The Plane Pendulum Problem in Classical Mechanics PhL 4.10.08

Comment: I did this analysis as an "exercise" after reading Goldstein through Chapter 9. In my current pass through his book (my first *complete* pass, really), I did not do any of his suggested problems, so this is an attempt to at least do something. I take this relatively simple plane pendulum problem and look at it in each of the formalisms that Goldstein has presented. The main motivation was to see it in the Hamilton-Jacobi theory and in particular the action-angle variables. I got sidetracked, however, in solving the actual problem which, for large angles, was much harder than I thought it would be, involving the Jacobi elliptic integrals and functions. Since I knew nothing about this Jacobi world, I had to go off and study *it*, and that is in a separate writeup called "elliptic functions". When I finished all this stuff, I took a quick look at what the quantum solution is to this same problem.

The notes below are summarized in a separate "executive summary" document which goes much lighter on the technical details. I would recommend that to the casual reader.

1. Setup **2**
2. Compute the Hamiltonian **3**
3. Write down Hamilton's Equations of motion **3**
5. Initial conditions **4**
7. Now let's try this method **5**
8. Take small angle limit of the exact solution **10**
9. Examine the exact solution **11**
10. Question: How can we make our solution show "rotation"? **14**
11. Comment on the arcsin function **17**
12. Phase Space **20**
13. Hamilton-Jacobi theory applied to this problem. **21**
14. Action-angle variables applied to this problem. **23**
15. What are the classical waves doing in this problem? **27**
16. What is the solution to the quantum plane pendulum problem? **28**

Contents of this very long document:

1. **Setup** for the pendulum problem
2. Compute **Lagrangian** and **Hamiltonian** and **canonical momenta**.
3. **Hamilton's Equations of Motion**, $\omega^2 = g/\ell$. Mass does not affect anything.
4. Statement of the **key equation** of motion: $\ddot{\theta} = -\omega^2 \sin\theta$, 2nd order *non-linear* (in θ) ODE. But the equation is trivially integrable twice.
5. Statement of the **initial conditions**:, start at bottom with $\ddot{\theta} = v$, then $C = \frac{1}{2} v^2 - \omega^2$
6. **General method** for solving such a non-linear ODE, called an "autonomous" ODE $\ddot{\theta} = f(\theta)$

You just integrate twice.

7. **Apply** this method to the case $f(\theta) = -\omega^2 \sin\theta$
   Implications of initial conditions, set the two integration constants.
   Look up integral in GR, find a useful form for **the "libration" solution**.
   Statement of problem solution as $\sin(\theta/2) = \sin(\theta_0/2)\ \mathrm{sn}(\omega t; k= \sin(\theta_0/2))$, web confirms.
   Here, $\boldsymbol{\theta_0}$ is max angle during swinging, sn is a certain **Jacobi** elliptic function, sine-like.

8. Taking the **small oscillation limit** of the general result, agrees with simple result

9. Examine the exact libration solution, plot for different values of θ_0.
   Comment on the **period** as a function of θ_0: $T = 4\mathbf{K}(k)$, $k= \sin(\theta_0/2) = v/(2\omega)$
   Here, $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

10. Looking for the **"rotation" type solution**
    Use a different GR integral, then realize this is just a "**transformation**" on the libration solution.
    The rotational solution is $\sin[\theta/2] = \mathrm{sn}(vt/2; k=2\omega/v)$
    Take the large v limit of this solution and get $\theta(t) = v\,t$.

11. Comment on the **arcsin** function and its infinite number of "branches" for plotting
    How to make Maple plot the true arcsin(x).
    How to make Maple plot the true $\theta = \arcsin(\sin\theta)$ as an ongoing straight line
    How to make **Maple** plot the rotation solution, example shown.

12. Plot of the **phase space** orbits picture, see both librations and rotations.

13. **Hamilton-Jacobi** theory applied to this problem.
    Write the HJE for function $W(\theta,E)$.
    Find that W is in fact an E -- second kind incomplete elliptic integral.
    Examine $t + \beta_1 = \partial W/\partial\alpha_1$, use known solution to verify this form is correct

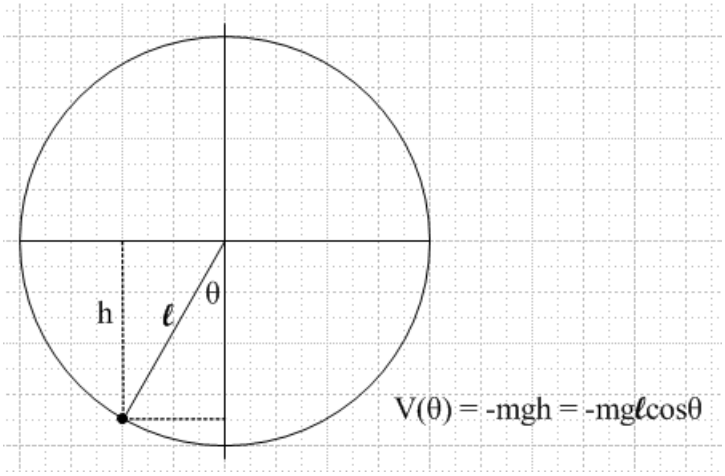
14. **Action-angle** variables applied to pendulum
    Compute J_θ and w_θ.
    Use the known general result that ν_θ = constant to solve the entire problem!
    Summary of "this little section", use HJ to derive the final result instead of confirm it.

15. What are the **classical waves** doing in this problem? Not much here.

16. Solve the same problem in **quantum mechanics**, get **Mathieu** functions, not Jacobi.
    The ODE is completely different, but problem is typical of a QM eigen problem.

