

Virtual Displacement in Lagrangian Dynamics

Subhankar Ray*

Dept of Physics, Jadavpur University, Calcutta 700 032, India and
C. N. Yang Institute for Theoretical Physics, Stony Brook, NY 11794

J. Shamanna†

Physics Department, Visva Bharati University, Santiniketan 731235, India

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The confusion and ambiguity encountered by students, in understanding virtual displacement and virtual work, is addressed in this article. A definition of virtual displacement is presented that allows one to express them explicitly for both time independent and time dependent constraints. It is observed that for time independent constraints the virtual displacements are the displacements allowed by the constraints. However this is not so for a general time dependent case. For simple physical systems, it is shown that, the work done on virtual displacements by the constraint forces is zero in both the situations. For allowed displacements however, this is not always true. It is also demonstrated that when constraint forces do zero work on virtual displacement, as defined here, we have a solvable mechanical problem. We identify this special class of constraints, physically realized and solvable, as *the ideal constraints*. The concept of virtual displacement and the principle of zero virtual work by constraint forces are central to both Lagrange's method of undetermined multipliers, and Lagrange's equations in generalized coordinates.

I. INTRODUCTION

Almost all graduate level courses in classical mechanics include a discussion of virtual displacement^{1,2,3,4,5,6,7,8,9,10,11} and Lagrangian dynamics^{1,2,3,4,5,6,7,8,9,10,11,12,13}. From the concept of zero work by virtual displacement the Lagrange's equations of motion are derived. However, the definition of virtual displacement is rarely made precise and often seems vague and ambiguous to students. In this article we attempt a more systematic and precise definition, which not only gives one a qualitative idea of virtual displacement, but also allows one to quantitatively express the same for any given constrained system. We show that in a number of natural systems, e.g., particle moving on a frictionless slope, pendulum with moving point of suspension, the work done by the forces of constraint due to virtual displacement is zero. We also demonstrate that this condition is necessary for the solvability of a mechanical problem. Hence we propose such systems as an important class of natural systems.

A. Ambiguity in virtual displacement

In the following we try to classify the difficulties faced by a student in understanding the definition of virtual displacement.

1. It is claimed that (i) *a virtual displacement $\delta \mathbf{r}$ is consistent with the forces and constraints imposed on the system at a given instant t^1* ; (ii) *a virtual displacement is an arbitrary, instantaneous, infinitesimal change of position of the system compatible with the conditions of constraint²*; (iii) *virtual displacements are, by definition, arbitrary displacements*

of the components of the system, satisfying the constraint³; (iv) *virtual displacement does not violate the constraints⁸*; (v) *we define a virtual displacement as one which does not violate the kinematic relations¹⁰*; (vi) *the virtual displacements obey the constraint on the motion¹¹*. These statements imply that the virtual displacements satisfy the constraint conditions, i.e., the constraint equations. However this is true only for time independent (scleronomous) constraints. We shall show that for time dependent (rheonomous) constraints, such as a pendulum with moving support, this definition would violate the zero virtual work condition.

2. It is also stated that (i) *virtual displacement is to be distinguished from an actual displacement of the system occurring in a time interval dt^1* ; (ii) *it is an arbitrary, instantaneous, change of position of the system²*; (iii) *virtual displacement $\delta \mathbf{r}$ takes place without any passage of time⁸*. (iv) *virtual displacement has no connection with the time - in contrast to a displacement which occurs during actual motion, and which represents a portion of the actual path⁹*; (v) *one of the requirements on acceptable virtual displacement is that the time is held fixed¹¹*. We even notice equation like ' $\delta x_i = dx_i$ for $dt = 0$ '⁸. The above statements are puzzling to a student. If position is a continuous function of time, a change in position during zero time has to be zero. In other words, this definition implies that the virtual displacement cannot possibly be an infinitesimal (or differential) of any continuous function of time. In words of Arthur Haas: *since its (virtual displacement) components are thus not functions of the time, we are not able to regard them as differentials, as we do for the components of the element*

of the actual path⁹. We shall show that virtual displacement can be looked upon as a differential, it is indeed a differential increment in virtual velocity over a time dt , Eq.(6).

