Surface Geometry Questions PhL 7.31.08

I looked in my interests/math and elsewhere for doc files on these subjects, but could find nothing. When I looked in my math books, Buck at least talks about vectors, gradients, divergence, and the Gauss and Stokes theorems, but no pictures of surfaces. Other books same thing. I found one undated pen-written page set about gradients and normals in my math notes which is now incorporated below.

My motivation for doing this was to understand the "classical waves" for moving particles as described at the start of Goldstein Chapter 9 Section 8. [On 10.19.10 I updated this doc by interpreting the parameter d in the equation of a plane, and by adding a TOC and standardizing the Overview.]

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Overview (5 pages)

0. What is the equation of a plane, and what is the normal to that plane?

If the plane is ax + by + cz + d = 0, the normal to the plane is **n** = (a,b,c). This result is shown several different ways. Also, d = np, where p is the distance from the plane to the origin.

0.1 Geometric meaning of the 3D gradient $\nabla_{3d}(g(\mathbf{r}))$. If g(x,y,z)= 0 defines a surface, and if **r** is a point on that surface, then :

(1) The vector $\nabla_{3d}(g(r))$ points in the direction of *a* normal to the surface, so that $\hat{\mathbf{n}} = \nabla_{3d}(g) / |\nabla_{3d}(g)|$.
(2) The function g has its maximal amount of positive change in this normal direction.

1. Why does a 2D gradient$\nabla_{2D}f$ to a surface z = f(x,y) point uphill, and what is the uphill vector?

Reason is explained, and $\mathbf{v_{up}} = (f_x, f_y, f_x^2 + f_y^2)$.

1.1 More comments on the 2D and 3D gradients just mentioned above.

Consider z = f(x,y) and let g(**r**) = f(x,y) - z. A point on the surface z = f(x,y) is a point on the surface g = 0. The $\nabla_{3d}(g)$ vector is given by

$$\nabla_{3d}(g) = (g_x, g_y, g_z) = (f_x, f_y, -1) = (\nabla_{2D} f, -1).$$

2. Imagine a surface z = f(x,y) in R^3. What is the normal n at any point on the surface?

We already know the answer as just given above, but I did this section first, and got the same result from 3 equations in 3 unknowns. There are always two normals ± . I show that $[\mathbf{n} \times \nabla f_{2D}]_z = 0$ and explain why that is so. Finally, I answer this question: what is the equation of the **tangent plane** which exists at some point $\mathbf{r_1}$ on the surface f(x,y) = z. The answer is this: $z = f_{x1}(x - x_1) + f_{y1}(y - y_1) + z_1$. As noted above, the normal at this point is $\mathbf{n} = \pm (f_x, f_y, -1)$ and we find $\mathbf{n} \bullet (\mathbf{r} - \mathbf{r_1}) = 0$ for **r** on the tangent plane.

3. What then is a 3D vector which points "across the hill"?

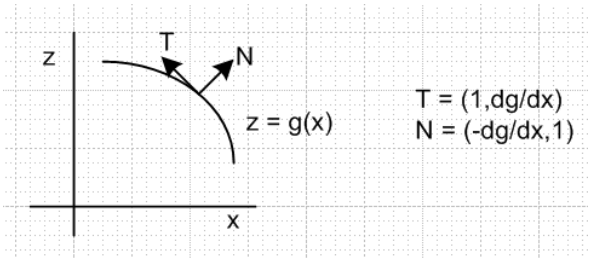
You can construct the following little orthogonal set of vectors:

$\mathbf{n} = (-f_x, -f_y, 1)$ // not normalized
$\mathbf{v_{up}} = (f_x, f_y, f_x^2 + f_y^2)$ // not normalized
$\mathbf{v_{across}} \equiv \mathbf{n} \times \mathbf{v_{up}} = (-f_y, f_x, 0)$ // not normalized

where the latter is "horizontal" in our usual construction, lying parallel to the x-y plane at some $z = z_1$ value. Easy to show that the three dot products of the above vectors are 0 which makes me think they are right.

4. Reduce all the above to the corresponding 2D problem.

Nothing very exciting, I just set y=0 and reduce the 3D stuff to 2D stuff by restricting to the y=0 plane. This picture says it all:



5. Consider the 2D line y = mx + b passing through a point x_1, y_1. What is the equation of the line through this same point that is perpendicular to the original line?

There are several ways to give the answer, here is my favorite:

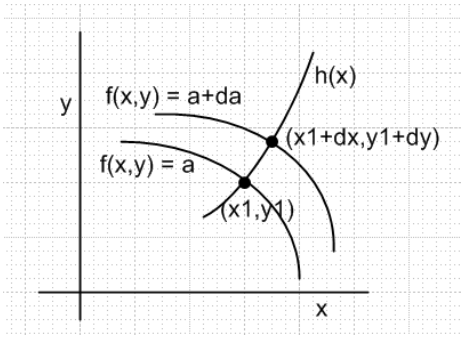
line through point $\mathbf{r}_1$: $y = m(x-x_1) + y_1$ $\quad b = y_1 - mx_1$

perp line thru point $\mathbf{r}_1$: $y = (-1/m)(x-x_1) + y_1$ $\quad b_\perp = y_1 + x_1/m$

You can see that point $\mathbf{r}_1$ in fact lies on both these lines, and we have the simple $m_\perp = -1/m$ rule.

6. Consider a the curve described by f(x,y) = a, it might be a circle or an ellipse for example.

Here I am trying to learn about the "normal curve" I call h(x) and I drew this projection picture:



Then I started exploring a set of mini questions:

(6a) What is the tangent line at $\mathbf{r}_1$? The answer is $y = m(x-x_1) + y_1$ with $m = -f_{x1}/f_{y1}$, easy to show.

(6b) What is the equation of the perp line at point 1, which is of course has slope there of our h(x)? Again from above, the answer is trivially $y = m_\perp (x-x_1) + y_1$ with $m_\perp = -1/m = f_{y1}/f_{x1}$

(6c) If we are given dx, how can we estimate dy (see picture)? Answer: $dy = [\, f_{y1}/f_{x1}]\, dx$. This is just coming from the slope of the normal line. At this point, I did not realize that the equation $dy/dx = f_y/f_x$ could be *solved* to find y(x) = h(x), the explicit normal curve, so I was just learning about h' and h".

(6d) What can we say about the $h'(x_1)$? Answer: $h'(x_1) = f_{y1}/f_{x1}$

(6e) By just repeating the above discussion, we know that $h'(x_2) = f_{y2}/f_{x2}$ where $f = f(x_2, y_2)$.

(6f) What is $h''(x_1)$? Answer: $h''(x_1) = \{ \; (f_{yx}/f_x)\,[1 - (f_y/f_x)^2 \;] + (f_y/f_x^2)\,[\, f_{yy} - f_{xx}] \;\}$

I then verified the above formulas for h' and h" using $h(x) = K\, x^{b/a}$ for $f = ax^2+by^2$. When I first wrote this section, I did not know this solution for h(x) so I did not verify anything except a circle situation.

