Binary Star Orbits PhL 11.13.16

In the equivalent 1-body problem the solution for $\mathbf{r} = r,\theta$ is this from (3-46),

$$r = (\ell^2/\mu k)\,/(1 - \varepsilon\cos\theta) \qquad \varepsilon = \varepsilon(E,\ell,\mu,k)$$

Here I use a minus sign in the denominator so that the ellipse goes off to the right instead of to the left. See just written document on Ellipse in Polar Coordinates. (math/geometry).

I like to think of the planet as being off to the right!

The standard form for an ellipse going to the right is this, where a is the semi-major axis,

$$r = a(1-\varepsilon^2)\,/(1 - \varepsilon\cos\theta) \qquad \varepsilon = \varepsilon(E,\ell,\mu,k) \qquad (1)$$

Question: What is the value of c and b for this ellipse?

Answer: (1) $c = \varepsilon a$ from the definition of ε.
(2) $b = a\sqrt{1-\varepsilon^2}$ since $b^2 = a^2-c^2 = a^2 - \varepsilon^2a^2 = a^2(1-\varepsilon^2)$

Now, comparing our two ellipse formulas we conclude that,

$$a(1-\varepsilon^2) = (\ell^2/\mu k) \quad \Rightarrow \quad a = \ell^2/[\mu k(1-\varepsilon^2)]$$

We have an expression for ε but I won't use it here. Think of it as the free variable instead of E.

We also know that

$$1/\mu = 1/m_1+1/m_2 = (m_1+m_2)/(m_1m_2) \qquad \mu = (m_1m_2)/(m_1+m_2)$$

$k = m_1m_2G$ G = gravitational constant

Then

$$\mu k = G\,(m_1m_2)^2/(m_1+m_2)$$

We also know that if we set $\mathbf{R} = 0$ to force the center of mass to be at the origin, then

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \qquad \mathbf{R} = (m_1\mathbf{r}_1+m_2\mathbf{r}_2)/(m_1+ m_2) = 0$$

has the solution

$$\mathbf{r}_1 = [m_2/(m_1+ m_2)]\,\mathbf{r} \qquad \text{and} \quad \mathbf{r}_2 = -\,[m_1/(m_1+ m_2)]\,\mathbf{r}$$

Another way to write this uses $m_2/(m_1+ m_2) = (1/m_1)\,(m_1m_2)/(m_1+m_2) = (1/m_1)\,\mu = \mu/m_1$, so

$\mathbf{r_1} = (\mu/m_1)\,\mathbf{r}$ and $\mathbf{r_2} = -(\mu/m_2)\,\mathbf{r}$

In polar coordinates. these leading scalar factors simply scale down the r-vector and leave the θ coordinate untouched, except the minus sign takes θ to θ+π. I restate this just below.

Now suppose we have an ellipse for $\mathbf{r} = (r,\theta)$ as shown above in (1).

And suppose we know that $\mathbf{r_1} = (\mu/m_1)\,\mathbf{r}$. What does this tell us about the orbit of $\mathbf{r_1}$? Every point of the **r** orbit is "scaled toward the origin". Every point has $\theta_1 = \theta$, but $r_1 = (\mu/m_1)r$ so then

$r_1,\theta_1 = (\mu/m_1)r, \theta$

$r_2,\theta_2 = (\mu/m_2)r, \theta+\pi$ // since a minus sign

So now I can write the equations for these two individual ellipses for the 2 bodies,

$r = (\ell^2/\mu k)\,/(1 - \varepsilon\cos\theta)$ // the 1-body problem ellipse

$r_1 = (\mu/m_1)(\ell^2/\mu k)\,/(1 - \varepsilon\cos\theta_1) = (\ell^2/m_1 k)\,/(1 - \varepsilon\cos\theta_1)$ right side

$r_2 = (\mu/m_2)(\ell^2/\mu k)\,/(1+ \varepsilon\cos\theta_2) = (\ell^2/m_2 k)\,/(1 + \varepsilon\cos\theta_2)$ left side

$k = m_1 m_2 G$

Now if you want to use the semimajor axes a, a_1 and a_2 you have to rewrite as

$r = a(1-\varepsilon^2)\,/(1 - \varepsilon\cos\theta)$ // the 1-body problem ellipse

$r_1 = a_1(1-\varepsilon^2)\,/(1+ \varepsilon\cos\theta_1)$ // left side ellipse for mass m_1 object

$r_2 = a_2(1-\varepsilon^2)\,/(1 - \varepsilon\cos\theta_2)$ // right side ellipse for mass m_2 object