3. It is also stated that (i) *virtual displacements do not necessarily conform to the constraints*⁴; (ii) *the virtual displacements δq have nothing to do with actual motion. They are introduced, so to speak, as test quantities, whose function it is to make the system reveal something about its internal connections and about the forces acting on it*²; (iii) *the word “virtual” is used to signify that the displacements are arbitrary, in the sense that they need not correspond to any actual motion executed by the system*³; (iv) *it is not necessary that it (virtual displacement) represents any actual motion of the system*⁶; (v) *it is not intended to say that such a displacement (virtual) occurs during the motion of the particle considered, or even that it could occur*⁹; (vi) *virtual displacement is any arbitrary infinitesimal displacement not necessarily along the constrained path*⁵. From the above we understand that the virtual displacements do not always satisfy the constraint equations, and they need not be the ones actually realized. We shall see that these statements are consistent with physical situations, but they cannot serve as a satisfactory definition of virtual displacement. Statements like: “not necessarily conform to the constraints” or “not necessarily along the constrained path” only tell us what virtual displacement is not, they do not tell us what it really is. Reader should note that there is a conflict between the claims under items 1 and 3. It is not clear from the above, whether the virtual displacements satisfy the constraints, i.e., the constraint equations or not.
4. Virtual displacement is variously described as: *arbitrary, virtual, and imaginary*^{1,2,3,5,6}. These adjectives make the definition somewhat mysterious to a student.

Together with the above ambiguities, students are often confused as to whether it suffices to understand virtual displacement as an abstract concept, or they need to have a quantitative definition. Some students appreciate that the virtual displacement as a vector should not be ambiguous. The principle of zero virtual work is used to derive Lagrange’s equations. For a particle under constraint this means that the virtual displacement is always orthogonal to the force of constraint.

At this stage a student may get further puzzled. Should he take the forces of constraint as supplied, and then the principle of zero virtual work as a definition of virtual displacement? In that case the principle reduces merely to a definition of a new concept, namely virtual displacement. Or should the virtual displacement be defined from the constraint conditions independently

? The principle of zero virtual work may then be used to calculate the forces of constraint that ensure constraint condition throughout the motion.

II. VIRTUAL DISPLACEMENT AND FORCES OF CONSTRAINT

A. Constraints and Virtual displacement

Let us consider a system of constraints that are expressible as equations involving positions and time. They represent some geometric restrictions (holonomic) either independent of time (scleronomic) or explicitly dependent on it (rheonomic). Hence for a system of N particles moving in three dimensions, a system of (s) holonomic, rheonomic constraints are represented by functions of $\mathbf{r}_k$ and (t),

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0, \quad i = 1, 2, \dots, s \quad (1)$$

Each constraint of this form imposes a restriction on the possible or *allowed velocities*, which must satisfy,

$$\sum_{k=1}^N \left(\frac{\partial f_i}{\partial \mathbf{r}_k} \right) \cdot \mathbf{v}_k + \frac{\partial f_i}{\partial t} = 0, \quad i = 1, 2, \dots, s \quad (2)$$

It is worth noting at this stage that there are many, in fact infinitely many, allowed velocities, since we have imposed only (s) number of constraints on ($3N$) scalar components of the allowed velocity vectors. An infinitesimal displacement over time (dt) due to allowed velocities will be called the *allowed infinitesimal displacement* or simply allowed displacement.

$$d\mathbf{r}_k = \mathbf{v}_k dt \quad k = 1, 2, \dots, N \quad (3)$$

Allowed displacements $d\mathbf{r}_k$ together with differential of time (dt) satisfy constraint equations similar to Eq.(2).

$$\sum_{k=1}^N \left(\frac{\partial f_i}{\partial \mathbf{r}_k} \right) \cdot d\mathbf{r}_k + \frac{\partial f_i}{\partial t} dt = 0, \quad i = 1, 2, \dots, s \quad (4)$$

As there are many allowed velocities we have many allowed infinitesimal displacements. We propose to define *virtual displacement* as the difference between any two such (unequal) allowed displacements,

$$\delta \mathbf{r}_k = d\mathbf{r}_k - d\mathbf{r}'_k, \quad k = 1, 2, \dots, N \quad (5)$$