7. Consider a surface z = f(x,y) in $\mathbf{R}^3$.

This is a long section. We know we can "go uphill" in 3D according to

$$d\mathbf{r}_{3D} = ds\ \hat{\mathbf{v}}_{up,3D} = ds\ (f_x, f_y, f_x^2 + f_y^2)/R \quad \text{where} \quad R^2 = (f_x^2 + f_y^2)\ (1 + f_x^2 + f_y^2)$$

Since $d\mathbf{r} = (dx,dy,dz)$ we can relate each differential to ds:

$$dx = (f_x/R)\ ds \qquad dy = (f_y/R)\ ds \qquad dz = [(f_x^2 + f_y^2)/R]\ ds$$

In the 2D projection world, we can "go up" in 2D according to

$$d\mathbf{r}_{2D} = ds\ \hat{\mathbf{v}}_{up,2D} = ds\ \nabla_{2D}f\ /R = ds\ (f_x, f_y)/R \qquad R = \sqrt{f_x^2 + f_y^2} = |\nabla_{2D}f|$$

and this leads to

$$dx = (f_x/R)\ dx \qquad dy = (f_y/R)\ ds$$

Notice that $dy/dx = f_y/f_x$ in either of these "worlds". I show that $dy/dx = f_y(x,y)/f_x(x,y)$ is the equation you must solve if you want to know $y(x) = h(x)$ = the 2D normal curve which is a projection on the x-y plane of the 3D uphill curve. You can compute arc length along this h(x) curve like so:

$$s(x) = \int_0^x \sqrt{1 + (f_y/f_x)^2}\ dx \qquad \text{// where need to replace } y = y(x) \text{ in the integrand.}$$

I then show two similar claims relating to the magnitude of gradients:

(2D) $d\mathbf{r} = \hat{\mathbf{n}}\ ds$ is along h(x) and therefore normal to the curve $z = f(\mathbf{r})$

$$df = \nabla_{2D}f \bullet d\mathbf{r} = |\nabla_{2D}f|\ ds \qquad \text{new curve is } z = f(\mathbf{r}+d\mathbf{r}) = f(\mathbf{r}) + df$$

$$\nabla_{2D}f = (f_x, f_y) \qquad |\nabla_{2D}f| = \sqrt{f_x^2 + f_y^2}$$

(3D) $d\mathbf{r} = \hat{\mathbf{n}}\ ds$ is normal to the surface $g(x,y,z) = 0$. We find that

$$dg = \nabla_{3D}g \bullet d\mathbf{r} = |\nabla_{3D}g|\ ds \qquad \text{new surface is } g(\mathbf{r}+d\mathbf{r}) = g(\mathbf{r}) + dg = dg$$

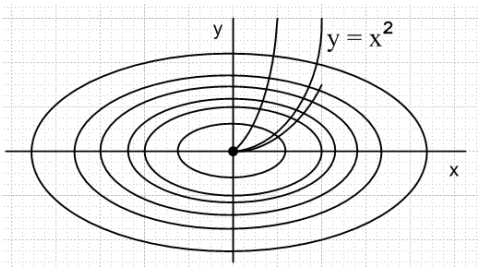
$$\nabla_{3D}g = (g_x, g_y, g_z) = (f_x, f_y, -1) \qquad |\nabla_{3D}g| = \sqrt{f_x^2 + f_y^2 + 1}$$

This second one appears as (9-80) on page 308 of Goldstein where $g(x,y,z) = W(x,y,z) = W(q_i)$.

I next solve the equation $dy/dx = f_y/f_x$ for the case $f = ax^2 + by^2$ and I find that the normal line function in the 2D plane is this

$$y = K\ x^{b/a} = h(x)$$

which I then show graphically for three values of K:



In this case, I compute the arc length s along h(x) to be a hypergeometric function.

I then attempt to compute the arc length along the 3D "up the hill" curve and get a formula for same, and then I do the example case and for a=b I get a result which agrees with something on the web. It is the arc length along a parabola and is not a simple function.

8. Consider a surface z = f(x,y).

Here I attempted to compute the change in the 2D gradient vector as you move ds along the normal curve h(x). The result may not be correct, but here is what I got

$$\delta\nabla f(\mathbf{r}) = (2/R)\,(f_x f_{xx} + f_y f_{yx},\ f_x f_{yx} + f_y f_{yy})\,ds \qquad R = |\nabla f(r)| = \sqrt{f_x^2 + f_y^2}$$

9. Confirm that the $h(x) = Kx^{b/a}$ curve really is normal to the ellipses $ax^2 + by^2 = z$.

I was having some discrepancies so I wanted to see if this h(x) really is the right "normal curve" solution for my prototype problem. Although I could not in general solve for the intersection of these two curves for general a,b,K, I know there will be one, and I am able to show that at such intersections, the curves are perpendicular.

THE DOCUMENT PROPER BEGINS HERE

(everything above is overview)

0. What is the equation of a plane, and what is the normal to that plane?

A plane is given by

$$z = Ax + By + C$$

A plane parallel to this through the origin is just z = Ax + By. A point on this parallel plane would be

$$\text{point} = (x, y, Ax + By)$$

This is also a vector to that point from the origin. This must be perp to the normal vector, so

$n_x x + n_y y + n_z(Ax+By) = 0 \;\; = \;\; [n_x + n_z A]\, x \;\; + [\, n_y + n_z B]\, y$

But this must be 0 for all x and y on our plane, so conclude that

$n_x = -n_z A \qquad\qquad n_Y = -n_z B$

Thus, our normal vector is

$\mathbf{n} = (\, -A, -B, 1)\, n_z \;\; = \;\; (\, -A, -B, 1) \qquad\qquad$ // not normalized

A more standard way to do all this is as follows.

plane: $ax + by + cz + d = 0 \qquad\qquad$ // $\mathbf{v}\bullet\mathbf{r} = -d$

parallel plane through origin: $ax + by + cz = 0$

define $\mathbf{v} = (a,b,c)$. Then we have $\mathbf{v}\bullet\mathbf{r} = 0$. Thus, $\mathbf{v}$ must be the normal vector, so

$\mathbf{n} = (a,b,c) \qquad\qquad$ // not normalized

Parallel translating back from the origin plane to the original one does not change the normal.

In the following section we will show that $\mathbf{n} = \nabla_{3d}(g)$. For $g = ax + by + cz + d = 0$, we see at once that $\mathbf{n} = \nabla_{3d}(g) = (a,b,c)$.

Note added: In the above, what is the interpretation of constant d? Let **p** = point on plane that lies closest to the origin. This means that p is "the distance from the plane to the origin", which means the closest distance between any point on the plane and the origin. The vector $\mathbf{p} = \mathbf{p} - \mathbf{0}$ is perp to the plane, which is why it is the closest point. For any point **r** on the plane, we then know that $(\mathbf{r}-\mathbf{p})\bullet\mathbf{n} = 0$. This tells us that $\mathbf{n}\bullet\mathbf{r} = \mathbf{n}\bullet\mathbf{p}$. But $\mathbf{n}\bullet\mathbf{r} = -d$, so $\mathbf{n}\bullet\mathbf{p} = -d$. But **n** and **p** are antiparallel, so this says np = d. If we were to normalize **n**, then d would *be* the distance from the origin to the plane. Without doing this normalization, we have $d = p\sqrt{a^2+b^2+c^2}$.

0.1 Geometric meaning of the 3D gradient $\nabla_{3d}(g)$.

Suppose you have a function g(x,y,z). Then we know that for any point in space, the following is true (to first order in d**r**),

$g(\mathbf{r}+d\mathbf{r}) = g(\mathbf{r}) + \nabla_{3d}(g) \bullet d\mathbf{r}$

Now suppose we consider the equation g(x,y,z) = 0 which defines a surface in R^3. Then for any point on this surface, we can say

$g(\mathbf{r}+d\mathbf{r}) = \nabla_{3d}(g) \bullet d\mathbf{r}$

Imagine our point on the surface surrounded by a little sphere of possible directions to move making d**r**. If we move in the direction $\nabla_{3d}(g)$, then we will get the largest possible change in the value of g for a given |d**r**|. Therefore, for a point on the surface g(x,y,z) = 0, $\nabla_{3d}(g)$ points in the direction of the maximum change of g. On the other hand, suppose we move in a direction tangent to the surface, such that d**r** = ds $\hat{\mathbf{t}}$. In this case, since g = 0 at all points on the surface, we must have g(r + ds $\hat{\mathbf{t}}$) = 0. But from our first statement above, we know then that 0 = g(r + ds $\hat{\mathbf{t}}$) = $\nabla_{3d}(g) \bullet$ ds $\hat{\mathbf{t}}$. Since this dot product is 0, this tells that the vector $\nabla_{3d}(g)$ must be perpendicular to all possible $\hat{\mathbf{t}}$. Thus, we may identify $\nabla_{3d}(g(r))$ as the normal to the surface g(r) for **r** being a point on the surface g = 0. In other words, $\hat{\mathbf{n}} = \nabla_{3d}(g) / |\nabla_{3d}(g)|$.

Our conclusions are then these

For a point **r** located on the surface g(**r**) = 0:

(1) The vector $\nabla_{3d}(g)$ points in the direction of the normal to the surface, so that $\hat{\mathbf{n}} = \nabla_{3d}(g) / |\nabla_{3d}(g)|$.
(2) The function g has its maximal amount of change in this direction.