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1. Setup . Here is a picture:



2. Compute the Hamiltonian as a function of its correct arguments:

When the pendulum is at rest straight down, I would like to have energy E = 0. Then E > 0 for all possible motions. This means we have V(θ) = mgh' where h' = ℓ-h and picture shows h = ℓcosθ. Therefore

$V(\theta) = mg[\ell - \ell\cos\theta] = mg\ell(1-\cos\theta)$ ok

$H = T + V = p^2/2m + mg\ell(1-\cos\theta)$

$L = T - V = p^2/2m - mg\ell(1-\cos\theta)$ ok

Since $r = \ell$ = constant, there is really only one coordinate in this problem, which is θ.

$v_\theta = \ell\dot{\theta}$ $\quad L = \frac{1}{2} m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$

$p_\theta = \partial L/\partial\dot{\theta} = m\ell^2\dot{\theta}$

$T = \frac{1}{2} mv^2 = \frac{1}{2} m (\ell\dot{\theta})^2 = \frac{1}{2} m (p_\theta/m\ell)^2 = p_\theta^2/(2m\ell^2)$

$H(q_i, p_i) = H(\theta, p_\theta) = p_\theta^2/(2m\ell^2) + mg\ell(1-\cos\theta)$ = Hamiltonian.

3. Write down Hamilton's Equations of motion which are $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$. Can remember the sign from free particle where $H = p^2/2m$ so $\dot{q} = p/m = V$. In our case then

$\dot{\theta} = \partial H/\partial p_\theta = p_\theta/(m\ell^2)$ $\quad \dot{p}_\theta = -\partial H/\partial\theta = -mg\ell\sin\theta$

4. Let's try to solve the right equation with θ as variable of interest:

$m\ell^2\ddot{\theta} = -mg\ell\sin\theta$ or $\ell\ddot{\theta} = -g\sin\theta$ $\quad \ddot{\theta} = -(g/\ell)\sin\theta$ $\quad \ddot{\theta} = -\omega^2\sin\theta$

where we have defined $\omega^2 = g/\ell$. For small angles, get $\ddot{\theta} = -\omega^2\theta$ and get traditional result of $\theta = A \sin(\omega t + B)$.

Note 1.27.17: Belendez et all agree with the above basic equation, which they write as

$$\frac{d^2\theta}{dt^2} + \omega_0^2 sin\theta = 0, \qquad (1)$$

where θ is the angular displacement, t is the time and ω_0 is defined as

$$\omega_0 = \sqrt{\frac{g}{l}}. \qquad (2)$$

so we are "OK to here". At least I have the correct starting equation.

5. Initial conditions. Suppose our start condition is $\theta=0$ and $\dot{\theta} = v$. Then we have $B = 0$ and $\dot{\theta}(0) = \omega A\cos(\omega t)\ ^{t=0}\ = \omega A = v$, so solution is , and $\omega t = \arcsin[\ \omega\theta/v\ ]$ and $\theta(t) = (v/\omega)\sin(\omega t)$. So our assumption should be that $(v/\omega) << 1$ for this approximation to work. [small osc only]

In general, what do I know about $\ddot{x} = -\omega^2 \sin x$ as a second order ODE ? Not obvious how to solve this thing. Let's ask Maple. Maple shows a mess. Web has help:

6. General method of solving this kind of non-linear ODE: [It is an "autonomous" equation because the variable t does not appear explicitly]

$d^2x/dt^2 = f(x(t))$

Let $w(x(t)) = dx(t)/dt$. Then $d^2x/dt^2 = dw/dt = dw/dx * dx/dt = w\ dw/dx$. Then have

$w\ dw/dx = f(x) \quad => \quad wdw = f(x)\ dx \quad => \quad \tfrac{1}{2}\ w^2 = \int^x f(x')dx' + C$ ok

// note that $w(x(t)) = dx(t)/dt = v(t)$, and this will be $\dot{\theta}$ in our application ok

Then we have $(dx/dt)^2 = 2\ [\int^x f(x')dx' + C\ ]$ which says

$$dx/dt = \pm \sqrt{2\ [\int^x f(x')dx' + C\ ]} = \pm\sqrt{2}\ \sqrt{\int^x f(x')dx' + C} \equiv \pm\sqrt{2}\ h(x)$$ ok

$$h(x) \equiv \sqrt{\int^{x} f(x')dx' + C}$$ ok

and then

$$dx/h(x) = \pm \sqrt{2}\, dt$$ ok

so get

$$\pm t = (1/\sqrt{2}) \int^{x} dx'' \left[1/\sqrt{\int^{x''} f(x')dx' + C}\right] + C' \equiv g(x)$$ ok

But this only tells you ± t = g(x). To find x(t), you need then to solve ±t – g(x) = 0. In other words, you need the "roots" x(t) of the equation ±t – g(x) = 0. This is what Maple is trying to say. ok

7. Now let's try this method for f(x) = –ω²sinx. We find // $\ddot{\theta} = -\omega^2 \sin\theta$ ok

$$\tfrac{1}{2} w^2 = \int^{x} f(x')dx' + C$$ ok

$$\tfrac{1}{2} w^2 = \int^{x} f(x')dx' + C = \int^{x} -\omega^2 \sin x'dx' + C = \omega^2 \cos x + C$$ ok

So end up with

$$\pm t = (1/\sqrt{2}) \int^{x} dx'' \left[1/\sqrt{\omega^2 \cos x'' + C}\right] + C' \equiv g(x)$$ ok

<u>Initial Conditions:</u> Suppose we "launch" our pendulum from θ(0) = 0 but with some velocity $\dot{\theta}(0) = v$. We had at one point, where x = θ and w = $\dot{\theta}$: // unusual initial condition.

$$\tfrac{1}{2} w^2 = \int^{x} f(x')dx' + C = \omega^2 \cos x + C$$ ok

At t=0 the LHS is ½ v² while the RHS is ω²+ C. Therefore,

$$\tfrac{1}{2} v^2 = \omega^2 + C \qquad C = \tfrac{1}{2} v^2 - \omega^2 = \tfrac{1}{2}[v^2 - 2\omega^2]$$ ok

The energy E of this pendulum is E = T + V = ½ m $(\ell\dot{\theta})^2$ + mgℓ(1-cosθ) and E(t=0) = ½ mℓ²v² ok

so ½ $v^2 = E/(m\ell^2)$ and then $C = E/(m\ell^2) - \omega^2$. ok

Now assume that our initial v can kick the pendulum up to angle θ_0.