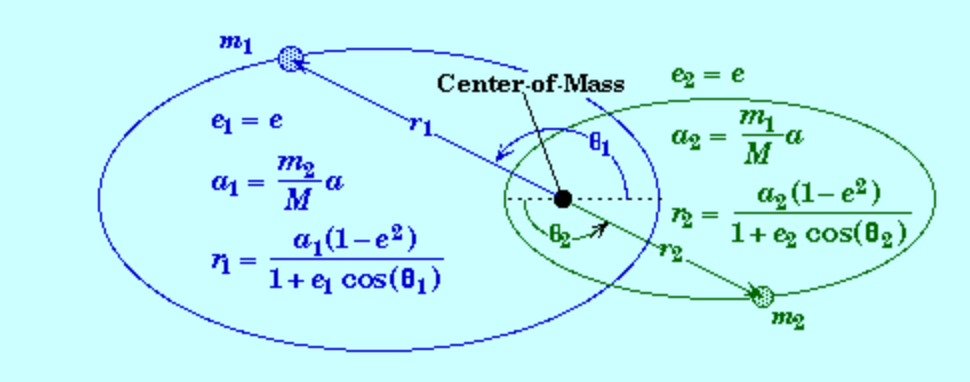
where

$$a = \frac{\ell^2}{\mu k(1-\varepsilon^2)} \qquad a_1 = \frac{\ell^2}{m_1 k(1-\varepsilon^2)} \qquad a_2 = \frac{\ell^2}{m_2 k(1-\varepsilon^2)}$$

$$a_1 = (\mu/m_1)a \qquad a_2 = (\mu/m_2)a \qquad \Rightarrow\ a_1/a_2 = m_2/m_1 \qquad (2)$$

$$a_1+a_2 = a\mu(1/m_1+1/m_2) = a\,\mu\,(1/\mu)\,1\ a \qquad \Rightarrow\quad a = a_1+ a_2 \qquad (3)$$

I am now in good agreement with this clip I put into Goldstein raw notes,

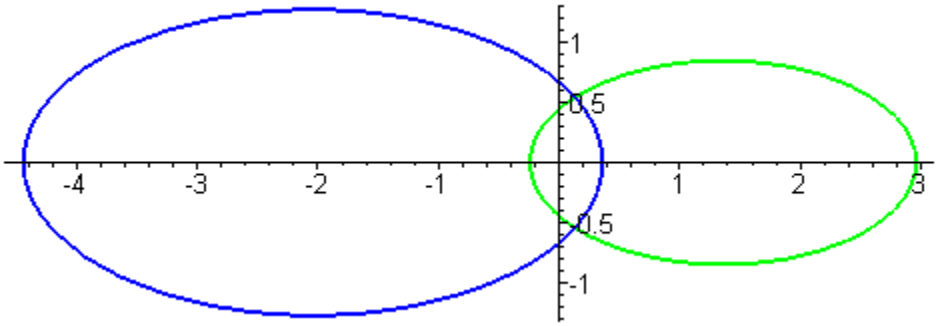


except I have an expression for a and for ε.

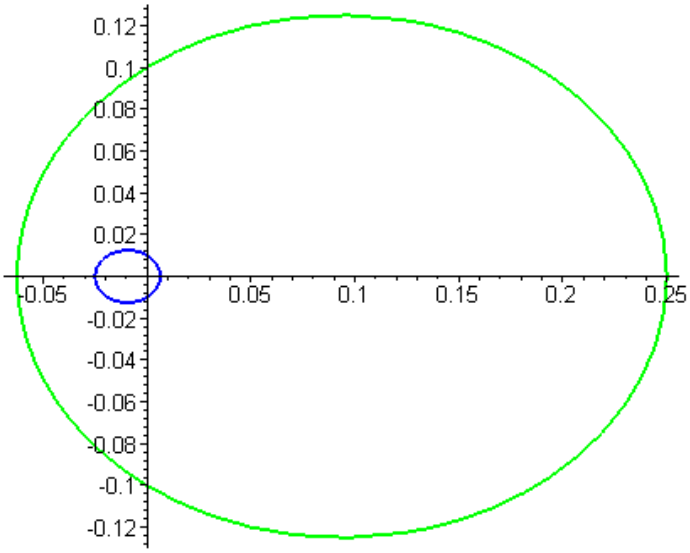
Now suppose m_1 is very large (like the sun). Then $a_1 = (\mu/m_1)a$ is very small, so the left ellipse has a very small major axis and this eventually shrinks down around the origin.

The Maple file "binary star orbit.mws" is how set to replicate the above plot.

e := .85; a := 1;m1 := 1; m2:= 1.5;



e := .6; a := 1;m1 := 10; m2:= 1;



So there is your sun on the left doing a small path. If this is a sun/earth orbit, what you really observe from the earth is $\mathbf{r} = \mathbf{r}_{sun}-\mathbf{r}_{earth}$ and that vector does the 1-body ellipse. You could observe that with a transparent earth by taking a shot every sidereal day.

Has anyone every seen a binary star close enough to verify these orbits?

Question: What can an astronomer learn by observing a binary star?

Answer: If you know how far away d the binary is (by other means), and if you know the period T of the orbits, and if you know where the CMS is after observing, and if it is a "visual binary" so you can measure the various angles, then you can determine the two masses m_1 and m_2, as described below. This is a direct measurement really, and does not depend on things like color or HR diagram location (if you know d).

An "optical double" is a fake, two distant stars on the same line of sight. An "eclipsing binary" lets you get the period from the light periodicity. A "spectrum binary" can be detected by the two stars having different spectra. A "spectroscopic binary" is detected by periodic Doppler shifts of the spectra.