This definition is motivated by the possibility of identifying a special class of ‘*ideal constraints*’ (sec. IIc), and verifying ‘*the principle of zero virtual work*’ in common physical examples (sec. III). It may be noted that, by this definition, virtual displacement $\delta \mathbf{r}_k$ is not a change in position in zero time. It is rather the difference of any two allowed displacements during a time dt .

$$\delta \mathbf{r}_k = (\mathbf{v}_k - \mathbf{v}'_k) dt, \quad k = 1, 2, \dots, N \quad (6)$$

The difference of two allowed velocities $\tilde{\mathbf{v}}_k = \mathbf{v}_k - \mathbf{v}'_k$ may be defined as the virtual velocity.

The virtual displacements thus defined satisfy the homogeneous part of the constraint equation Eq.(4) (i.e., with $\partial f_i / \partial t = 0$).

$$\sum_{k=1}^N \frac{\partial f_i}{\partial \mathbf{r}_k} \cdot \delta \mathbf{r}_k = 0, \quad i = 1, 2, \dots, s \quad (7)$$

The absence of the $(\partial f_i / \partial t)$ in the above equation, Eq.(7), gives the precise meaning to the statement that *virtual displacements are the allowed displacements in the case of frozen constraints*. The constraints are frozen in time in the sense that we make the $(\partial f_i / \partial t)$ term zero, though the $\partial f_i / \partial \mathbf{r}_k$ term still involves time. In the case of stationary constraints, i.e., $f(\mathbf{r}_1, \dots, \mathbf{r}_N) = 0$, the virtual displacements are identical with allowed displacements as $(\partial f_i / \partial t)$ is zero.

B. Existence of forces of constraints

In the case of an unconstrained system of N particles described by position vectors $(\mathbf{r}_k)$ and velocity vectors $(\mathbf{v}_k)$, the motion is governed by Newton's Law,

$$m_k \mathbf{a}_k = \mathbf{F}_k(\mathbf{r}_l, \mathbf{v}_l, t), \quad k, l = 1, 2, \dots, N \quad (8)$$

where m_k is the mass of the k th particle, a_k is its acceleration and F_k is the total external force acting on it. However, for a constrained system, the equations of constraint, namely Eq.(1), impose the following restrictions on the allowed accelerations,

$$\sum_{k=1}^N \frac{\partial f_i}{\partial \mathbf{r}_k} \cdot \mathbf{a}_k + \sum_{k=1}^N \frac{d}{dt} \left(\frac{\partial f_i}{\partial \mathbf{r}_k} \right) \mathbf{v}_k + \frac{d}{dt} \left(\frac{\partial f_i}{\partial t} \right) = 0, \quad i = 1, 2, \dots, s \quad (9)$$

Given $\mathbf{r}_k$, $\mathbf{v}_k$ one is no longer free to choose all the accelerations $\mathbf{a}_k$ independently. Therefore in general the accelerations $\mathbf{a}_k$ allowed by Eq.(9) are incompatible with Newton's Law,

$$m_k \mathbf{a}_k = \mathbf{F}_k, \quad k = 1, 2, \dots, N$$

This implies that during the motion the constraint condition cannot be maintained by the external forces alone. Physically some additional forces, e.g., normal reaction from the surface of constraint, tension in the pendulum string, come into play to ensure that the constraints are satisfied. Hence one is compelled to introduce forces of constraints $\mathbf{R}_k$ and modify the equations of motion as,

$$m_k \mathbf{a}_k = \mathbf{F}_k + \mathbf{R}_k, \quad k = 1, 2, \dots, N \quad (10)$$

Now the problem is to determine the motion of N particles, namely their positions $(\mathbf{r}_k(t))$, velocities $(\mathbf{v}_k(t))$ and the forces of constraints $(\mathbf{R}_k)$, for a given set of external

forces $\mathbf{F}_k$, constraint equations $(f_i(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0, i = 1, 2, \dots, s)$ and initial conditions $(\mathbf{r}_k(0), \mathbf{v}_k(0))$. It is important that the initial conditions are also compatible with the constraints.