Example: Suppose g = ax + by + cz + d, which is a plane in R^3 . If we are at a point on this plane, then if we move in the direction $\nabla_{3d}(g)$ = (a,b,c), we will get a maximum change in g, and this is **n** .

I guess I did not appreciate the fact that the normal to a surface g = 0 points in the direction of max change.

1. Why does a 2D gradient∇_{2D}f to a surface z = f(x,y) point uphill, and what is the uphill vector?

Imagine z = f(x,y). Imagine we are at some point x,y in ground coordinates and we move some small $d\mathbf{s}_{2D} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$. Then the change in the height will be

$$dz = df = f_x\, dx + f_y dy = \nabla_{2D} f \bullet d\mathbf{s}_{2D}$$

The distance dz will be a maximum when $d\mathbf{s}_{2D}$ aligns with ∇_{2D}f because that is what dot products do. Therefore, ∇_{2D}f must point in that direction in the x-y plane which is "uphill" on the surface.

While we are here, what *3D* vector then points up the hill? We know that $\nabla_{2D} f = (f_x, f_y)$. So let's imagine a small displacement in the ground plane x,y along this up direction so $ds_{2D} = (f_x ds, f_y ds)$. Then we know the vertical distance is going to be $df = f_x dx + f_y dy = f_x^2 ds + f_y^2 ds$. Thus, our up-hill 3D vector is this one:

$$dv_{up} = (f_x ds, f_y ds, f_x^2 ds + f_y^2 ds) = (f_x, f_y, f_x^2 + f_y^2)\, ds$$

so an un-normalized vector pointing up the hill is this:

$$\mathbf{v}_{up} = (f_x, f_y, f_x^2 + f_y^2) \qquad \text{// not normalized}$$

Notice that if you are at the top of a hill, $\mathbf{v}_{up} = \mathbf{0}$.

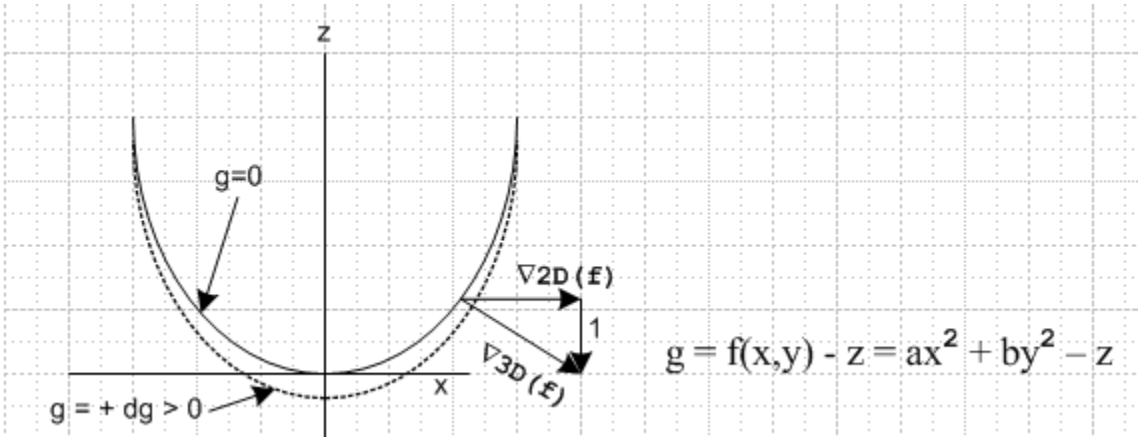
Example: Suppose $f = x^2$ so $f_x = 2x$. Then

$\mathbf{v_{up}} = (2x, 0, 4x^2) = 2x(1,0,2x) \doteq (1,0,2x)$ rise/run = 2x/1 = 2x, correct

1.1 More comments on the 2D and 3D gradients just mentioned above. Consider z = f(x,y) so that g = f(x,y) - z. So a point on the surface z = f(x,y) is a point on the surface g = 0. The $\nabla_{3d}(g)$ vector is given by

$\nabla_{3d}(g) = (g_x, g_y, g_z) = (f_x, f_y, -1) = (\nabla_{2D} f, -1)$.

This vector is the normal to the surface g=0 [which is the same as the surface z = f(x,y)], and points in the direction of maximum change of g. Below we talk about a paraboloid example which looks like an "oval cup". Here is a cross section in the z,x plane:



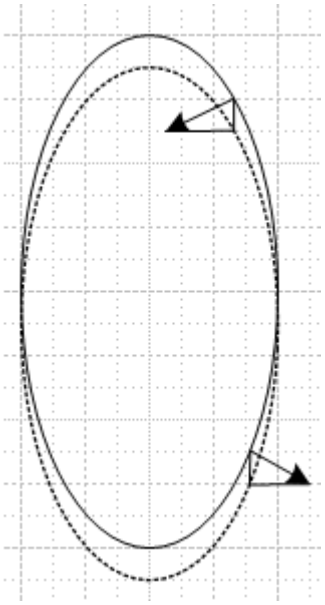
This picture shows a lot of things.

(a) First, it shows $\nabla_{3d}(g)$ being the normal to the surface at a certain point I have picked. I have then drawn this normal vector as the sum of two vectors since $\nabla_{3d}(g) = (\nabla_{2D} f, -1)$. You can see that $\nabla_{2D} f$ is correctly pointing in that direction in the x-y plane which makes you go "uphill" the surface, as was proven in the previous section.

(b) The dotted line shows g = dg > 0 and is of course a surface near the g=0 surface. Because our sample g is linear in z, the dotted curve happens to be an exact duplicate of the solid curve, just lowered by distance dg. We have $ax^2 + by^2 - z = dg$ or $z = ax^2 + by^2 - dg = f(x,y) - dg$ which shows this fact.

(c) For any point **r** on g=0 such as the one I have chosen, the $\nabla_{3d}(g)$ normal points in the direction in which Δg is largest for a given $\Delta\mathbf{r}$. This is in the direction of the dotted line surface which in our example is on the positive g-change side of the solid surface. To the extent dg is small, $\nabla_{3d}(g)$ points in the direction of the least distance between the solid and dotted surfaces, just because $\nabla_{3d}(g)$ is the normal, and because the two surfaces are "locally parallel" for small dg and some small patch region around our selected point.

(d) There is one other point we can bring out. Consider this situation:



If we stick with our sign choice that g = f(x,y) - z, then the normal vector defined by $\nabla_{3d}(g) = (\nabla_{2D}f, -1)$ has to always point toward the dotted g = dg > 0 surface, and it always has a negative z component. This means that on the bottom of the above ellipsoid, this normal points "out" from the surface, but on the top it points "in". In both cases, notice that the horizontal component which is $\nabla_{2d}(f)$ always points in a direction corresponding to "up the hill" on the surface.

(e) Of course if we define g' = -g = z - f(x,y), then the dotted surface is g = dg < 0 and all the normals defined by $\nabla_{3d}(g')$ are the reverse of those shown above. There are always two normals at a point for the two sides of the surface.

2. Imagine a surface z = f(x,y) in R^3 . What is the normal n at any point on the surface?

It discourages me a lot that I have sat for 30 minutes trying to answer this 8th grade question, and now time is up for a day or two while Kent arrives. // Kent is gone, more fiddling and here is a solution:

> Since doing the below solution, I realized from added item 0.1 above that the normal is going to be given by $\nabla_{3d}(g) = (g_x, g_y, g_z) = (f_x, f_y, -1) = (\nabla_{2D}f, -1)$ where we define g = g = f(x,y) - z. Of course if you define g' = -g = z - f(x,y), you find that your normal is $\nabla_{3d}(g') = (- \nabla_{2D}f, 1)$ which is the result which appears in my derivation below. Obviously a surface has two normals **n** and -**n** and the direction $\nabla_{3d}(g)$ points depends on whether you use g or g' = -g.

