2\theta\ \theta'^2) = -2g\ C(1-\cos\theta)$

or

$C(1 - \cos\theta)^2\theta'^2 + C(\sin^2\theta\ \theta'^2) = -2g\ (1-\cos\theta)$

or

$C\theta'^2[\ (1 - \cos\theta)^2 + \sin^2\theta] = -2g\ (1-\cos\theta)$

or

$C\theta'^2[\ 1 + \cos^2\theta - 2\cos\theta + \sin^2\theta] = -2g\ (1-\cos\theta)$

or

$C\theta'^2[2 - 2\cos\theta\ ] = -2g\ (1-\cos\theta)$

or

$2C\theta'^2[1 - \cos\theta\ ] = -2g\ (1-\cos\theta)$

or

$C\theta'^2 = -g$

or

$\theta'^2 = -g/C$

or

$\theta' = -\sqrt{g/(-C)}$

This seems perhaps reasonable. Then

$\theta(t) = -\sqrt{g/(-C)}\ t + \theta(0)$

But $\theta(0) = 0$, so conclude that

$\theta(t) = -\sqrt{g/(-C)}\ t$

and finally I have the connection between $\theta(t)$ and t and it is completely trivial. Marion does not mention this equation. Maple verifies this now

```
KE := (1/2)*m*(vx^2+vy^2): simplify(%): KE := factor(%);
```

$$KE := -m\,C^2\,tp^2\,(-1+\cos(\theta(t)))$$

```
PE := m*g*y;
```

$$PE := m\,g\,C\,(1-\cos(\theta(t)))$$

```
solve(KE+PE = 0, tp);
```

$$\frac{\sqrt{-C\,g}}{C}, -\frac{\sqrt{-C\,g}}{C}$$

So I can now write out everything with $\theta(t) = -\sqrt{g/(-C)}\ t$:

$x = C(\theta - \sin\theta)$
$y = C(1-\cos\theta)$.

$v_x = C(1 - \cos\theta)\theta'$
$v_y = C(\sin\theta\ \theta')$

$a_x = C[(1 - \cos\theta)\theta'' + \sin\theta\ \theta'^2]$
$a_y = C(\cos\theta\ \theta'^2 + \sin\theta\ \theta'')$

$F_x = ma_x$
$F_y = ma_y$

$N_x = F_x = ma_x$
$N_y = F_y + g = ma_y + g$

$dy/dx = v_y/v_x = \sin\theta/(1-\cos\theta)$

This is everything you ever wanted to know. But I want something to determine C from my assumed point on the trajectory which is say x_1, y_1 :

$x = C(\theta - \sin\theta)$
$y = C(1-\cos\theta)$.

Want to eliminate the angle. Second says

$y/C = 1-\cos\theta$

or

$\cos\theta = 1-y/C$

or

$\theta = \cos^{-1}(1-y/C)$.

Now for our range $-2\pi < \theta < 0$ you have to be careful about the $\cos^{-1}$ function. But ignore for a moment and continue

$$\sin\theta = -\sqrt{1-\cos^2\theta} = -\sqrt{1-(1-y/C)^2} = -\sqrt{2y/C - (y/C)^2}$$

Then you get

$x = C(\cos^{-1}(1-y/C) - \sqrt{2y/C - (y/C)^2})$

which you would have to solve for C in terms of x,y . Try other

$x^2 = C^2(\theta - \sin\theta)^2$
$y^2 = C^2(1 - \cos\theta)^2$

$(x^2+y^2)/C^2 = (\theta - \sin\theta)^2 + (1 - \cos\theta)^2$

$= \theta^2 + \sin^2\theta - 2\theta\sin\theta + 1 + \cos^2\theta - 2\cos\theta$

$= \theta^2 - 2\theta\sin\theta + 2 - 2\cos\theta$ = not very simple

I think it is only numerically solvable, a "transcendental equation". Maple cannot solve it, for example,

```
restart;
eq1 := x = C*(theta - sin(theta));
```

$$eq1 := x = C\,(\theta - \sin(\theta))$$

```
eq2 := y = C*(1 - cos(theta));
```

$$eq2 := y = C\,(1 - \cos(\theta))$$

```
solve({eq1,eq2},{C,theta});
_SolutionsMayBeLost;
```

$$true$$

Maple can only do it numerically.

Question: what is the duration of a Bracky train trip on a flat earth?

Answer: $\theta(t) = -\sqrt{g/(-C)}\ t$ so set $\theta = -2\pi$ to get

$2\pi = \sqrt{g/(-C)}\ t \quad \Rightarrow \quad t = 2\pi\sqrt{-C/g}$

But the article quoted above says

> cycloid is the fastest possible journey. If you work it out, connecting two places a distance d apart by a cyloid takes a time

$$T = \sqrt{\frac{2\pi d}{g}}$$

Now for the special problem where y = 0, we do know that

$y = C(1-\cos\theta) = 0 \ \Rightarrow\ \cos\theta = 1 \ \Rightarrow \theta = -2\pi$ for our situation

$x = C(\theta - \sin\theta) = C(-2\pi - 0) = 2\pi(-C)$

In this case we know that $C = -(x/2\pi)$ so my result above gives

$$t = 2\pi\sqrt{-C/g} = 2\pi\sqrt{x/2\pi g} = \sqrt{2\pi x/g}$$

and this agrees with the quoted result. In this case we have

$$y = -(x/2\pi)(1-\cos\theta)$$

and this takes its larges neg value at $\theta = -\pi$ so we get

$$y_{min} = -(x/2\pi)(2) = -x/\pi$$ which is about 1/3 of x

H