How high does the pendulum swing for libration solutions? Set $E(0) = \frac{1}{2} m\ell^2 v^2 = mg\ell(1-\cos\theta_0)$ ok which says $\frac{1}{2} v^2 = (g/\ell)(1-\cos\theta_0)$ and find that $v^2 = 2\omega^2(1-\cos\theta_0)$ so that ok

$C = \frac{1}{2}[v^2 - 2\omega^2] = \frac{1}{2}[2\omega^2(1-\cos\theta_0) - 2\omega^2] = \omega^2(1-\cos\theta_0 - 1) = -\omega^2\cos\theta_0$ ok

Meanwhile, when t = 0, we know θ = 0 (which is x above), so we know C' = 0 [assuming the lower endpoint of the integral below is 0] . Our solution is then

$$\pm t = (1/\sqrt{2}) \int^{x} dx'' \left[1/\sqrt{\omega^2\cos x'' + C}\right]$$ ok

$$\pm\omega t = (1/\sqrt{2}) \int^{\theta} d\theta \left[1/\sqrt{\cos\theta - \cos\theta_0}\right]$$ where $v^2 = 2\omega^2(1-\cos\theta_0)$ ok

But $(1-\cos\theta_0) = 2\sin^2(\theta_0/2)$ so $v^2 = 4\omega^2 \sin^2(\theta_0/2)$ and so $v = 2\omega \sin(\theta_0/2)$ ok

This integral fits the mold of GR page 154 #4 which says

$$\int^{\theta} d\theta \left[1/\sqrt{\cos\theta + a}\right] = 2/\sqrt{1+a}\ F[x/2, \sqrt{2/(1+a)}]$$ $a = -\cos\theta_0$ ok

Here it is from GR7 and set b = 1:

4. $$\int \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{\sqrt{a+b}} F\left(\frac{x}{2}, r\right) = \sqrt{\frac{2}{b}} F\left(\gamma, \frac{1}{r}\right)$$

$[a > b > 0, \quad 0 \le x \le \pi]$

$\left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$

BY (289.00)

$$\alpha = \arcsin\sqrt{\frac{1-\sin x}{2}}, \qquad \beta = \arcsin\sqrt{\frac{b(1-\sin x)}{a+b}},$$

$$\gamma = \arcsin\sqrt{\frac{b(1-\cos x)}{a+b}}, \qquad \delta = \arcsin\sqrt{\frac{(a+b)(1-\cos x)}{2(a-b\cos x)}}, \qquad r = \sqrt{\frac{2b}{a+b}}$$

But $\sqrt{1+a} = \sqrt{2}\sin(\theta_0/2)$ and $r = \sqrt{2/(1+a)} = \sqrt{2} / [\sqrt{2}\sin(\theta_0/2)] = \csc(\theta_0/2)$ ok

so our result is this

$$\int_{}^{\theta} d\theta \,[1/\sqrt{\cos\theta - \cos\theta_0}\,] = 2/(\sqrt{2}\sin(\theta_0/2))\, F[\, x/2, \sqrt{2/(1+a)}\,]$$ ok

or

$$(1/\sqrt{2})\int_{}^{\theta} d\theta \,[1/\sqrt{\cos\theta - \cos\theta_0}\,] = \csc(\theta_0/2)\, F[\, x/2, \csc(\theta_0/2)\,]$$ ok

But I don't like this form because the second argument k > 1 ! So let's try the other form that GR give in the same place which is this

$$\int_{}^{\theta} d\theta \,[1/\sqrt{\cos\theta + a}\,] = \sqrt{2}\; F(\gamma, \sqrt{1+a}/\sqrt{2}) \qquad \gamma = \sin^{-1}\sqrt{(1-\cos\theta)/(1-\cos\theta_0)}$$ ok

$$= \sqrt{2}\; F(\sin^{-1}\sqrt{(1-\cos\theta)/(1-\cos\theta_0)}\;, \sin(\theta_0/2))$$ ok

Our problem's solution is then this

$$\pm\omega t = (1/\sqrt{2})\int_{}^{\theta} d\theta \,[1/\sqrt{\cos\theta - \cos\theta_0}\,] = (1/\sqrt{2})\sqrt{2}\; F(\sin^{-1}\sqrt{(1-\cos\theta)/(1-\cos\theta_0)}\;, \sin(\theta_0/2))$$ ok

$$= F[\, \sin^{-1}\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right), \sin(\theta_0/2)] = F(\varphi,k) = u$$ // This is the GR7 F ok

Recall that if someone hands you F(φ,k) = u and asks you to find φ(u), the answer is sin(φ) = sn(u). ok

We can restate this as φ = am(u) and sn(u) = sin(φ)

How does GR7 support these claims? First, here is their definition of F(φ,k) from page 860:

$$F(\varphi,k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)\,(1-k^2x^2)}}$$