A general ds_{3D} along a tangent patch would be $d\mathbf{s}_{3D} = (dx,dy,df) = (dx,dy,f_x dx + f_y dy)$. Our desired normal vector **n** has to have this property : $\mathbf{n} \bullet d\mathbf{s}_{3D} = 0$ for any combination of dx and dy. Write things out: [f_x means $\partial f/\partial x$]

$$0 = \mathbf{n} \bullet d\mathbf{s}_{3D} = n_x\,dx + n_y dy + n_z(f_x dx + f_y dy) \quad = [\,n_x + n_z f_x]\,dx + [\,n_y + n_z f_y]\,dy$$

Therefore, each square bracket has to vanish. So we then have these three equations in three unknowns:

$$n_x + n_z f_x = 0$$
$$n_y + n_z f_y = 0$$
$$n_x^2 + n_y^2 + n_z^2 = 1$$

Use the first two to eliminate n_x and n_y from the last to get

$$f_x^2 + f_y^2 + 1 = 1/n_z^2$$

and solve that to get

$$n_z = \pm\, 1/R \qquad\qquad R \equiv \sqrt{f_x^2 + f_y^2 + 1}$$

Then our solution is [and agrees with earlier derivation as commented on above]

$$\mathbf{n} = (n_x, n_y, n_z) = \pm\,(-f_x, -f_y, 1)/R = (-\nabla_{2D} f, 1)/R \qquad \text{// normalized}$$

These are the normals pointing out the two sides of the surface. I have chosen the + sign for the discussion here, see discussion above. So,

$$\mathbf{n} = (n_x, n_y, n_z) = (-f_x, -f_y, 1)/R = (-\nabla_{2D} f, 1)/R \qquad \text{// normalized}$$

which agrees with the normal we found earlier using ∇_{3D}. We can then verify

$$\mathbf{n} \bullet d\mathbf{s}_{3D} = [\,n_x + n_z f_x]\,dx + [\,n_y + n_z f_y]\,dy = [-f_x + f_x]\,dx + [-f_y + f_y]\,dy = 0 + 0 = 0$$

Example: Suppose you are at the top of a hill. Then $f_x = f_y = 0$ and $\mathbf{n} = (0,0,1)$ which is correct. (And this is why I chose the sign the way I did; these normals point "in" on the oval cup bottom, as noted above.)

Extra note: if we add the first two of our three equations above in a way to eliminate n_z we find

$$-n_x / f_x = -n_y/f_y \;=> \qquad n_x f_y = n_y f_x \qquad => \; n_x f_y - n_y f_x = 0 \; => \qquad [\mathbf{n} \times \nabla f_{2D}]_z = 0$$

This has a simple geometric interpretation. Since $\nabla_{2D} f$ points up the hill, and $\mathbf{n}$ points out, the cross product most point "across the hill" and thus has no z component. This is pretty obvious if you just think of a square patch of surface aligned with the up direction. And of course we have just derived it.

Extra note: What is the equation of a plane through the origin which is parallel to the tangent plane at the point x_1, y_1, f_1 where $f_1 = f(x_1, y_1)$? It is given by $\mathbf{n} \bullet \mathbf{r} = 0$ or

$$(-f_{x1}, -f_{y1}, 1) \bullet (x,y,z) = 0 \quad => \qquad -f_{x1}x - f_{y1}y + z = 0 \quad => \quad z = f_{x1}x + f_{y1}y$$

Now, what is the equation of the tangent plane itself? It will be of the form $z = f_{x1}x + f_{y1}y + K$ for some K. But our surface point is (x_1,y_1,f_1) so that must lie on this new plane, so

$f_1 = f_{x1}x_1 + f_{y1}y_1 + K \Rightarrow \qquad K = f_1 - f_{x1}x_1 - f_{y1}y_1$

So the result is:

$$z = f_{x1}x + f_{y1}y + K = f_{x1}x + f_{y1}y + f_1 - f_{x1}x_1 - f_{y1}y_1 = f_{x1}(x-x_1) + f_{y1}(y-y_1) + f_1$$

So: tangent plane equation is: $z = f_{x1}(x-x_1) + f_{y1}(y-y_1) + f_1$

Verify it contains our point x_1,y_1,f_1: trivial. Has the form of a plane. Finally, verify perp to **n**: Just remove the constant to get parallel plane through origin so $z = f_{x1}x + f_{y1}y$. Point on plane is $(x,y, f_{x1}x + f_{y1}y)$. Verify then that $(-f_{x1}, -f_{y1}, 1) \bullet (x,y, f_{x1}x + f_{y1}y) = 0$. Done.

Here is another way to write this tangent plane. We know that $\nabla_{3D}g \bullet d\mathbf{r} = 0$ for any d**r** lying in the tangent plane near some point $\mathbf{r}_1$. Thus, we know that for **r** on the tangent plane, $\nabla_{3D}g \bullet (\mathbf{r} - \mathbf{r}_1) = 0$. But from above we know that $\nabla_{3d}(g_1) = (g_{x1}, g_{y1}, g1_z) = (f_{x1}, f_{y1}, -1) = (\nabla_{2D,1}f, -1)$ where we define $g = f(x,y) - z$. Therefore, we have $(\nabla_{2D}f, -1) \bullet (\mathbf{r} - \mathbf{r}_1) = 0$ which says

$f_{x1}(x-x_1) + f_{y1}(y-y_1) - 1\,(z-z_1) = 0$ giving $\qquad z = f_{x1}(x-x_1) + f_{y1}(y-y_1) + z_1$

which agrees with the above, since $f_1 = z_1$. That is, **r** = (x,y,z) which satisfy this equation lie on the plane which is tangent to the surface at a point $\mathbf{r}_1$ lying on the surface.

3. What then is a 3D vector which points "across the hill"?

It is $\mathbf{v}_{across} \equiv \mathbf{n} \times \mathbf{v}_{up}$. We compute from the above

$\mathbf{n} = (-f_x, -f_y, 1)/R$ where $R \equiv \sqrt{f_x^2 + f_y^2 + 1}$ // **n** is normalized

$\mathbf{v}_{up} = (f_x, f_y, f_x^2 + f_y^2)$ // not normalized

So

$\mathbf{v}_{across} \equiv \mathbf{n} \times \mathbf{v}_{up}$:

$\mathbf{v}_{across,x} = n_y v_{up,z} - n_z v_{up,y} = (-f_y[f_x^2 + f_y^2] - 1*f_y)/R = -f_y R^2/R = -f_y R$

$\mathbf{v}_{across,y} = n_z v_{up,x} - n_x v_{up,z} = (1*f_x - (-f_x)[f_x^2 + f_y^2])/R = f_x R^2/R = f_x R$

$\mathbf{v}_{across,z} = n_x v_{up,y} - n_y v_{up,x} = ((-f_x)(f_y) - (-f_y)f_x)/R = 0$ as expected

Thus, our non-normalized across vector is this one

$\mathbf{v_{across}} = (-f_y, f_x, 0)$ $\mathbf{v_{up}} = (f_x, f_y, f_x^2 + f_y^2)$ $\mathbf{n} = (-f_x, -f_y, 1)$

Now let's verify that this is perpendicular to **n**: (and might as well do all three dots while at it)

$\mathbf{n} \bullet \mathbf{v_{across}} = (-f_x, -f_y, 1) \bullet (-f_y, f_x, 0) = (-f_x * -f_y + -f_y * f_x + 0) = 0$
$\mathbf{n} \bullet \mathbf{v_{up}} = (-f_x, -f_y, 1) \bullet (f_x, f_y, f_x^2 + f_y^2) = -f_x^2 - f_y^2 + (f_x^2 + f_y^2) = 0$
$\mathbf{v_{up}} \bullet \mathbf{v_{across}} = (f_x, f_y, f_x^2 + f_y^2) \bullet (-f_y, f_x, 0) = -f_x f_y + f_y f_x + 0 = 0$

Summary:

$\mathbf{n} = (-f_x, -f_y, 1)$ // not normalized

$\mathbf{v_{up}} = (f_x, f_y, f_x^2 + f_y^2)$ // not normalized

$\mathbf{v_{across}} \equiv \mathbf{n} \times \mathbf{v_{up}} = (-f_y, f_x, 0)$ // not normalized

4. Reduce all the above to the corresponding 2D problem.

Imagine that we have z = f(x,y) but the equation does not really depend on y. This describes a surface in 3D space which is a surface of extrusion in the y direction. And of course $f_y = 0$. Let's now look at all of our results above:

$\mathbf{v_{up}} = (f_x, f_y, f_x^2 + f_y^2) = (f_x, 0, f_x^2) \doteq (1,0,f_x)$ // not normalized

$\mathbf{n} = (-f_x, -f_y, 1) = (-f_x, 0, 1)$ // not normalized

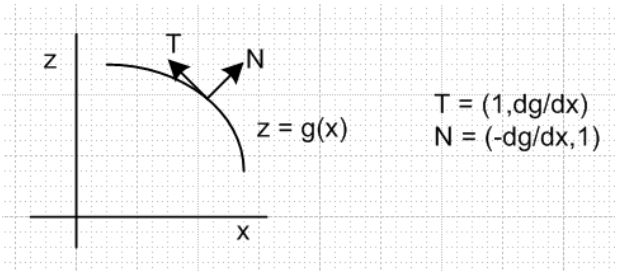
$\mathbf{v_{across}} = (-f_y, f_x, 0) = (0,f_x,0) \doteq (0,1,0)$ // not normalized

So let's now think of the y=0 plane only, and think of z = f(x,y) = g(x) as a 2D problem only. Then in this 2D world we have vectors of the form $\mathbf{a} = (a_x, a_z)$. So

$\mathbf{v_{up}} = (1,g_x) = (1,dg/dx) \doteq (dx,dg) = (run, rise)$ m = rise/run

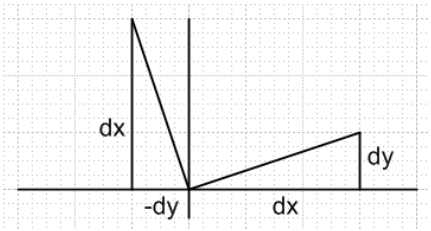
$\mathbf{n} = (-g_x, 1) = (-dg/dx, 1) \doteq (-dg,dx) = (-rise, run)$ m' = run/(-rise) = -1/m as in 5 below

Notice that $\mathbf{v_{up}} \bullet \mathbf{n} = 0$ trivially. Here is a summary picture



5. Consider the line y = mx + b passing through a point x_1, y_1. What is the equation of the line through this same point that is perpendicular to the original line?

Lemma: Consider a line y = mx through the origin. What line is perpendicular to this through the origin? Try the line y = $m_\perp$x. A point on the first line is (x, mx) and this is also a vector on this line from the origin. Similarly, a point on the second line is (x,$m_\perp$x) and this is also a vector on this line from the origin. We want these two vectors to have a zero dot product. So need $x^2 + mm_\perp x^2 = 0$ or $1 + mm_\perp = 0$ or $mm_\perp = -1$ or $m_\perp = -1/m$, a "famous result". Get same thing thinking of (dx,dy) and (-dy,dx) and this little picture:



Now to the original question. The perp line will have the form y = $m_\perp$x + $b_\perp$ where we know $m_\perp = -1/m$. So y = (-1/m)x + $b_\perp$. But then we have y_1= (-1/m)x_1 + $b_\perp$ because this line has to pass through our point of interest. Therefore, $b_\perp$= y_1+ x_1/m . So here is the answer to our question:

line through x_1y_1: y = mx + b

perp line through x_1y_1: y = – (1/m) x + [y_1+ x_1/m] = $m_\perp$x + $b_\perp$

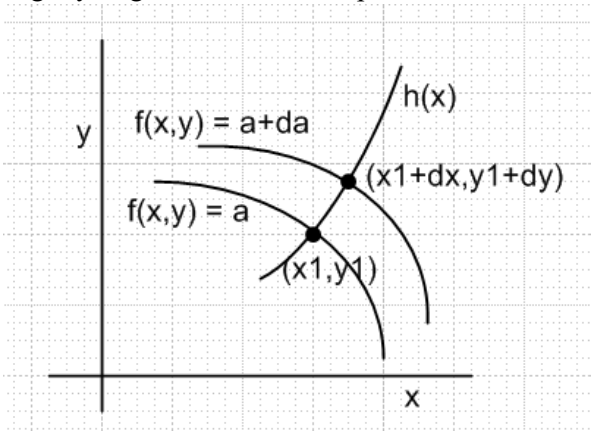
Now, from the first line we know that b = y_1 - mx_1 since it is said to pass through point 1. Thus we could write:

line through point 1: y = mx + [y_1 - mx_1] = mx + b = m(x-x_1) + y_1

perp line thru point 1: y = – (1/m) x + [y_1+ x_1/m] = $m_\perp$x + $b_\perp$ = (-1/m)(x-x_1) + y_1

When written in the last form, all you have to do is replace m with -1/m to get the perp line.

6. Consider a the curve described by f(x,y) = a, it might be a circle or an ellipse for example. Imagine that we draw our curve, and then we draw it again for f(x,y) = a+da, which might be an ellipse of slightly larger size. There is a picture:



Our long-term goal is to come up with an equation for the curve h(x) which is always normal to the family of curves, but right now we have much simpler goals because that problem is too hard for me at the moment. I have to "build up" to it.

(6a). What is the equation of the line tangent at point 1 ? [Note these dx,dy are not those shown in pic.]

As we move along the lower curve which is f(x,y) = a, we know that

$$da = 0 = f_{x1}dx + f_{y1}dy \qquad \Rightarrow \qquad dy/dx = m = -f_{x1}/f_{y1}$$

So from above the tangent line is :

$$y = [-f_{x1}/f_{y1}]\ (x-x_1) + y_1$$

(6b) What is the equation of the perp line at point 1, which is of course has slope there of our h(x)?

$$y = [\ f_{y1}/f_{x1}]\ (x-x_1) + y_1 \qquad \text{since } m_{\perp} = -1/m = f_{y1}/f_{x1}$$

(6c) If we are given dx, how can we estimate dy ? Just differentiate the above normal line:

$$dy = [\ f_{y1}/f_{x1}]\ dx$$

{ The above in the form $dy = [f_y/f_x]\ dx$ is the equation you can solve for the curve y(x) = h(x) that is perp to all the curves denoted by f(x,y) = a for varying a. But I did not realize this at the time and so did not try to solve this equation for my simple ellipses example. This is however done later below. }

(6d) What can we say about the $h'(x_1)$?

$$h'(x_1) = f_{y1}/f_{x1}$$

Notice that this is not something you can integrate to get a solution for h(x). { But all I had to do was write this at $dh/dx = f_y(x,y=h)/f_x(x,y=h)$ and solve for h(x) ! }

(6e) By just repeating the above discussion, we know that

$h'(x_2) = f_{y2}/f_{x2}$ where $x_2 = x_1 + dx$

(6f) Make the following claim: (get rid of dy using normal slope condition $dy/dx = f_y/f_x$)

$$f_{y2} = f_y(x+dx,y+dy) = f_y(x,y) + f_{yx}\,dx + f_{yy}dy = f_{y1} + dx\,[\,f_{yx} + f_{yy}\,(f_{y1}/f_{x1})\,]$$
$$f_{x2} = f_x(x+dx,y+dy) = f_x(x,y) + f_{xx}\,dx + f_{xy}dy = f_{x1} + dx\,[\,f_{xx} + f_{xy}\,(f_{y1}/f_{x1})\,]$$

Therefore we can write

$$f_{y2}/f_{x2} = \{\, f_{y1} + dx\,[\,f_{yx} + f_{yy}\,(f_{y1}/f_{x1})\,]\}\,/\,\{\, f_{x1} + dx\,[\,f_{xx} + f_{xy}\,(f_{y1}/f_{x1})\,]\}$$

Now suppress the 1 labels and rewrite this as (we should have had them as well on the second derivatives)

$$f_{y2}/f_{x2} = \{\, f_y + dx\,[\,f_{yx} + f_{yy}\,(f_y/f_x)\,]\}\,/\,\{\, f_x + dx\,[\,f_{xx} + f_{xy}\,(f_y/f_x)\,]\}$$

Factor f_y from the numerator and f_x from the denominator

$$f_{y2}/f_{x2} = (f_y/f_x)\,\{\,1 + dx\,[\,f_{yx} + f_{yy}\,(f_y/f_x)\,]/f_y\,\}\,/\,\{\,1 + dx\,[\,f_{xx} + f_{xy}\,(f_y/f_x)\,]/f_x\}$$

