Chasle's Theorem PhL 5.31.12

Goldstein on page 143 just says that a general rigid body displacement can be represented as a combination of a rotation and a translation. Here I want to be more specific.

Imagine we take a copy of our original rigid object and put it at some arbitrary position and orientation relative to the original rigid object. Think of these then as two different but identical rigid objects which we want to "line up" by doing certain transformations. What is necessary to line them up?

Forget translation for the moment, and apply a general triple Euler angle rotation to the copy rigid object using some arbitrarily selected origin $\mathbf{x}_0$ which we use to define our Euler rotations. The group theory claim is that if you apply $R_z(\psi)R_y(\theta)R_z(\varphi)$, you can put a rigid object into an arbitrary orientation. That is what the Euler angles are all about, and this is just one choice of parameterization. So let's do that as a first step, then the second step is to translate so that the "two objects" then line up. So here is how you transform any point $\mathbf{x}$ located in the copy object into corresponding point $\mathbf{x}^*$ in the original object :

$$\mathbf{x}^* = R_z(\psi)R_y(\theta)R_z(\varphi)\,(\mathbf{x}-\mathbf{x}_0) + \mathbf{a}_0 \qquad \text{// spherical origin located at } \mathbf{x}_0$$

Now suppose you pick some *other* origin $\mathbf{x}'_0$ for your rotations. Then there will be some different solution which we write as

$$\mathbf{x}^* = R_z(\psi')R_y(\theta')R_z(\varphi')\,(\mathbf{x}-\mathbf{x}'_0) + \mathbf{a}_0' \qquad \text{// spherical origin located at } \mathbf{x}'_0$$

The point $\mathbf{x}$ has to end up at the same point $\mathbf{x}^*$ by either method. So let's condense notation :

$$\mathbf{x}^* = R\,(\mathbf{x}-\mathbf{x}_0) + \mathbf{a}_0 \qquad \text{// spherical origin located at } \mathbf{x}_0$$
$$\mathbf{x}^* = R'\,(\mathbf{x}-\mathbf{x}'_0) + \mathbf{a}'_0 \qquad \text{// spherical origin located at } \mathbf{x}'_0$$

Now set these equal to find that

$$R\,(\mathbf{x}-\mathbf{x}_0) + \mathbf{a}_0 = R'\,(\mathbf{x}-\mathbf{x}'_0) + \mathbf{a}'_0$$

This equation must be true for all points $\mathbf{x}$ in space because we could always just "add" any point $\mathbf{x}$ to our rigid object definition if that point happened not to lie within our rigid object. In particular, the above equation must be true at $\mathbf{x} = \mathbf{x}_0$ and at $\mathbf{x} = \mathbf{x}_0'$. So we then obtain these two equations

$$\mathbf{a}_0 = R'(\mathbf{x}_0-\mathbf{x}'_0) + \mathbf{a}'_0 \qquad // \ \mathbf{x} = \mathbf{x}_0 \qquad (*)$$

$$R(\mathbf{x}'_0-\mathbf{x}_0) + \mathbf{a}_0 = \mathbf{a}'_0 \qquad // \ \mathbf{x} = \mathbf{x}_0'$$

Subtract these equations to get

$$-R(\mathbf{x}'_0-\mathbf{x}_0) = R'(\mathbf{x}_0-\mathbf{x}'_0)$$

or

$$R(\mathbf{x}'_0-\mathbf{x}_0) = R'(\mathbf{x}'_0-\mathbf{x}_0)$$

or

$\mathbf{R}\mathbf{b} = \mathbf{R}'\mathbf{b}$ $\qquad\qquad$ $\mathbf{b} \equiv (\mathbf{x}'_0 - \mathbf{x}_0)$

One obvious solution to this equation is R' = R, meaning the primed Euler angles are the same as the unprimed ones. In this case, equation (*) becomes

$$\mathbf{a}_0 = \mathbf{R}(\mathbf{x}_0 - \mathbf{x}'_0) + \mathbf{a}'_0$$

or

$$\mathbf{a}'_0 = \mathbf{a}_0 - \mathbf{R}(\mathbf{x}_0 - \mathbf{x}'_0)$$

Suppose we start off by specifying some origin $\mathbf{x}_0$ and we then find R and $\mathbf{a}_0$ that brings our two rigid bodies into alignment. We then specify some new origin $\mathbf{x}'_0$ and we then want to find the R' and $\mathbf{a}'_0$ that work for that new origin. A solution is that R' = R, and then $\mathbf{a}'_0 = \mathbf{a}_0 - \mathbf{R}(\mathbf{x}_0 - \mathbf{x}'_0)$, problem solved.

Comment 1: Now go back to the starting situation,

$$\mathbf{x}^* = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_z(\varphi)\,(\mathbf{x} - \mathbf{x}_0) + \mathbf{a}_0 \qquad (**)$$

One is certainly allowed to select $\mathbf{x}_0 = 0$. Then we get some $\mathbf{a}$ such that

$$\mathbf{x}^* = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_z(\varphi)\,\mathbf{x} + \mathbf{a}$$

Comment 2: On page 334, Lai writes (**) this way:

$$\mathbf{x}^* = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_z(\varphi)\,(\mathbf{x} - \mathbf{x}_0) + \mathbf{a}_0 \qquad (**)$$

$$\mathbf{x}^* = \mathbf{Q}\,(\mathbf{x} - \mathbf{x}_0) + \mathbf{c}$$

If the copy rigid object is moving in time, then you could write

$$\mathbf{x}^* = \mathbf{Q}(t)\,(\mathbf{x}(t) - \mathbf{x}_0) + \mathbf{c}(t)$$

You could consider both objects moving in time, and then you would have

$$\mathbf{x}^*(t) = \mathbf{Q}(t)\,(\mathbf{x}(t) - \mathbf{x}_0) + \mathbf{c}(t)$$

At any instant of time t, for a given origin $\mathbf{x}_0$, we know there is some solution Q(t) and $\mathbf{c}(t)$ that solves this equation. On page 337 top Lai wants to assume that the two objects have the same orientation at the special time $t = t_0$ so that Q(t) = 1.

Comment 3. In Lai, instead of thinking of the "copy object" as described above, one thinks of a moving camera platform or frame of reference that is observing the original rigid body. This then is the passive view of the same problem, so what that moving camera sees is in effect the "copy object" but back-rotated and back-translated in the usual active/passive sense. As the camera moves to the right, the copy object moves to the left, for example, relative to the original view the camera had of the object.