Then many pages down they define sn this way

8.144

1. $$u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}}$$ SI 21(23)

which I can write as

$$u = F(\sin^{-1}(sn(u)),k)$$

Now set u = $sn^{-1}x$ to get

$sn^{-1}(x) = F(\sin^{-1}(x),k)$

Now let $\sin^{-1}(x) = \varphi$ to get

$sn^{-1}(\sin\varphi) = F(\varphi,k)$

Now someone hands you an equation $u = F(\varphi,k)$. You can conclude then that

$u = sn^{-1}(\sin\varphi)$ or $sn(u) = \sin(\varphi)$

What about the am(u) function? Here is what GR7 say

8.14 Jacobian elliptic functions

8.141 Consider the upper limit φ of the integral

$$u = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

as a function of u. Using the notation

$$\varphi = \operatorname{am} u$$

So this says $u = F(\varphi,k) = F(am(u),k)$ and $\varphi = am(u)$. So both claims are OK

Now notice that the second argument $k = \sin(\theta_0/2)$ has our desired property that $k \leq 1$. Next, we know that

$\varphi = am(u) = \sin^{-1}\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right)$ ok

$sn(u; k= \sin(\theta_0/2)) = \sin\varphi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$ ranges from 0 to 1 during motion ok

Inserting $\pm\omega t$ for u, we find that

$\sin(\theta/2) = \sin(\theta_0/2)\ sn(\pm\omega t; k= \sin(\theta_0/2))$ ok

$v^2 = 2\omega^2(1-\cos\theta_0)$ $v = 2\omega \sin(\theta_0/2)$ ok

$\theta(0)=0$ and $\dot{\theta}(0) = v$ ok

$\omega^2 = g/\ell$ ok

Now which sign of $\pm\omega t$ should I take ? Expansion in GR7 says that sn(x) is odd in its argument. So I guess either sign is fine, so I take the + sign.

I have confirmation from the web that this is the correct solution of the plane pendulum, see last equation of this paragraph taken from http://www.mai.liu.se/~halun/complex/elliptic/ :

New site: http://users.mai.liu.se/hanlu09/complex/elliptic/

An application: the pendulum

The Jacobi sn function appears, for example, in the solution of the pendulum equation $d^2\theta/dt^2+\omega^2\sin(\theta) = 0$. For small oscillations, the approximation $\sin(\theta) \approx \theta$ reduces this equation to the harmonic oscillator, with trig solutions of period $2\pi/\omega$. Here is a quick sketch of how to derive the exact solution, without the approximation of small oscillations. Assume for simplicity that the energy is low enough that the pendulum doesn't swing past the upright position, so that there is a maximal angle $\theta_{max}\in[0,\pi]$, and assume also that $\theta = 0$ when $t = 0$. Let $z = \omega t$. Then $d^2\theta/dz^2+\sin(\theta) = 0$. Multiply this equation by $d\theta/dz$ and integrate: $(1/2)(d\theta/dz)^2-\cos(\theta) = E$, where $E = -\cos(\theta_{max}) = 2k^2-1$, if we define $k = \sin(\theta_{max}/2)\in[0,1]$. Then $w(z) = (1/k)\sin(\theta(z)/2)$ will satisfy the defining differential equation for the sn function (including initial conditions), so that $w(z) = sn(z,k)$. Consequently, the solution of the pendulum equation is $\theta(t) = 2\,\arcsin[k\,sn(\omega t,k)]$.

I go to that site again and blow up and it does confirm my solution for $\theta(0) = 0$.

An application: the pendulum

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When $k = 0$ the pendulum just hangs straight down without moving, while the case $k = 1$ is a pendulum which performs a single swing between upright positions in the limits $t\to\pm\infty$. The larger the modulus k, the larger the period $4K(k)/\omega$ and the amplitude $\theta_{max} = 2\arcsin(k)$.

Comments: The above solution is for $\theta(0) = 0$ and θ_0 as the max. Same source above says

The number $K(k)$ is half the width of the rectangles in the plot, and the **right-left period** of the sn function is consequently $4K(k)$.

Thus K(k) is a quarter period. So if you time shift my solution by a quarter period, but starting it at the max value, you will get

$\sin(\theta/2) = \sin(\theta_0/2)\ \ sn(\omega t;\ k= \sin(\theta_0/2))$ // my original solution

$\theta/2\ = \sin^{-1}[\ \sin(\theta_0/2)\ \ sn(\omega t;\ k= \sin(\theta_0/2))\ ]\ =\ \sin^{-1}[\ k\ \ sn(\omega t;\ k)\ ]\quad k =\ \sin(\theta_0/2)$

Here is a plot:

Replace t by t' = t - K(k). Then at t' = 0 you will have t = K(t) = one quarter cycle in! Then

$\sin(\theta/2) = \sin(\theta_0/2)\ \ sn(\omega[K(k)+t']; k= \sin(\theta_0/2))$ // my original solution

1. This source is a paper just on this subject

[PDF] Exact solution for the nonlinear pendulum - Sociedade Brasileira de ...
sbfisica.org.br/rbef/pdf/070707.pdf ▾
by A Beléndez - 2007 - Cited by 51 - Related articles
This paper deals with the nonlinear oscillation of a **simple pendulum** and ... of the Jacobi elliptic **function sn**(u;m) using the following initial conditions: the initial ...

and their solution is this,

$$\sin(\theta/2) = \sin\frac{\theta_0}{2}\, sn\left[K\left(\sin^2\frac{\theta_0}{2}\right) - \omega_0 t\,;\, \sin^2\frac{\theta_0}{2}\right] \quad (30) \qquad \omega_0 = \sqrt{\frac{g}{l}}. \quad (2)$$

whereas *my* solution is

$\sin(\theta/2) = \sin(\theta_0/2)\ \ sn(\omega t; k= \sin(\theta_0/2))$

we seem to disagree on both arguments of the sn function! Let's find another source.

