Line Integrals as Used in Goldstein PhL 7.29.08

Goldstein writes integrals of the form $\oint pdq$ done using a contour in the q-p plane, which I can think of as $\oint p(q)dq$. I am unfamiliar with this kind of integral, so let's look at it more closely. Change variables

$I = \oint y(x)dx$ // contour is in the x-y plane now

Imagine this is around a closed contour of some arbitrary smooth shape (possibly convex). I want to claim that this integral gives the *area* inside the contour (it is not the arc length!), and all the examples I have played with seem to suggest that fact is true.

Here is a way to prove this using the much more powerful and general Stokes theorem which I write as:

$$\int_S \text{curl } \mathbf{F} \bullet d\mathbf{A} = \oint_C \mathbf{F} \bullet d\boldsymbol{\ell} \qquad (1)$$

Here we use a CCW contour and we have $d\mathbf{A}$ naively in the RHR direction.

Consider the following very simple vector function **F**:

$\mathbf{F}(x,y,z) = \hat{\mathbf{x}}\, y$ so $F_x = y$ and $F_y = F_z = 0$

Then on the RHS of Stokes we have

$$\oint \mathbf{F} \bullet d\boldsymbol{\ell} = \oint y\, dx$$

Now (doing LHS) for our contour take a planar loop lying in the x,y plane, and for our surface S, take the plane region inside this contour. We then have $d\mathbf{A} = dA\, \hat{\mathbf{z}}$ and

$\text{curl } \mathbf{F} \bullet d\mathbf{A} = (\text{curl } \mathbf{F})_z\, dA = (\partial_x F_y - \partial_y F_x)\, dA = -dA$

Then our LHS in (1) above is

$\int_S [-dA] = -$ Area of Loop

Thus, my conclusion is this:

$\oint y\, dx$ = Area of loop // if contour is done Clockwise!

This agrees with any simple example you make, such as a first quadrant square. The loop referred to here is a loop in the x-y plane.

Example: suppose the loop is a circle of radius a. We know then that $x^2 + y^2 = a^2$ so $y(x) = \pm \sqrt{a^2 - x^2}$, where we use the + sign on the upper half of the contour and the – sign on the lower. Or, we could parameterize the loop with an angle θ and say $y = a \sin\theta$ and $x = a\cos\theta$ and this would work over the entire loop, no need to select signs. Then $dx = -a\sin\theta d\theta$ and we have for our CCW loop

$$\oint y\,dx = \int_0^{2\pi} a\sin\theta\ [-a\sin\theta d\theta] = -a^2 \int_0^{2\pi} \sin^2\theta\, d\theta = -4a^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = -4a^2\,(\pi/4) = -\pi a^2$$

and we do in fact obtain the area of a circle.

The subject of line integrals in general is discussed in my yellow Calculus II book and many other places.