There are a total of $(6N)$ scalar unknowns, namely the components of $\mathbf{r}_k(t)$ and $\mathbf{R}_k$, connected by $(3N)$ equations of motion, Eq.(10), and (s) equations of constraints, Eq.(1). For $(6N > 3N + s)$ we have an under-determined system. Hence to solve this problem we need $(3N - s)$ additional scalar relations.

C. Solvability and ideal constraints

In simple problems with stationary constraints, e.g., motion on a smooth stationary surface, we observe that the allowed displacements are tangential to the surface. The virtual displacement being a difference of two such allowed displacements, is also a vector tangential to it. The force of constraint, so called 'normal reaction', is perpendicular to the surface. Hence the work done by the constraint forces on allowed as well as virtual displacement is zero,

$$\sum_{k=1}^N \mathbf{R}_k \cdot d\mathbf{r}_k = 0, \quad \sum_{k=1}^N \mathbf{R}_k \cdot \delta \mathbf{r}_k = 0$$

When the constraint surface is in motion, the allowed velocities, and hence the allowed displacements are no longer tangent to the surface (see sec. III). The virtual displacement remains tangent to the constraint surface. If the forces of constraint can still be assumed normal to the instantaneous position of the surface, we have zero virtual work. However note that the work by constraint forces on allowed displacements is not zero.

$$\sum_{k=1}^N \mathbf{R}_k \cdot d\mathbf{r}_k \neq 0, \quad \sum_{k=1}^N \mathbf{R}_k \cdot \delta \mathbf{r}_k = 0 \quad (11)$$

In a number of physically interesting simple problems, such as, motion of a pendulum with fixed and moving support, motion of a particle along a stationary and moving slope, we observe that the above interesting relation between the force of constraint and virtual displacement holds (see sec. III). Out of the above $3N$ virtual displacements, only $n = 3N - s$ are independent. If the (s) dependent quantities are expressed in terms of remaining $n = 3N - s$ independent objects we get

$$\sum_{j=1}^n \tilde{R}_j \cdot \delta \tilde{x}_j = 0 \quad (12)$$

where $\tilde{x}_j$ are the independent components of $\mathbf{r}_k$. $\tilde{R}_j$ are the coefficients of $\delta \tilde{x}_j$, and are composed of different $\mathbf{R}_k$. Since the above components of virtual displacements $\delta \tilde{x}_j$ are independent, one can equate each of their coefficients to zero ($\tilde{R}_j = 0$). This brings in $(3N - s)$ new scalar

conditions or equations and the system is solvable (not under-determined) again.

Thus we find a special class of constraints which is observed in nature (see sec. III) and which gives us a solvable system. We call this special class of constraints, satisfying zero virtual work principle by constraint forces, i.e., $\sum_k \mathbf{R}_k \cdot \delta \mathbf{r}_k = 0$, the *ideal constraint*.

Our interpretation of the principle of zero virtual work, as a definition of an ideal class of constraints, find support in Sommerfeld. In his words, “a general postulate of mechanics: in any mechanical systems the virtual work of the reactions equals zero. Far be it from us to want to give a general proof of this postulate, rather we regard it practically as definition of a mechanical system”.

III. EXAMPLES OF VIRTUAL DISPLACEMENTS

A. Simple Pendulum with stationary support

The motion of the pendulum is confined to a plane and the bob moves at a fixed distance from the point of suspension. The equation of constraint by Eq.(1) therefore is,

$$f(x, y, t) \doteq x^2 + y^2 - r_0^2 = 0$$

Whence

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial t} = 0$$

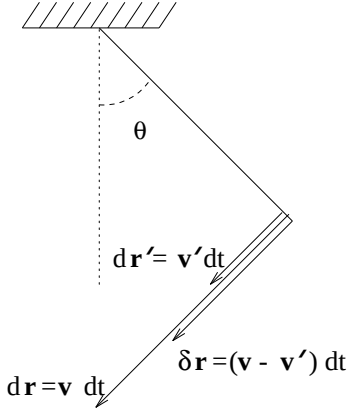


FIG. 1: Allowed and virtual displacements for a pendulum with stationary support

Hence the constraint equation for allowed velocities (compare Eq.(2)) is,

$$x \cdot v_x + y \cdot v_y = 0$$

Hence the allowed velocity (v_x, v_y) is orthogonal to the instantaneous position (x, y) of the bob relative to stationary support. The same may also be verified taking a plane polar coordinate.