2. Here is another solution from another paper

$$\vartheta(t) \;=\; 2\,\arcsin\left\{\sin\frac{\vartheta_0}{2}\,\mathrm{sn}\left[\frac{\omega_0}{\omega}\left(\frac{\pi}{2} - \omega t\right)\,\middle|\,\sin\frac{\vartheta_0}{2}\right]\right\}, \quad (6)$$

At least we agree on the second argument, but notice that vertical bar notation

8. Take small angle limit of the exact solution. Now what can we say about sn(ωt;k) for small k?

Before going on, we need to learn something about the famous K and K', to wit,

$K = K(k) \approx \pi/2$ for small k GP page 905

$K' = K'(k) = K(\sqrt{1-k^2}) \to \infty$ as $k \to 0$ because K(1) diverges.

Therefore, in the small k regime, we know that the "nome" q = exp(-πK'/K) ≈ exp(-2K') = very small << 1. Therefore we can use the very first term in the GR page 911 expansion for snu which becomes

$$sn(u; k) \approx 2\pi/(kK)\ \sqrt{q}\ /(1-q)\ \sin(\pi u/2K) = (4\sqrt{q}\ /k)\ \sin(u).$$

But in the k→ 0 limit, we need to study $\sqrt{q}\ /k \approx \exp(-K')/k$. This is a 0/0 limit. We know that

$$K' = K'(k) = K(k') \text{ where } k' = \sqrt{1-k^2} \text{ so we are near } k' = 1.$$

Use the expansion #2 shown page 905 GR [item 8.113.3]. Swap k and k' to get

$$K'(k) = K(k') = \ln(4/k) + [...]k^2 + ... \qquad \text{near } k = 0$$

Thus we approximate K(k') ≈ ln(4/k) for k near 0 and we then have

$$\lim_{k\to 0}\ [\sqrt{q}\ /k] = \lim_{k\to 0}\ [\exp\{-K'\}/k] = \exp\{-\ln(4/k)\}/k = (k/4)/k = 1/4$$

Thus our we get desired result that $(4\sqrt{q}\ /k)\to 1$ and then our limit is

$$sn(u; k) \approx \sin(u) \qquad \text{as } k\to 0 \qquad \text{// but of course I knew that}$$

so that our problem solution becomes

$$\sin(\theta/2) = k\ sn(\omega t; k= \sin(\theta_0/2)) \approx k\ \sin(\omega t) = \sin(\theta_0/2)\ \sin(\omega t)$$

and for the small angles as we are assuming, this says

$$\theta = \theta_o\ \sin(\omega t) \qquad \theta(0) = 0 \qquad \text{// this seems correct !!!}$$

$$\dot{\theta} = \omega\ \theta_o\ \cos(\omega t) \qquad \dot{\theta}(0) = \omega\theta_0$$

$$v^2 = 2\omega^2(1-\cos\theta_0) = \omega^2\ \sin^2(\theta_0/2)/4 \qquad v = (\omega/2)\ \sin(\theta_0/2) \approx \omega\ \theta_0$$

and we finally recover our small angle result. [that only took me a whole day and a half!]

In the above, I wanted to see an expansion so I could in theory look at other terms in the approximation sn(u; k) ≈ sin(u). The series I used is sort of a Fourier harmonic expansion.

9. Examine the exact solution. We have confirmed that this is the complete pendulum solution:

$$\sin(\theta/2) = \sin(\theta_0/2)\ \ sn(\omega t; k=\sin(\theta_0/2))$$

where we use the notation sn(u,k) given on GR page 904. We can also express this as

$$\omega t = F[\ \sin^{-1}\left(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right)\ ,\ \sin(\theta_0/2)] = F(\varphi,k)$$

Thus we could say that we first "relate" φ to our physical angle θ, and then

$$\sin\varphi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \qquad \text{and} \qquad F(\varphi,k) = \omega t \qquad k = \sin(\theta_0/2)$$

But consider the first form above which says this:

$$\sin(\theta/2) = \sin(\theta_0/2)\ \ sn(\omega t;\ k)$$

We know that $sn(\omega t;\ k)$ is "sinusoidal" in a general sense and goes between the limits of +1 and -1. We know that for small k at least, as can approximate

$$sn(\omega t;\ k) \approx \sin(k'\omega t)$$

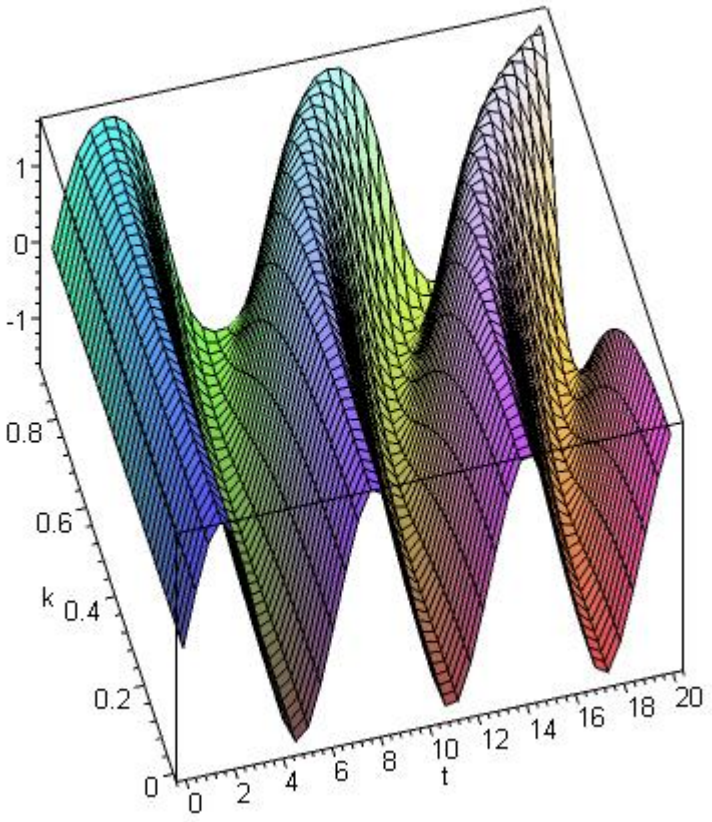
So what exactly is the above saying? It says θ goes back and forth between the two limits θ_0 in some sine like fashion.

