Confusion about CT's PhL 8.4.08

Suppose we have the HJE and we have some S that is not an explicit function of time. Then we know that K = H. [these notes relate to Goldstein Chapter 8 on canonical transformations I call CT's]

1. What exactly do we mean by "K = H" ?

We know for example that this happens when we use $S = \Sigma_i q_i P_i$. The CT equations in this case say p = P and q = Q, so this is a confusing case. Consider some other S like that shown page 245 bottom, where $S = \Sigma_i q_i Q_i$. The equations say p = Q and P = -q.

I think "the meaning" of K = H is this:

$K(P,Q) = H(p,q)$ in value, if you evaluate K at P,Q and H at p,q.

I do NOT think these two functions have the same functional form. So we are saying,

$f_1(P(p,q),Q(p,q) = f_2(p,q)$

so f_1 and f_2 are different functions, but are equal when evaluated at corresponding phase space points. So in our last example, we would have

$K(P,Q) = H(p,q) = K(-q, p)$

If we let -q = x and p = y, then we have

$K(-q, p) = K(x,y) = H(y,-x)$

Suppose H(x,y) = x + 5y. Then K(x,y) = H(y,-x) = y - 5x. So clearly H and K are "different functions" of their arguments.

2. Given H(p,q) and p and q and some specific S(q,P), how do you find the P and Q?

The CT equations in this case tell us that $p_i = \partial S/\partial q_i$ and $Q_i = \partial S/\partial P_i$. So this at least tells us what Q_i is, but what are the P_i?

Let's try an example:

$H(p,q) = ap^2 + bq^2$ //harmonic oscillator

$S(q,P) = 5qP$

The CT equations say $p = \partial S/\partial q = 5P$ so we think P = p/5. Other says $Q = \partial S/\partial P = 5q$ so we conclude that

$P = p/5 \qquad Q = 5q$

$K(P,Q) = H(p,q) = H(5P,Q/5) = a(5P)^2 + b(Q/5)^2 = 25aP^2 + bQ^2/25$

Let's check Hamilton's equations. We have

$\dot{p} = -\partial H/\partial q = 2bq$ and $\dot{q} = \partial H/\partial p = 2ap$

$\dot{P} = -\partial K/\partial Q = 2bQ/25$ and $\dot{Q} = \partial K/\partial P = 50aP$

Are these consistent?

$\dot{P} = 2bQ/25 \Rightarrow \dot{p}/5 = 2b(5q)/25 \Rightarrow \dot{p} = 2bq$ yes

$\dot{Q} = 50aP \Rightarrow 5\dot{q} = 50\,a(p/5) \Rightarrow \dot{q} = 2ap$ yes

---

So I was saying $Q_i = \partial S/\partial P_i$. So this at least tells us what Q_i is, but what are the P_i? If we write the other equation which is $p_i = \partial S(q_i,P_i)/\partial q_i = G(q_i,P_i)$, we have a system of n equations i = 1..n which we can hopefully solved for the n variables P_i!

Conclusion: the two CT equations tell you what P and Q will be!

Another example, this time with W, as per page 286 G: assume a separable form such that

$W(q,P) = \Sigma_i W_i(q_i, P_i) = W_1(q_1, P_1) + W_2(q_2, P_2) + \ldots$

Also assume that H(p,q) is cyclic in all coordinates except q_1. From Hamilton's equations, we know that $\dot{p}_i = \partial H/\partial q_i = 0$ constants for i = 2...n. so we know that $p_i = \alpha_i$ constant for i=2..n. Now that we know the p_i are constants, we can use the CT equation $p_i = \partial W/\partial q_i = \partial W_i/\partial q_i = \alpha_i$ and conclude that we have $W_i = \alpha_i q_i$ for i=2..n. We then have $W = W_1(q_1, P_1) + \Sigma_i \alpha_i q_i$.

Now, what can we say about P_i and Q_i corresponding to this W and H? We have already used one of the CT equations to find that $p_i = \alpha_i$ fir i=2..n . The other equation says $Q_i = \partial W/\partial P_i = \partial W_i/\partial P_i$. This tells us then that $Q_1 = \partial W_1/\partial P_1$ and $Q_i = 0$ for i = 2...n. But what are the P_i ? We know $K(P,Q) = K(P,Q_1)$ since all the other Q's are 0. This K is cyclic in all the other Q_i and so we should have $P_i = \gamma_i$ = constants for i=2..n. Can I show that $\gamma_i = \alpha_i$ as suggested by the page 286 footnote??