> 2. *Visual Binary orbits and the P-e-correlation.* The visual binaries for which orbital elements have been catalogued now number about 600. Most of the periods are between 20 and 250 years. Longer periods can sometimes be derived on the basis of numerous and accurate data; they are not obtainable for very close pairs as the observational errors then are too large,

Jupiter is 12 years around the sun, near the low end above

First, we need to extract more equations from the above work. Recall from above that

$r_1 = a_1(1-\varepsilon^2) /(1+ \varepsilon \cos\theta_1)$ // left side ellipse for mass m_1 object

$r_2 = a_2(1-\varepsilon^2)/(1 - \varepsilon \cos\theta_2)$ // right side ellipse for mass m_2 object

At some instant in time we have

$r_1/r_2 = a_1/a_2\ (1 - \varepsilon\cos\theta_2)/\ (1 + \varepsilon\cos\theta_1)$

where θ_i are the "natural polar angles" one uses to parameterize each ellipse. They are both measured up from the right-directed x axis. But at any instant, we know that $\theta_2 = \theta_1 \pm \pi$, so the last ratio is one and

$r_1(t)/r_2(t) = a_1/a_2$ at any time t

But from (2) we know more, so we end up with

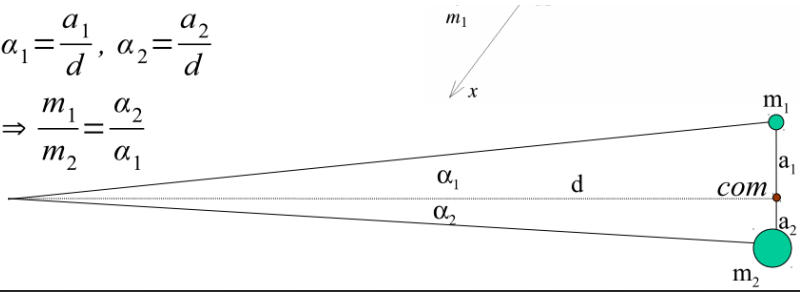
$r_1(t)/r_2(t) = a_1/a_2 = m_2/m_1$

in agreement with my PDF

$$\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1}$$

Suppose you knew the distance to the binary star, and suppose you knew you were looking at it flat as this picture shows, and suppose you had enough observations to know where the CMS is located. Then consider

$$\alpha_1 = \frac{a_1}{d},\ \alpha_2 = \frac{a_2}{d}$$
$$\Rightarrow \frac{m_1}{m_2} = \frac{\alpha_2}{\alpha_1}$$



You know d, you measure angles α_1 and α_2 so then you know a_1 and a_2 and you then know m_1/m_2.

Kepler 3 from Goldstein p 80 says this

$\tau = 2\pi a^{3/2}\sqrt{\mu/k}$

where I already know about from 1-body equivalent that

$\mu = (m_1m_2)/(m_1+m_2)$ $k = m_1m_2G$

so then

$\tau^2 = 4\pi^2 a^3 [\quad [(m_1 m_2)/(m_1+m_2)] / m_1 m_2 G]$

$= 4\pi^2 a^3 / [G(m_1+m_2)]$

and this also agrees with my PDF,

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3$$

Now with the above measurements, you know a_1 and a_2 and by (3) you know $a = a_1+a_2$. You know G. You then have these equations

$a_1/a_2 = m_2/m_1$

$P^2 = 4\pi^2(a_1+a_2)^3 / [G(m_1+m_2)]$

You have measured a_1, a_2 and if it moves fast enough, you measure P, so you can then determine the two masses m_1 and m_2 ! Game over says the guy. So this works with the conditions I state above.

PDF author goes on to show you can adjust for a tilted binary orbit plane relative to the line of site. You can detect the tilt because focus will appear to be in the wrong place. Gives some math.

What do I know about velocities of the two stars? Recall

$r_1(t) = (m_2/m_1) r_2(t)$

so

$v_1(t) = (m_2/m_1) v_2(t)$

at some instant in time. With spectroscopy you can measure the "radial" component of each velocity toward or away from the observer on Earth. So you can measure what the PDF calls v_{1r} and v_2r . You can then factor in the tilt of the plane as discussed earlier in the PDF and then again you get two equations in two unknowns

$$\frac{m_1}{m_2} = \frac{v_{2r}}{v_{1r}}$$

$$m_1 + m_2 = \frac{P}{2\pi G} \frac{(v_{1r} + v_{2r})^3}{\sin^3 i} = \frac{P v_{1r}^3}{2\pi G} \frac{\left(1 + \frac{v_{2r}}{v_{1r}}\right)^3}{\sin^3 i}$$

You know the tilt angle i and you measure v_{1r} and v_{2r}

The PDF author is David Golimowski who was at JHU in 1995 but no longer. His notes are for "physics 371" perhaps at JHU. He is now at the Space Telescope Science Institute STScI. Hubble went up in 1990. He is listed there at STScI today 11.14.16. But where are the rest of his notes?

*https://www2.onu.edu/~j-pinkney/PHYS3471/.../phys371_w3Binary_Stars.**pdf***

I took in a few teaching PDF's on astronomy in general.