Lagrange paper bug PhL 9.20.16

This paper has been out for quite a while now. I did not take time to really check out its logic too much, and today I have a question.

In my proof of Theorem 1 for my sample case, I start by "eliminating" the three variables x_1, x_2, x_3. I then reach this point

$$\begin{aligned} f(X^1(x_4, x_5, x_6), X^2(x_4, x_5, x_6), X^3(x_4, x_5, x_6), x_4, x_5, x_6) &\equiv F(x_4, x_5, x_6) \\ a(X^1(x_4, x_5, x_6), X^2(x_4, x_5, x_6), X^3(x_4, x_5, x_6), x_4, x_5, x_6) &\equiv A(x_4, x_5, x_6) \\ b(X^1(x_4, x_5, x_6), X^2(x_4, x_5, x_6), X^3(x_4, x_5, x_6), x_4, x_5, x_6) &\equiv B(x_4, x_5, x_6) \\ c(X^1(x_4, x_5, x_6), X^2(x_4, x_5, x_6), X^3(x_4, x_5, x_6), x_4, x_5, x_6) &\equiv C(x_4, x_5, x_6)\ . \end{aligned} \tag{3.4}$$

where now I can express all four of these equations in terms of just the residual variables. I then go on to look at derivatives and I end up with three 4x4 matrix equations. I then claim that all three of the matrices must have zero determinant otherwise we get a contradiction. In this way, I go part way on my proof that all 4x4 submatrices of the R matrix must have zero determinant. When I repeat this eliminating all other possible sets of three variables, I claim to have shown that all 4x4 submatrices are zero and therefore the R matrix has rank < 4 and this is then associated with df = 0 since df = 0 was used to obtain some of the equations.

But how do you know you can do this elimination? For example, how do you know you can eliminate x_1, x_2, x_3 for some arbitrary set of 3 constraint equations? I suggest this can always be done, but I suspect it cannot always be done.

Let's just look first at some simple cases.

1. Suppose one of the three constraints is $x_1 = 1$. Then you have $X^1(x_4, x_5, x_6) = 1$, so that is no problem.

2. Suppose a constraint is $x_1 + x_2 = 0$. You could eliminate x_1 everywhere it appears elsewhere by setting it equal to $-x_2$. So then I guess you have $x_1 = -X^2(x_4, x_5, x_6)$ and that would be OK.

What do I know *in general* about a set of N weird equations in M unknowns for M > N? Have I ever addressed this pure math question?

Go back to the general case of $a(x_1,x_2....x_N) = 0$. This equation describes an N-1 dimensional surface in E^N, as I note on page 4. How do I know this fact? I know that $a_1dx_1+a_2dx_2 + ... + a_Ndx_N = 0$. Well near (4.1.3) I claim that this surface fact is true, and moreover I claim that ∇a is normal to this surface at any point on the surface. But how do I know it is a surface? If I knew that I could solve the equation for just one of the variables x_s, then I could say $x_s = X^s(x_1....x_n)$ where x_s is missing from the list. This surely is the equation of a surface in E^n. Of course if x_2 did not appear in $a(x_1,x_2....x_N) = 0$, I certainly could not solve the equation for x_2 and make my surface statement based on X^2. But I know that at least some x_s appears in the equation.

Maybe better to look at $da = a_1dx_1+a_2dx_2 + ... + a_Ndx_N = 0$. There must be *some* derivative which is non-zero in this equation. Let x_s be such a term. Then solve to get

$dx_s = [a_1dx_1+a_2dx_2 + ... + a_ndx_n]/a_s$ // a_sdx_s is missing from the sum

This equation then says that if you move very slightly in all those other variables, a dx_s is implied, and if you move with this dx_s then you will stay on the surface which is defined by da = 0. So as long as none of the derivatives is infinite, this means you have a piece of surface.

Question: Consider then a smooth surface $a(x_1,x_2....x_n) = 0$. Can you solve this for x_1 ? There are several cases:

(a) Suppose x_1 does not appear in the equation. That means that the surface is an extruded surface along this axis. It is the extrusion of the surface $a(x_2.....x_n)$ of one lower dimension. You solution for x_1 is then that any value of x_1 is a solution. So you could take $x_1 = 1$ as your solution.

(b) otherwise, I would argue that locally at some point the surface is linear and as such, the surface is a plane in E^n defined by $a_1dx_1+a_2dx_2 + ... + a_Ndx_N = 0$. This is the tangent plane.

OK, now consider these two equations:

$a(x_1,x_2....x_N) = 0$
$b(x_1,x_2....x_N) = 0$

each of which describes a surface in E^n. The intersection of these surfaces is a surface of dimension n-2 in the space E^n. Consider some point r which lies on this dim n-2 surface. Near that point both those surfaces are linear, so this is the intersection of two dim n-1 planes.

Question: If you intersect two dim n-1 planes in E^n , you get an n-2 dim plane. If we select a point on the intersection as our coordinate origin, we have (vectors are derivative vectors, and **r** means d**r**)

$\mathbf{a} \bullet \mathbf{r} = 0$
$\mathbf{b} \bullet \mathbf{r} = 0$

$a_1x_1+a_2x_2 + ... + a_Nx_N = 0.$
$b_1x_1+b_2x_2 + ... + b_Nx_N = 0.$

Can you solve these equations for x_1 and x_2 ? Here are the cases:

(a) x_1 and x_2 appear in neither equation. Then solution is $x_1 = 0$ and $x_2 = 0$, fine.

(b) x_1 is missing from one of the equations. Solve that equation for $x_2 = X^2(x_3, ...x_n)$. Then put this into the other equation for x_2 and solve that equation for x_1. Done.

I have dealt with this little problem somewhere else recently! Where was it? The issue is the solvability of M linear equations in N unknowns and involves Cramer's rule. I search docs for Cramer and unknowns and I get a hit in my non-square matrix algebra stuff. I think this is the idea.

In the above equations, assume that x_3 ....x_N have some assigned values. Then our two equations are

$$a_1x_1+a_2x_2 = -\text{ stuff1}$$
$$b_1x_1+b_2x_2 = -\text{ stuff2}$$

Regardless of the values of stuff1 and stuff2, you can Cramer-solve these equations for x_1 and x_2 as long as

$$\det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \neq 0$$

That is, the rows have to be linearly independent. We are talking here about the intersection of two lines in E^2. The only way there could NOT be a solution for x_1, x_2 is if the lines are parallel which means that the normals are parallel which means that **a** = k **b** and so the rows are linearly dependent.

I am feeling better now that you really can do what I claim in Lagrange doc.