The allowed displacements are always collinear to allowed velocities. Virtual displacement being difference of two allowed displacements, is also a vector collinear to the allowed velocities, hence tangential to the line of suspension.

$$d\mathbf{r} = \mathbf{v} dt, \quad d\mathbf{r}' = \mathbf{v}' dt$$

$$\delta \mathbf{r} = (\mathbf{v} - \mathbf{v}') dt$$

We may assume that the string of the pendulum provides a tension ($\mathbf{T}$) but no shear (ideal string). We get zero work by tension due to allowed and virtual displacements,

$$\mathbf{T} \cdot d\mathbf{r} = 0, \quad \mathbf{T} \cdot \delta \mathbf{r} = 0$$

B. Simple Pendulum with moving support

Let us first consider the case when the support is moving vertically with a velocity u . The motion of the pendulum is still confined to a plane. The bob moves keeping a fixed distance from point of suspension. The equation of constraint is,

$$f(x, y, t) \doteq x^2 + (y - ut)^2 - r_0^2 = 0$$

where u is the velocity of the point of suspension along a vertical direction.

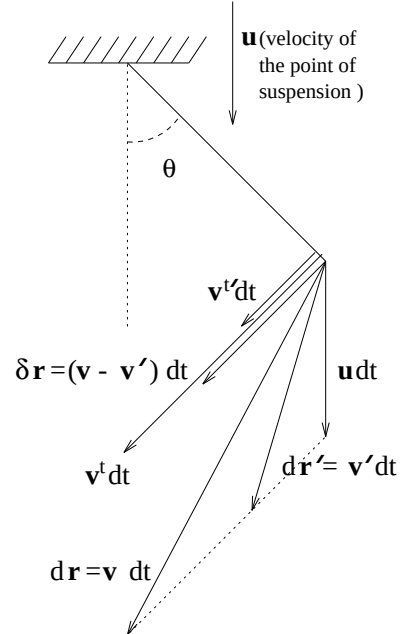


FIG. 2: Allowed and virtual displacements for a pendulum with moving support

Whence

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2(y - ut), \quad \frac{\partial f}{\partial t} = -2u(y - ut)$$

Hence the constraint equation gives,

$$x \cdot v_x + (y - ut) \cdot v_y - u(y - ut) = 0$$

or,

$$x \cdot v_x + (y - ut) \cdot (v_y - u) = 0$$

Hence the allowed velocities (v_x, v_y) and hence the allowed displacements, are not orthogonal to the instantaneous position $(x, y - ut)$ of the bob relative to the instantaneous position of the support. It is easy to verify from the above equation that the allowed velocity (v_x, v_y) is equal to the sum of a velocity vector $(v_x, v_y - u)$ perpendicular to the relative position of the bob with respect to the point of suspension $(x, y - ut)$, and the velocity of the support $(0, u)$.

$$\mathbf{v} = \mathbf{v}_t + \mathbf{u}$$

The allowed displacements are vectors collinear to allowed velocities. A virtual displacement being the difference of two allowed displacements, is a vector collinear to the difference of allowed velocities. Hence it is tangential to the instantaneous line of suspension.

$$d\mathbf{r} = \mathbf{v}dt = \mathbf{v}_t dt + \mathbf{u}dt$$

$$\delta\mathbf{r} = (\mathbf{v} - \mathbf{v}')dt = (\mathbf{v}_t - \mathbf{v}'_t)dt$$

At any given instant string provides a tension along its length, with no shear (ideal string). Hence the constraint force, tension, still does zero work on virtual displacement.

$$\mathbf{T} \cdot d\mathbf{r} \neq 0, \quad \mathbf{T} \cdot \delta\mathbf{r} = 0$$

If one considers the support moving in a horizontal (or in any arbitrary direction), one can show that the allowed displacement is not normal to the instantaneous line of suspension. But the virtual displacement as defined in this article always remains perpendicular to the instantaneous line of support.