$$\theta(t) = 2 \arcsin\left[\sin(\theta_0/2)\ \ sn(\omega t;\ k)\right]$$

This is the exact solution for any k. It is clear what happens for $k < 1$, meaning $\sin(\theta_0/2) < 1$ meaning $\theta_0/2 < \pi/2$ meaning $\theta_o < \pi$.

We can have Maple plot this for some values of k. Well, here is a 3D plot where we vary both t and k

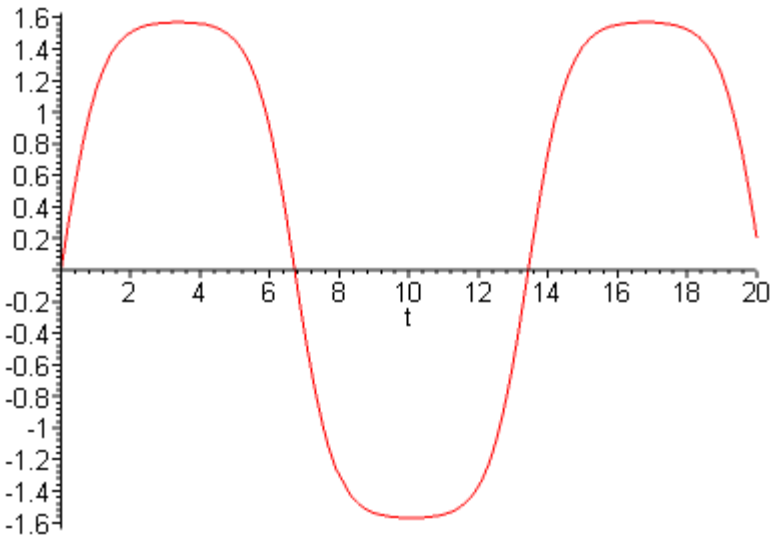
```
plot3d(2*arcsin( sin(theta0/2) * JacobiSN(t,k)), t = 0..20, k=0..0.9, grid = [40,40]);
```



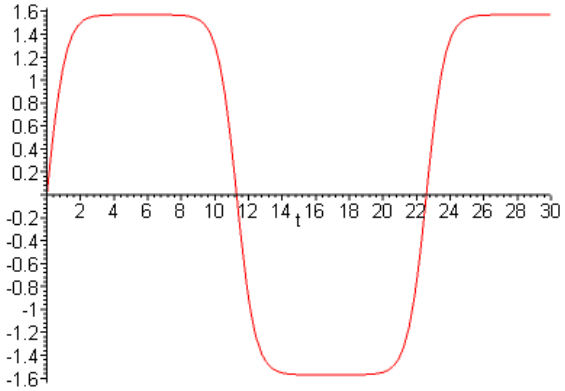
For small k we get the near curves which are very sine like, but as k increases, the tops of the sines flatten off as the pendulum starts to "hold" at the top of its swing! Also, we clearly see the pendulum period increasing according to K(k) as plotted above.

Here is a 2D plot for k = .99

plot(2*arcsin(sin(theta0/2) * JacobiSN(t,.99)), t = 0..20);




```
plot(2*arcsin( sin(theta0/2) * JacobiSN(t,.9999)), t = 0..30);
```



Fascinating!

Fact: The **period** of the librating pendulum is determined by $\omega T = 4*K(k)$ where $k = \sin(\theta_0/2)$ so the swinging frequency is given by $\omega' = 2\pi/T = [2\pi/4K(k)]\,\omega$. For example, if you have small amplitude, then $K(k) \approx K(0) = \pi/2$ as for sin, $\omega' = \omega$. But suppose the pendulum goes half way up to $\theta_0 = \pi/2$. Then we have $k = \sin(\pi/4) = 1/\sqrt{2}$ and then $K(1/\sqrt{2}) = 1.854074677$ so $[2\pi/4K(k)] = .847$ and our swinging frequency is about 15% slower than that of the small oscillations. By increasing θ_0, we can get the frequency to be arbitrarily low (in theory).

I just made a simple pendulum and measured 86% for the .847 above. Hard to get it to swing full up where I set it up, and there is air friction for the big swing since more velocity, but close enough!!!