The answer is no, you cannot show they are equal. But we know we can choose each set arbitrarily. So if we were to *select* $\gamma_i = \alpha_i$, then indeed we would have $P_i = p_i$ for i=2..n. In this case we could write

$W = W_1(q_1, P_1) + \Sigma_{i=2..n} P_i q_i$. and in this case we recognize the sum as the identify CT for those coordinates. So we would expect then that $P_i = p_i$ (which are then both α_i), and $Q_i = q_i$. But the q_i don't even show up in our H, so we can select them 0, and we have $Q_i = q_i = 0$, and indeed we have the identify situation for the n=2..n coordinates.

3. Clarification of the notion of "separable" in HJ theory.

Discussion starts on page 284 and I failed the first time to understand it. We know that the most general form of an F_2 W is W(**q**,**P**). We could of course imagine that there are also constants **α** and these have nothing to do with our previous uses of α_i. Then we might have W(**q**,**P**, **α**). Now let's first restrict to the case where we select W's which don't depend on P, so we have W = W(**q**, **α**) . Now let's second assume that we can write $W = \Sigma_i W_i(q_i, \boldsymbol{\alpha})$. That is, each W_i is a function only of coordinate q_i. This is certainly a notion of "separation of variables" but in a summation sense.

Now let's jump ahead to the 3D Kepler solution which starts on page 299. Since energy is conserved, we have our familiar constant $\alpha_1 = E$ and we shall regard this in fact as α_1 of our set **α**. So we have not said one of our constants is energy and we name that to be α_1. In the Kepler problem, we then write

$$W(r,\theta,\varphi, \boldsymbol{\alpha}) = W_r(r,\boldsymbol{\alpha}) + W_\theta(\theta,\boldsymbol{\alpha}) + W_\varphi(\varphi,\boldsymbol{\alpha})$$

By waving of the arms properly, we are able then to come up with three different equations which have this form

$f_\varphi(\varphi, \partial W_\varphi/\partial\varphi, \boldsymbol{\alpha}) = 0;$	// see 9-63a	which involves only α_φ, one of our **α**
$f_\theta(\theta, \partial W_\theta/\partial\theta, \boldsymbol{\alpha}) = 0;$	// see 9-63b	which involves α_φ and α_θ
$f_r(r, \partial W_r/\partial r, \boldsymbol{\alpha}) = 0;$	// see 9-63c	which involves α_θ and α_1

The various α_i constants *just appear* when you go through the normal separation of variables procedure with a multi-variable differential equation such as H = E for central potential. Traditionally we are doing perhaps QM and we have $H\psi = E\psi$ and we try ψ as the product of three functions, each in its own variable, and we call that "separation of variables". But our HJ equation is different, although we still write it as H = E. The HJE says $H(q_i, p_i = \partial W_i/\partial q_i, \boldsymbol{\alpha}) = E = \alpha_1$. In the case of 3D Kepler, we find that we are able to "separate" the HJE into the three equations shown above. These equations are violently different from each other, but we can combine the above three into one by saying:

$$f_i(q_i, \partial W_i/\partial q_i, \boldsymbol{\alpha}) = 0 \qquad i = r,\theta,\varphi$$

Although we started with a Hamiltonian H, these three equations are not themselves Hamiltonians, they are just functions. Suppose we take our three f_i equations, whatever they are, and define three new functions H_i (having nothing to do with Hamiltonians!)

$$H_i(q_i, \partial W_i/\partial q_i, \boldsymbol{\alpha}) \equiv f_i(q_i, \partial W_i/\partial i, \boldsymbol{\alpha}) + \alpha_i$$

Then of course we end up with these three equations

$H_i(q_i, \partial W_i/\partial q_i, \boldsymbol{\alpha}) \equiv \alpha_i$

and this is what we see on page 284 equation 9-23. Notice these equations are typically not linear, but they are "first order" since only the first derivative of W_i appears, and they are "ordinary" because there is only one variable q_i . Usually we get $(\partial W_i/\partial q_i)^2$ in the equation and we just solve it by taking a square root of all the other stuff.

In the Kepler example, we could interpret the three equations as follows:

H_φ $\quad\quad$ $\hat{L}_z\ (p_\varphi = \partial W_\varphi/\partial\varphi) = \alpha_\varphi = m$

H_θ $\quad\quad$ $\hat{L}^2\ (p_\varphi = m,\ p_\theta = \partial W_\theta/\partial\theta) = \alpha_\theta^2 = L^2$

H_r $\quad\quad$ $H\ (\ p_r = \partial W_r/\partial r,\ L^2) = E$

They are "like" Hamiltonians in that they all represent constants of the motion.

So, when G writes these equations as $H_i(q_i, \partial W_i/\partial q_i, \boldsymbol{\alpha}) \equiv \alpha_i$, he is just emphasizing that each of these equations is going to be associated with some conserved quantity, and he writes that on the right hand side as the corresponding α_i. They are *analogous* to $H = \alpha_1$. We don't have $H = \Sigma_i H_i$.

So, this then is the meaning of separability in HJ theory and this finally explains 9-23 page 284.