C. Motion along a fixed inclined plane

The constraint is more conveniently expressed in the polar coordinate. The constraint equation is,

$$f(r, \theta) \doteq \theta - \theta_0 = 0$$

where θ_0 is a constant. Hence the constraint equation for allowed velocities, Eq.(2), gives,

$$\sum_{k=1}^N \left(\frac{\partial f}{\partial \mathbf{r}_k} \right) \cdot \mathbf{v}_k + \frac{\partial f}{\partial t} \doteq \dot{\theta} + 0 = 0$$

Thus the allowed velocities are along the constant θ plane. Allowed velocity, allowed and virtual displacements are,

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}}, \quad d\mathbf{r} = \dot{r}\hat{\mathbf{r}}dt, \quad \delta\mathbf{r} = (\dot{r} - \dot{r}')\hat{\mathbf{r}}dt$$

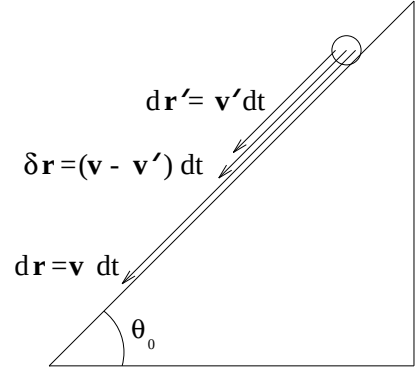


FIG. 3: Allowed and virtual displacements for a particle on a stationary slope

If the inclined slope is frictionless (ideal), the constraint force provided by the surface is the normal reaction; which is perpendicular to the plane. Hence the work done by this force on allowed as well as virtual displacement is zero.

$$\mathbf{N} \cdot d\mathbf{r} = 0, \quad \mathbf{N} \cdot \delta\mathbf{r} = 0$$

D. Motion along a moving inclined plane

For an inclined plane moving along the horizontal side, the constraint is given by,

$$\frac{(x + ut)}{y} - \cot(\theta_0) = 0$$

$$f(x, y) \doteq (x + ut) - \cot(\theta_0)y = 0$$

whence the constraint for allowed velocities Eq.(2) become,

$$(\dot{x} + u) - \cot(\theta_0)\dot{y} = 0$$

Hence the allowed velocity $(\dot{x}, \dot{y})$ is the sum of two vectors, one along the plane $(\dot{x} + u, \dot{y})$, and the other equal to the velocity of the plane itself $(-u, 0)$.

$$\mathbf{v} = \mathbf{v}_t + \mathbf{u}$$

Allowed displacements are vectors along the allowed velocities, however the virtual displacement is still a vector along the instantaneous position of the plane.

$$d\mathbf{r} = (\mathbf{v}_t + \mathbf{u})dt, \quad d\mathbf{r}' = (\mathbf{v}'_t + \mathbf{u})dt$$

$$\delta\mathbf{r} = (\mathbf{v} - \mathbf{v}')dt = (\mathbf{v}_t - \mathbf{v}'_t)dt$$

For the moving frictionless (ideal) slope, the constraint force provided by the surface is perpendicular to the plane. Hence the work done by the constraint force on virtual displacement is remains zero.

$$\mathbf{N} \cdot d\mathbf{r} \neq 0, \quad \mathbf{N} \cdot \delta\mathbf{r} = 0$$