Now expand for small dx to get

$$f_{y2}/f_{x2} = (f_y/f_x)\,\{\,1 + dx\,[\,f_{yx} + f_{yy}\,(f_y/f_x)\,]/f_y - dx\,[\,f_{xx} + f_{xy}\,(f_y/f_x)\,]/f_x\,\}$$

$$= (f_y/f_x) + dx\,\{\,[\,f_{yx} + f_{yy}\,(f_y/f_x)\,]/f_y*(f_y/f_x) - [\,f_{xx} + f_{xy}\,(f_y/f_x)\,]/f_x*(f_y/f_x)\,\}$$

$$= (f_y/f_x) + dx\,\{\,[\,f_{yx} + f_{yy}\,(f_y/f_x)\,]/f_x - [\,f_{xx} + f_{xy}\,(f_y/f_x)\,]\,(f_y/f_x^2)\,\}$$

$$= (f_y/f_x) + dx\,\{\,[\,f_{yx}/f_x + f_{yy}\,(f_y/f_x^2)\,] - [\,f_{xx}f_y/f_x^2 + f_{xy}\,(f_y^2/f_x^3)\,]\,\}$$

$$= (f_y/f_x) + dx\,\{\,f_{yx}[\,1/f_x - (f_y^2/f_x^3)\,] + (f_y/f_x^2)\,[\,f_{yy} - f_{xx}]\,\}$$

$$= (f_y/f_x) + dx\,\{\,(f_{yx}/f_x)\,[1 - (f_y/f_x)^2\,] + (f_y/f_x^2)\,[\,f_{yy} - f_{xx}]\,\}$$

Therefore,

$$h''(x_1) = [h'(x+dx) - h'(x)]/dx = [f_{y2}/f_{x2} - f_y/f_x]/dx$$

$$= \{\,(f_{yx}/f_x)\,[1 - (f_y/f_x)^2\,] + (f_y/f_x^2)\,[\,f_{yy} - f_{xx}]\,\}$$

Let's try this for a set of circles $f(x,y) = x^2 + y^2 = a$. Then

$f_x = 2x$ $f_y = 2y$ $f_{xx} = 2$ $f_{yx} = 0$ $f_{yy} = 2$

so get

$$h''(x_1) = \{ (f_{yx}/f_x)[1 - (f_y/f_x)^2] + (f_y/f_x^2)[f_{yy} - f_{xx}] \}$$

$$= \{ 0 + 0 \} = 0$$

which is at least the correct answer: we know for concentric circles, h" = 0 because the curve of normals is a straight line.

(6g) So far then we have accumulated these facts about h(x):

$h(x_1) = y_1$ // since goes through this point

$h'(x_1) = f_{y1}/f_{x1}$

$$h''(x_1) = \{ (f_{yx}/f_x)[1 - (f_y/f_x)^2] + (f_y/f_x^2)[f_{yy} - f_{xx}] \}$$

So this tells us a little about our curve h(x). At least for a set of concentric ellipses, we can see that the normal line is bending, because $[f_{yy} - f_{xx}] \neq 0$, so h(x) is curving.

But we are still very far away from having an expression for h(x) itself! [coming soon!]

Note added: I am not sure all the above detail is correct, but we can test it using the solution we find below which is that when $f = ax^2 + by^2$, we have $h(x) = K x^{b/a}$. So first let's compute the derivatives directly:

$h(x) = K x^{b/a}$
$h'(x) = K(b/a) x^{b/a - 1}$ $= (a/b) h(x)/x$
$h''(x) = K(b/a)(b/a - 1) x^{b/a - 2}$

Now let's see if the above formulas are right, where I will ignore the 1 label.

$h(x) = y = y(x) = K x^{a/b}$ // nothing really being checked here

Next,

$f_y = 2by$ $f_x = 2ax$ $h'(x_1) = f_{y1}/f_{x1} = by/ax = (b/ax)K x^{b/a}$ // agrees!

Next,

$f_{xy} = 0$ $f_{yy} = 2b$ $f_{xx} = 2a$ so

$h''(x_1) = \{ \ (f_{yx}/f_x)\,[1 - (f_y/f_x)^2\ ] + (f_y/f_x^2)\,[\,f_{yy} - f_{xx}]\ \}$

$= \{\ 0 + \qquad\qquad + (2by/[2ax]^2)\,[\,2b - 2a]\ \}$

$= by/(a^2x^2) * (b\text{-}a) = by/(ax^2) * (b/a - 1) = (b/a)\, y\, x^{-2}(b/a - 1)$

$= (b/a)\, K\, x^{b/a-2}\,(b/a - 1) \qquad\qquad$ // agrees!

So the ugly calculations above at least give the right answer for h" in this case.

7. Consider a surface z = f(x,y) in R^3.

We know that, starting from any point, there is an uphill path that defines a curve in 3D, and its projection onto the x-y plane is a 2D curve. In either case, we can parameterize the distance along the curve by a variable s, with s=0 at the starting point. In the 3D case, ds is along the 3D curve, and in the 2D case it is along the 2D curve, so ds is not the same in both cases.

3D Case: From above we know that

$\mathbf{v}_{up} = (f_x, f_y, f_x^2 + f_y^2)$

$v_{up}^2 = f_x^2 + f_y^2 + (f_x^2 + f_y^2)^2 = (f_x^2 + f_y^2)\,(1 + f_x^2 + f_y^2) \equiv R^2$

In this situation, we can relate differentials in the 3 directions all to ds. Let d$\mathbf{r}$ be a small step uphill:

$d\mathbf{r} = ds\ \hat{\mathbf{v}}_{up} = ds\ (f_x, f_y, f_x^2 + f_y^2)/R$

Then we have

$dx = (f_x/R)\ ds$
$dy = (f_y/R)\ ds$
$dz = [(f_x^2 + f_y^2)/R]\ ds$

Again, if we move a small amount ds along our parameterized hill-climb curve, the three differentials are as shown, and so all are relatable to ds.

2D Case:

$\mathbf{v}_{up} = (f_x, f_y)$

$v_{up}^2 = f_x^2 + f_y^2 \equiv R^2 \qquad\qquad R = \sqrt{f_x^2 + f_y^2} = |\nabla f|$

$d\mathbf{r} = ds\ \hat{\mathbf{v}}_{up} = ds\ (f_x, f_y)/R$

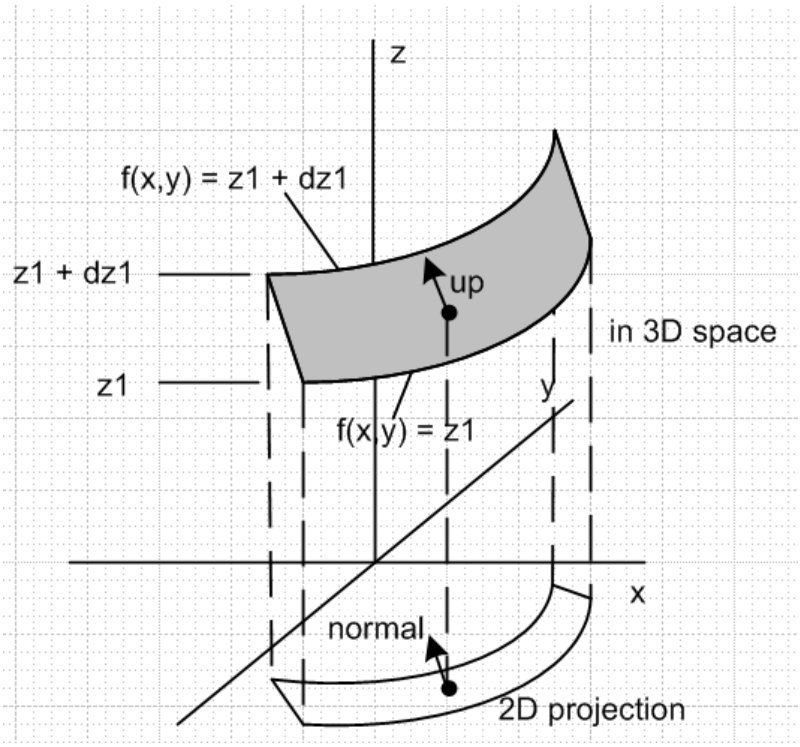
$dx = (f_x/R)\, ds$
$dy = (f_y/R)\, ds$

So here ds is a distance along the normal curve in the ground plane which is the projection of the 3D curve. Here is a ratio of interest:

$dy/dx = f_y/f_x$

Important point not yet stated. This $dy/dx = f_y/f_x$ is the slope of the ground-plane curve which is the projection of the up-hill-climb curve. Go up to some 3D point on the surface, call it **r**. Imagine the z=constant curve which passes through this point, sort of a topo line at this altitude. Call this z_1 . Then our curve has the equation $f(x,y) = z_1$. Just above this curve we have another topo curve $f(x,y) = z_1 + dz_1$. We can imagine drawing these two topo lines on the ground plane at z=0. This is what we would see if we were to project our two real topo lines onto the z=0 plane. And we could also project our 3D $\mathbf{v_{up}}$ vector onto the ground plane, where it becomes $\mathbf{v_{up2D}} = (f_x, f_y)$. We know that $\mathbf{v_{up}}$ tells us the direction of the shortest distance from one topo line to the next one above. That is, in 3D we know that $\mathbf{v_{up}}$ is perpendicular to its topo line which has the direction $\mathbf{v_{across}}$ mentioned above, where $\mathbf{v_{across}}$ lies entirely in the z_1 plan, ie, it is the horizontal tangent vector in our construction. If we were to rotate $\mathbf{v_{up}}$ around the $\mathbf{v_{across}}$ axis, we would sweep out a disk all of which is perpendicular to the topo line $\mathbf{v_{across}}$. In particular, when we rotated $\mathbf{v_{up}}$ into the horizontal position, it would still be perpendicular to the topo line. But this is exactly the direction that $\mathbf{v_{up2D}} = (f_x, f_y)$ has. Therefore, $\mathbf{v_{up2D}} = (f_x, f_y)$ is normal to our curve $f(x,y) = z_1$, whether we think of this curve as existing in 3D space, or being projected down onto the z==0 plane. That is, if we just think of $f(x,y) = z_1$ as a 2D curve in the x-y plane, then (f_x, f_y) will provide the normal vector to this curve. See picture below.

Well, that was quite long-winded and here is an easier and 2D-only way to see the point. Consider a curve $f(x,y) = z_1$ in the x-y plane. For a motion along the curve we can say $df = 0 = f_x dx + f_y dy$. This puts a condition on (dx,dy) = d**r** which will keep us on the curve if we start at a point on the curve. Ie, we will have $f(x+dx,y+dy) = z_1$. Thus, the slope of a tangent line will be $dy/dx = -f_x/f_y$. Therefore, we know that the slope of the normal line at that same point must be $dy/dx = +f_y/f_x$. This then explains why $dy/dx = f_y/f_x$ is an equation for the *normal* at some point on one of our curves $f(x,y) = z_1$.



In general, we treat the above equation as something you solve for y(x) which is the curve of interest. Once we have a solution y(x) we can compute arc length along the curve as follows:

$$ds = \sqrt{1 + (dy/dx)^2}\, dx = \sqrt{1 + (f_y/f_x)^2}\, dx$$

$$s(x) = \int_0^x \sqrt{1 + (f_y/f_x)^2}\, dx \qquad \text{// where need to replace } y = y(x) \text{ in the integrand.}$$

Note added: We know that $df = f_x dx + f_y dy$. If we insert from above, we get

$$df = f_x dx + f_y dy = f_x(f_x/R)\, ds + f_y(f_y/R)\, ds = (f_x^2 + f_y^2)\, ds/R = R\, ds = |\nabla_{2D} f|\, ds = \nabla_{2D} f \bullet d\mathbf{r} = dz$$

Let's restate this. Suppose $d\mathbf{r} = \hat{\mathbf{n}}\, ds$ is along h(x), the normal to our 2D curve f(x,y) = z. We already know that we can write $\hat{\mathbf{n}} = \nabla_{2D} f / |\nabla_{2D} f|$ so it follows that

$$dz = df = \nabla_{2D} f \bullet d\mathbf{r} = (\, |\nabla_{2D} f|\, \hat{\mathbf{n}}) \bullet d\mathbf{r} = (\, |\nabla_{2D} f|\, \hat{\mathbf{n}}) \bullet \hat{\mathbf{n}}\, ds = |\nabla_{2D} f|\, ds$$

This just says as you move that little normal ds in the x-y plane, you will move from the curve f(x,y) = z to another curve f(x,y) = z + dz where $dz = |\nabla_{2D} f|\, ds$.

There is a 3D version of this statement. It says this for g = f(x,y)-z , and assumes that ds is a 3D distance along the normal $\nabla_{3D} g$ taking you from g(x,y,z) = 0 to some nearby surface g(x,y,z) = dg. We can write $\hat{\mathbf{n}} = \nabla_{3D} g / |\nabla_{3D} g|$ so that $d\mathbf{r} = ds\, \hat{\mathbf{n}}$ and thus

$dg = \nabla_{3D} g \bullet d\mathbf{r} = |\nabla_{3D} g|\, \hat{\mathbf{n}} \bullet d\mathbf{r} = |\nabla_{3D} g|\, ds$ // Goldstein page 308 equation 9-80

2D Example: Suppose $z = f(x,y) = ax^2 + by^2$. Then we have $f_x = 2ax$ and $f_y = 2by$. The last equation then becomes

$$dy/dx = f_y/f_x = (b/a)\, y/x$$

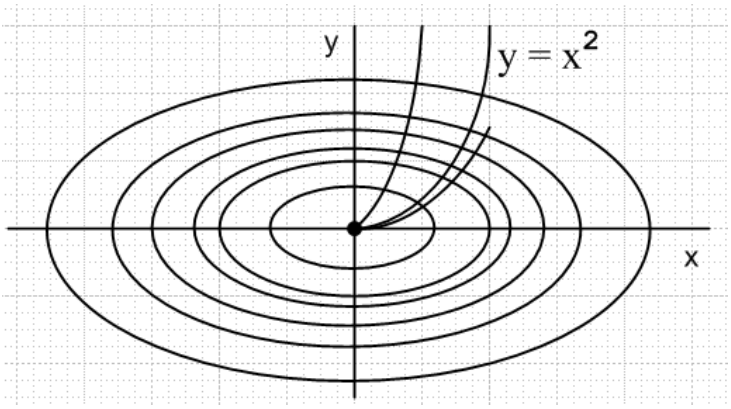
We can *integrate* this to obtain the equation for the 2D curve:

$$dy/y = (b/a)\, dx/x \quad \Rightarrow \quad \ln y = (b/a) \ln x + C = \ln [\, x^{b/a}\, e^C]$$

Therefore we find

$$y = K\, x^{b/a}$$

Physical interpretation: We have here a surface build from a stack of ellipses which scale up in size as we go up in z. The surface is a paraboloid which is different in the x and y directions (oval cup). The cross section at z = constant is an ellipse. If we were to choose a/b = 2, we get $y = Kx^2$. Here is a picture of this situation looking down on the x-y plane:



I show three curves, the left is roughly $y = 2x^2$ while the right one is $y = (1/2)\, y^2$. So as we vary K, we are generating all the curves which are normal to all the ellipses! This was the problem I started out today trying to solve.

If we wanted the distance along the curve (arc length) as a function of x, we could do this:

$$ds = \sqrt{1 + (dy/dx)^2}\, dx = (\ 1 + [\, (b/a)\, K\, x^{b/a-1}]^2\ )^{1/2}\, dx = (1 + (Kb/a)^2\ x^{2(b/a)-2}\ )^{1/2}\, dx$$

Then we get

$$s(x) = \int_0^x (1 + (Kb/a)^2\ t^{2(b/a)-2}\ )^{1/2}\, dt = \int_0^x (1 + \alpha^2\ t^{\beta}\ )^{1/2}\, dt$$

$= F(-1/2, 1/\beta; 1+1/\beta; -\alpha t^{\beta})$ where $\alpha = Kb/a$ and $\beta = 2(b/a - 1)$ // Maple

Suppose now a = b so we are doing circles. We then get y = Kx which are, as expected, straight lines through the set of concentric circles. The arc length integral is then

$$s(x) = \int_0^x (1 + \alpha^2)^{1/2}\, dt = \sqrt{1 + K^2}\ x$$

which seems pretty reasonable: line y = Kx, length along line = $\sqrt{x^2 + y^2}$.