10. Question: How can we make our solution show "rotation"?

Our libration solution was this:

$\theta(t) = 2 \arcsin [k\, sn(\omega t; k)]$ $\quad v = 2\omega \sin(\theta_0/2) = 2\omega k \quad k = \sin(\theta_0/2)$

$\sin(\theta/2) = \sin(\theta_0/2)\ sn(\omega t; k)$

But let's go back and resolve this problem for general constant C. We find

$$\tfrac{1}{2}\dot{\theta}^2 = \omega^2\cos\theta + C \quad \text{so} \quad \dot{\theta} = \sqrt{2}\sqrt{\omega^2\cos\theta + C} = \sqrt{2}\,\omega\sqrt{\cos\theta + C/\omega^2} = d\theta/dt$$

$$\Rightarrow \quad d\theta/\sqrt{\cos\theta + C/\omega^2} = \pm\sqrt{2}\,\omega\,dt$$

and as usual the ± just means our solution can run either way in time, so take the +. Then

$$\sqrt{2}\,\omega\,t = \int_0^\theta d\theta/\sqrt{\cos\theta + C/\omega^2}$$

where I continue to assume that at t=0, θ = 0 which is fine. Repeating what we said above: This integral fits the mold of GR page 154 #4 which says

$$\int_0^\theta d\theta\,[1/\sqrt{\cos\theta + a}\,] = 2/\sqrt{1+a}\ F[\,x/2, \sqrt{2/(1+a)}\,] \qquad a = C/\omega^2$$

valid $a > 1$ hence $C > \omega^2$

AND valid $0 \le x \le \pi$

Now in the libration case, we had $C = -\omega^2\cos\theta_0$ which was always negative, so we had $a = -\cos\theta_0$ and then we had $k = \sqrt{2/(1+a)} = \csc\theta_0$ and I complained that this was > 1 and so I used a different GR integral form. But now we have $C = -\omega^2 + \tfrac{1}{2}v^2$ and we will be looking beyond the critical point which we can see is $v = \sqrt{2}\,\omega$. Beyond this point C will be positive and the nature of the solution is going to change! The energy of this pendulum is always $E = \tfrac{1}{2}m\ell^2v^2$ since V = 0 at the bottom. We find that

$k = \sqrt{2/(1+a)} = 2\omega/v$ which is rather convenient.

Now our integral says valid only if $C > \omega^2$ which means $-\omega^2 + \tfrac{1}{2}v^2 > \omega^2$ or $\tfrac{1}{2}v^2 > 2\omega^2$ or $v > 2\omega$. This means that $k = 2\omega/v \le 1$ which is what we expect.

So here is the solution I think I have found using this GR integral:

$$\sqrt{2}\,\omega\,t = \int_0^\theta d\theta/\sqrt{\cos\theta + C/\omega^2} = 2/\sqrt{1+a}\ F[\,\theta/2, \sqrt{2/(1+a)}\,] = \sqrt{2}\ k\,F(\theta/2,k) \qquad k = 2\omega/v$$

Thus we have for our rotational solution

$\omega t/k = (v/2)t = F(\theta/2,k)$ $\quad k = 2\omega/v$ $\quad$ valid only for $0 \le \theta \le \pi$

Notice that our libration solution was this:

$$\omega t = F[\sin^{-1}(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}), \sin(\theta_0/2)] = F(\varphi,k)$$

Recall that if someone hands you $F(\varphi,k) = u$ and asks you to find $\varphi(u)$, the answer is $\sin(\varphi) = sn(u)$. We can restate this as $\varphi = am(u)$ and $sn(u) = \sin(\varphi)$

$\sin(\theta/2) = \sin(\theta_0/2)\ sn(\omega t; k)$
$\theta(t) = 2 \arcsin [k\ sn(\omega t; k)]$ $\quad v = 2\omega \sin(\theta_0/2) = 2\omega k$ $\quad k = \sin(\theta_0/2) = v/(2\omega)$

In our rotation situation we have this for our solution:

$\omega t/k = (v/2)t = F(\theta/2,k)$ $\quad k = (2\omega/v)$ $\quad$ valid only for $0 \le \theta \le \pi$

Recall that if someone hands you $F(\varphi,k) = u$ and asks you to find $\varphi(u)$, the answer is $\sin(\varphi) = sn(u)$. We can restate this as $\varphi = am(u)$ and $sn(u) = \sin(\varphi)$

In our present situation we have $\varphi = \theta/2$ and so we know that $\sin(\theta/2) = sn((v/2)t)$. So here is our answer:

$\sin[\theta/2] = sn(vt/2; k=2\omega/v)$ $\quad$ valid only for $0 \le \theta \le \pi$

For very large v, we get $k \approx 0$ so we get

$\sin[\theta(t)/2] = \sin(vt/2)$

The conclusion is then

$\theta(t)/2 = vt/2$ $\quad \Rightarrow \quad$ $\theta(t) = v\,t$ $\quad \dot{\theta} = v$

This is the limit really of $g = 0$ and $\omega = 0$, a hard knock. The initial energy will be as before which was $E = \frac{1}{2} m\ell^2 v^2$. All this energy goes into rotational motion, $E = \frac{1}{2} \omega'^2 I$ where I is the moment of inertia which is $m\ell^2$ so we get $\frac{1}{2} m\ell^2 v^2 = \frac{1}{2} \omega'^2 * m\ell^2$ so $\omega'^2 = v^2$ and $\omega' = v$. This is pretty obviously the right limit for a very hard kick.

Furthermore, we know that **the period** of the above sn function is 4K(k). How much time goes by for this function sn(vt/2; k=2ω/v) to repeat itself ?

$\Delta(vt/2) = 4K(k) = (v/2)\Delta t \quad \Rightarrow \quad \Delta t = (8/v)K(k)$ $\quad k = 2\omega/v$

During this time Δt, the function $\sin(\theta/2)$ makes a complete cycle as well, so this means that the variable $\theta/2$ has gone 2π, which means θ has gone 4π. But this would be 2 cycles of θ or two "periods of the rotational motion", so we would say

$2T_{rot} = \Delta t = (8/v)K(k)$

and we then get this result for the period of our rotational motion

$T_{rot} = 4K(k)/v \qquad k = 2\omega/v$

and the frequency and angular frequency are

$\nu_{rot} = 1/T_{rot} = v/(4K(k)) \qquad \omega_{rot} = 2\pi\,\nu_{rot} = v\,(\pi/2)/K(k)$

Let's check this in the high energy limit. In this case, $k \to 0$ and $K = \pi/2$ so we get $\omega_{rot} = v = \dot{\theta}(0)$ which is what we expect.