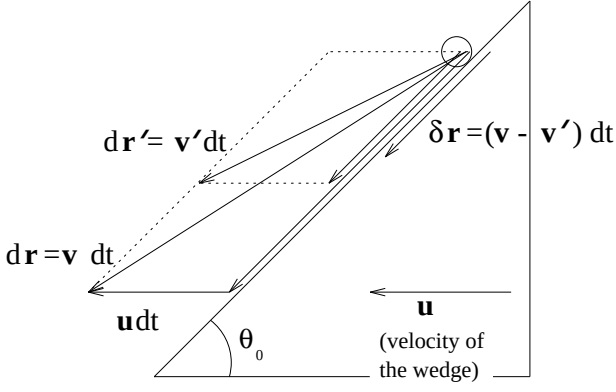


FIG. 4: Allowed and virtual displacements for a particle on a moving slope

IV. LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

A constrained system of particles follow the equation of motion given by,

$$m_k \mathbf{a}_k = \mathbf{F}_k + \mathbf{R}_k, \quad k = 1, 2, \dots, N$$

where m_k is the mass of the k th particle, $\mathbf{a}_k$ is its acceleration. $\mathbf{F}_k$ and $\mathbf{R}_k$ are the total external force and force of constraint on the particle. If the constraints are *ideal*, we can write

$$\sum_{k=1}^N \mathbf{R}_k \cdot \delta \mathbf{r}_k = 0 \quad (13)$$

whence we obtain,

$$\sum_{k=1}^N (m_k \mathbf{a}_k - \mathbf{F}_k) \cdot \delta \mathbf{r}_k = 0 \quad (14)$$

If the components of $\delta \mathbf{r}_k$ were independent, we could recover Newton's Law for unconstrained system from this equation. However for a constrained system $\delta \mathbf{r}_k$ are dependent through the constraint equations,

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0, \quad i = 1, 2, \dots, s \quad (15)$$

or,

$$\delta f_i = \sum_{k=1}^N \frac{\partial f_i}{\partial \mathbf{r}_k} \delta \mathbf{r}_k = 0, \quad i = 1, 2, \dots, s \quad (16)$$

We multiply the above equations, Eq.(16), successively by s scalar multipliers $(\lambda_1, \lambda_2, \dots, \lambda_s)$, called the Lagrange's multipliers, and subtract them from the zero virtual work equation, Eq.(13).

$$\sum_{k=1}^N \left(\mathbf{R}_k - \sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k} \right) \delta \mathbf{r}_k = 0 \quad (17)$$

Explicitly in terms of components,

$$\sum_{k=1}^N \left(\left[R_{k,x} - \sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial x_k} \right] \delta x_k + [Y]_k \delta y_k + [Z]_k \delta z_k \right) = 0 \quad (18)$$

where $[Y]_k$ and $[Z]_k$ denote the coefficients of δy_k and δz_k respectively.

The constraint equations Eq.(15) allows us to write the (s) dependent virtual displacements in terms of the remaining $n = 3N - s$ independent ones. We choose (s) multipliers $(\lambda_1, \lambda_2, \dots, \lambda_s)$ such that the coefficients of (s) dependent components of virtual displacement vanish. The remaining virtual displacements being independent, their coefficients must vanish as well. Thus it is possible to choose $(\lambda_1, \lambda_2, \dots, \lambda_s)$ such that all coefficients $([X]_k, [Y]_k, [Z]_k)$ of virtual displacements $(\delta x_k, \delta y_k, \delta z_k)$ in Eq.(18) vanish. Hence we have the forces of constraint in terms of the Lagrange's multipliers.

$$\mathbf{R}_k = \sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k}, \quad k = 1, 2, \dots, N \quad (19)$$

Thus the problem of mechanics reduces to finding solution of equations of motion,

$$m_k \mathbf{a}_k = \mathbf{F}_k + \sum_{i=1}^s \lambda_i \frac{\partial f_i}{\partial \mathbf{r}_k}, \quad k = 1, 2, \dots, N \quad (20)$$

with the constraints,

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0, \quad i = 1, 2, \dots, s \quad (21)$$

Thus we have to solve $3N + s$ scalar equations in $3N + s$ unknown scalar quantities $(x_k, y_k, z_k, \lambda_i)$. After solving this system we can obtain the forces of constraint $\mathbf{R}_k$ from Eq.(19).

V. GENERALIZED COORDINATES AND LAGRANGE'S EQUATIONS OF MOTION

For the sake of completeness we discuss very briefly Lagrange's equations in generalized coordinates (for detail see^{1,2,3,4,5,6,7,8,9,10,11,12,13}). Consider a system of N particles under s holonomic, rheonomous constraints of the form given by Eq.(1). We can in principle express s of these coordinates in terms of the remaining $3N - s$ independent ones. Or we may express all the $3N$ scalar components of position in terms of $n = 3N - s$ independent parameters $q_1, q_2, \dots, q_n$ and time (t) .