3D Example: Take the same function as in the 2D example. We have this projection onto the ground plane:

$$y = K\, x^{b/a}$$

At each point (x,y) on this 2D curve, the z value is

$$z = ax^2 + by^2 = ax^2 + b\, K^2\, x^{2b/a}$$

so we know all about this 3D curve then. To get arc length along this curve, we do

$$ds = \sqrt{1 + (dy/dx)^2 + (dz/dx)^2}\ dx$$

Recall from above that

$$dx = (f_x/R)\ ds$$
$$dy = (f_y/R)\ ds$$
$$dz = [(f_x^2 + f_y^2)/R]\ ds$$

so we have

$$dy/dx = (f_y/f_x)$$
$$dz/dx = ((f_x^2 + f_y^2)/f_x = (f_x + f_y/f_x)$$

Then

$$ds = \sqrt{1 + (f_y/f_x)^2 + (f_x + f_y/f_x)^2}\ dx$$

and then arc length is given by

$$s = \int_0^x \sqrt{1 + (f_y/f_x)^2 + (f_x + f_y/f_x)^2}\ dx$$

Looking at our special paraboloid case, we have $z = f(x,y) = ax^2 + by^2$. Then we have $f_x = 2ax$ and $f_y = 2by$, so this becomes

$$s = \int_0^x \sqrt{1 + (by/ax)^2 + (2ax + by/ax)^2}\ dx \qquad \text{where } y = K\, x^{b/a}$$

which I suppose Maple could do. If we set a = b for circles, then

$$s = \int_0^x \sqrt{1 + (y/x)^2 + (2ax + y/x)^2}\ dx \qquad \text{where } y = K\, x$$

$$= \int_0^x \sqrt{1 + K^2 + (2ax + K)^2}\ dx$$

If we take a path in the y=0 plane, this corresponds to K = 0 and we then get

$$s = \int_0^x \sqrt{1 + (2ax)^2}\ dx = (1/2a) \int_0^{2ax} \sqrt{1 + t^2}\ dt$$

$$= (1/2a)\ \{\ \tfrac{1}{2} q \sqrt{1+q^2} + \tfrac{1}{2} \ln(q + \sqrt{1+q^2})\ \} \qquad q = 2ax$$

$$= (x/2) \sqrt{1 + (2ax)^2} + (1/4a) \ln (2ax + \sqrt{1 + (2ax)^2}\ )$$

This agrees with something on line which uses a = 1/(2p) or p = (1/2a) or 1/p = 2a

> equation $y = x^2/2p$ for some constant *parameter* p. The length element ds along the parabola is such that $(ds)^2 = (dx)^2 + (dy)^2$, or $ds/dx = \sqrt{(1+(dy/dx)^2)} = \sqrt{(1 + x^2/p^2)}$. The length s of the arc of parabola from the apex (0,0) to the point $(x, y = x^2/2p)$ is simply the following integral of this (in which we may eliminate x or p, using $2py = x^2$).
>
> $$s = (x/2) \sqrt{1 + x^2/p^2} + (p/2) \ln(\sqrt{1 + x^2/p^2} + x/p\ )$$

The following section does not add much, but I keep it anyway.

8. Consider a surface z = f(x,y): $\nabla_{2D} f = (f_x, f_y)$ points in the x,y direction of "uphill"

at any point on the surface. If we start at some point and walk uphill for a while, curving around as needed to always be going up, we map out a curve in 3D, and the projection of this curve in the x,y plane is a 2D curve. Both curves are of interest.

At any given point on our walk, we know that $\nabla_{2D} f = (f_x, f_y)$ points uphill. It would seem not too complicated to "thread" these little vectors together to come up with an equation for the curve in the x,y plane, yet I have spent 30 minutes trying to do this and have been unable to do it!

(7a) If you move along a small amount, how does the gradient change?

$$d\mathbf{r} = (dx,dy) = ds\, \nabla f(r)/R = ds\, \hat{\mathbf{n}} \qquad R = |\nabla f(r)| = \sqrt{f_x^2 + f_y^2}$$

Therefore

$$\nabla f(\mathbf{r} + d\mathbf{r}) = \nabla[\, f(\mathbf{r} + ds\, \nabla f(r)/R)\,]$$

But in general

$$f(\mathbf{r} + d\mathbf{r}) = f(\mathbf{r}) + \nabla f(r) \bullet d\mathbf{r}$$

Therefore

$$f(\mathbf{r} + ds\, \nabla f(r)/R) = f(\mathbf{r}) + \nabla f(r) \bullet ds \nabla f(r)/R = f(\mathbf{r}) + ds\, [\, f_x^2 + f_y^2]\, /R = f(\mathbf{r}) + ds\, R$$

Therefore

$$\nabla f(\mathbf{r} + d\mathbf{r}) = \nabla[\, f(\mathbf{r} + ds\, \nabla f(r)/R)\,] = \nabla[\, f(\mathbf{r}) + ds\, R\,]$$

$$= \nabla f(\mathbf{r}) + ds\, \nabla[\, \sqrt{f_x^2 + f_y^2}\,]$$

This tells us then how the gradient changes as we walk,

$$\delta\nabla f(\mathbf{r}) = ds\, \nabla[\, \sqrt{f_x^2 + f_y^2}\,]$$

We have,

$$\delta\nabla f(\mathbf{r})_x = ds\, \partial_x [\sqrt{f_x^2 + f_y^2}\,] = ds\, [\, f_x f_{xx} + f_y f_{yx}]/R$$
$$\delta\nabla f(\mathbf{r})_y = ds\, \partial_y [\sqrt{f_x^2 + f_y^2}\,] = ds\, [\, f_x f_{yx} + f_y f_{yy}]/R$$

Thus,

$$\delta\nabla f(\mathbf{r}) = (2/R)\, (f_x f_{xx} + f_y f_{yx},\ f_x f_{yx} + f_y f_{yy})\, ds$$

I think this result is correct, but not very useful right now. I have already solved my problem in item 7 above, so I will let this path 8 just end here.

9. Confirm that the $h(x) = Kx^{b/a}$ curve really is normal to the ellipses $ax^2 + by^2 = z$.

Our first task is to find the point of intersection. We have

$$ax^2 + b[Kx^{b/a}]^2 = z \qquad ax^2 + bK^2 x^{2b/a} = z$$

Cannot solve for x in the general case! Rats. But don't give up yet. Whatever the solution point x is, it has to satisfy this equation above.

What is the slope of h(x) at a point x ?

$$h(x) = Kx^{b/a}$$
$$h'(x) = (b/a)Kx^{b/a-1}$$

What is the slope of our ellipse?

$$y(x) = (1/\sqrt{b}) \sqrt{z-ax^2}$$

$$y'(x) = -(1/\sqrt{b})ax/\sqrt{z-ax^2}$$

The requirement of perpendicularity is that $h'(x) = -1/y'(x)$, which here says

$$(b/a)Kx^{b/a-1} = \sqrt{b}\sqrt{z-ax^2}/(ax)$$

Square both sides to get

$$(b/a)^2K^2 x^{2b/a-2} = b(z-ax^2)/(ax)^2$$

$$b^2K^2 x^{2b/a} / (ax)^2 = b(z-ax^2)/(ax)^2$$

$$bK^2 x^{2b/a} = (z-ax^2)$$

$$ax^2 + bK^2 x^{2b/a} = z \qquad \text{// result of insisting slopes be perp}$$

But our equation of intersection from above reads like so,

$$ax^2 + bK^2 x^{2b/a} = z \qquad \text{// x is intersection of two curves}$$

These are seen to be the same. Thus, the same point x which marks a point where the curves intersect also marks a point where they are perpendicular! And this is true for any real K (assuming the intersection

equation really does have a real solution). As shown in our earlier picture, changing K changes the h(x) curve, but our "perp at intersections" fact is still true at any intersection on any of these curves.