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \dots, q_n, t), \quad k = 1, 2, \dots, N \quad (22)$$

The allowed and virtual displacements are given by,

$$\begin{aligned} d\mathbf{r}_k &= \sum_{j=1}^n \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j + \frac{\partial \mathbf{r}_k}{\partial t} dt, \\ \delta \mathbf{r}_k &= \sum_{j=1}^n \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j, \quad k = 1, 2, \dots, N \end{aligned} \quad (23)$$

From the Eq.(14) we obtain,

$$\sum_{k=1}^N m_k \frac{d\dot{\mathbf{r}}_k}{dt} \left(\sum_{j=1}^n \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j \right) - \sum_{k=1}^N \mathbf{F}_k \left(\sum_{j=1}^n \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j \right) = 0 \quad (24)$$

Introducing the expression of kinetic energy,

$$T = \frac{1}{2} \sum_{k=1}^N m_k \dot{\mathbf{r}}_k^2$$

and that of the generalized force,

$$Q_j = \sum_{k=1}^N \mathbf{F}_k \frac{\partial \mathbf{r}_k}{\partial q_j} \quad j = 1, 2, \dots, n \quad (25)$$

After some simple algebra one finds,

$$\sum_{j=1}^n \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0 \quad (26)$$

since the q_j are independent coordinates, coefficient of each δq_j must be zero separately.

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \quad j = 1, 2, \dots, n \quad (27)$$

In problems where forces $\mathbf{F}_k$ are derivable from a scalar potential $\tilde{V}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$,

$$\mathbf{F}_k = -\nabla_k \tilde{V}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N), \quad k = 1, 2, \dots, N \quad (28)$$

we can write the generalized force as,

$$Q_j = -\nabla_k \tilde{V} \cdot \left(\frac{\partial \mathbf{r}_k}{\partial q_j} \right) = -\frac{\partial V}{\partial q_j}, \quad j = 1, 2, \dots, n \quad (29)$$

Where V is the potential $\tilde{V}$ expressed as a function of $(q_1, q_2, \dots, q_n)$. In addition if the potential V does not depend on the generalized velocities, we obtain from Eq.(27),

$$\frac{d}{dt} \frac{\partial (T - V)}{\partial \dot{q}_j} - \frac{\partial (T - V)}{\partial q_j} = 0, \quad j = 1, 2, \dots, n \quad (30)$$

At this stage one introduces the Lagrangian function $L = T - V$ and in terms of the Lagrangian, the equations of motion Eq.(30) take up the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, 2, \dots, n \quad (31)$$

VI. CONCLUSION

In this article we make an attempt to present a quantitative definition of the virtual displacement. We show that for certain simple cases the virtual displacement does zero work on forces of constraint. We also demonstrate that this zero work principle allows us to have a solvable class of problems. Hence we define this special class of constraint, *the ideal constraint*. We demonstrate in brief how one can solve a general mechanical problem by: i) Lagrange's method of undetermined multiplier and ii) Lagrange's equations in generalized coordinates.

In Lagrange's method of undetermined multipliers we have to solve a larger number $(3N + s)$ of equations, than in the case of Lagrange's equations $(3N - s)$ in generalized coordinates. However we can immediately derive the forces of (ideal) constraints in the former case.

It is interesting to note that both the abovementioned methods require the zero virtual work by constraint forces as a crucial starting point. In the case of Lagrange's method of undetermined multipliers we start with the ideal constraint condition Eq.(13). From there we write down Eq.(14), Eq.(17), Eq.(18) and express the constraint forces in terms of Lagrange's multipliers, Eq.(19). For Lagrange's equations in generalized coordinates we start with the ideal constraint, Eq.(13). We work our way through Eq.(14), Eq.(24), Eq.(26) and finally obtain Lagrange's equations in generalized coordinates, Eq.(27) and Eq.(31).

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* Electronic address: subho@juphys.ernet.in

† Electronic address: jshamanna@rediffmail.com

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