Review of the two solution regimes: [$v = 2\omega \sin(\theta_0/2)$, $E = \frac{1}{2} m\ell^2 v^2$]

libration	$\sin(\theta/2) = \sin(\theta_0/2)\ \mathrm{sn}(\omega t; k)$ $\omega t = F[\sin^{-1}(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}), k]$ period = 4K(k)	$k = \sin(\theta_0/2) = v/(2\omega)$
rotation	$\sin(\theta/2) = \mathrm{sn}(vt/2; k)$ $\omega t/k = (v/2)t = F(\theta/2, k)$ period = 4K(k)	$k = \csc(\theta_0/2) = (2\omega/v)$

One is appropriate for small v, the other for large v. The solutions are the same and have the same period when k = 1 which is the critical point between the two solution modes, corresponding to $\theta_0 = \pi$.

The solutions are not the same! Perhaps they are related by a transformation of the sn function, however. Yes! Write the first as

$\sin(\theta/2) = k\,\mathrm{sn}(\omega t; k) \qquad k = v/2\omega$

Use this transformation rule

$\mathrm{sn}(ku, 1/k) = k\,\mathrm{sn}(u,k)$

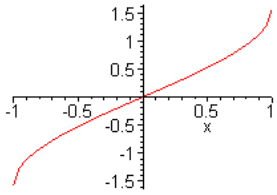
using $u = \omega t$. The LHS is then

$\mathrm{sn}(v/2\omega\,\omega t, 2\omega/(v)) = \mathrm{sn}(vt/2, 2\omega/v)$

which agrees!

11. Comment on the arcsin function.

Maple provides a single-valued "function" like so:



It cannot continue this because then we don't really have a function with a single value at each point. But we can define a more generalized function like so. We would like to define another function to plot at the same time as the above, which will be a continuation. Use these variable names

$x = \sin(\theta)$

$\theta = \arcsin(x)$

For θ in range $(-\pi/2,\pi/2)$ we can use the Maple function arcsin. n=0

For θ in range $(\pi/2,3\pi/2)$ we need to instead use $\theta = -\arcsin(x) + \pi$ n=1

For θ in range $(3\pi/2,5\pi/2)$ we need to instead use $\theta = +\arcsin(x) + 2\pi$ n=2

In general then , for θ in range $(-\pi/2 + n\pi, \pi/2 + n\pi)$ $\theta = +(-1)^n \arcsin(x) + n\pi$

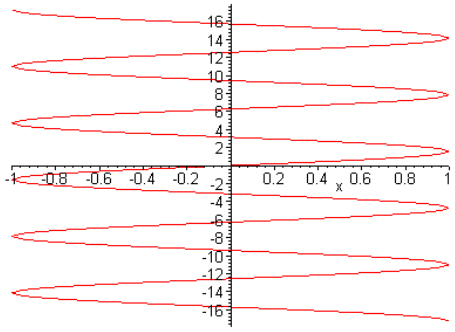
So here is a set of functions

$\theta_n(x) = (-1)^n \arcsin(x) + n\pi$ n = any integer

OK, here is how to make Maple plot a bunch of these functions together:

```
plot([seq((-1)^n*arcsin(x)+n*Pi, n=-5..5)], x=-1..1, color = [seq(red,n=-5..5)]);
```

We are just doing a plot of multiple functions and we enumerate them using the seq(f_n, n=1..3) type construction. Something inside a [...] in Maple is called a **list**, whereas {...} denotes a **set**. Here we just want lists of things. If we don't specify colors as shown, the various curves come out different colors, but I want them all the same color. Here is what the above command yields:

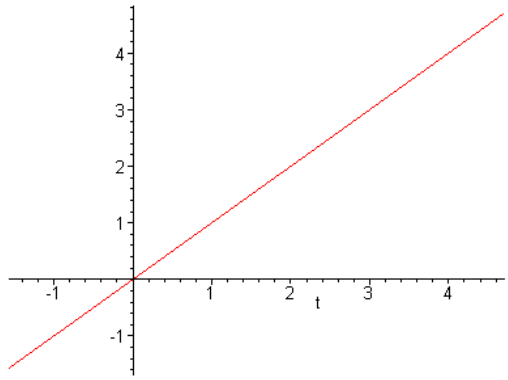


which is exactly what I want.

Comment on plotting $u = \sin^{-1}[\, w=\sin\theta \,]$

What we want of course is a plot that shows u = θ, but this is tricky to obtain because the $\sin^{-1}$ function acts on a domain of w = (-1,1) and a range of u = (-π/2,π/2). We want it to work out of this range, so we have to do a lot of work. Here is my solution to this problem to plot two adjacent regions:

```
> K := Pi/2:
> q := Pi/2:
> f1 := arcsin(sin(t)):
> f2 := arcsin(sin(t-2*K))+2*q:
> plot(piecewise(t>=-K and t<K,f1,t>K and t<=3*K, f2, 0),t=-K..3*K);
```



I have to make a separate piecewise function for each of the two ranges of t. Quantity K is the quarter period of the periodic function which is the argument of the arcsin, while q is the quarter period of the sin function. Using this idea, we can examine our "rotation problem" solution

sin[θ/2] = sn(vt/2; k=2ω/v)

θ = 2*arcsin[sn(vt/2; k=2ω/v)]

I will select v = 2 to make the scale simple, so we get

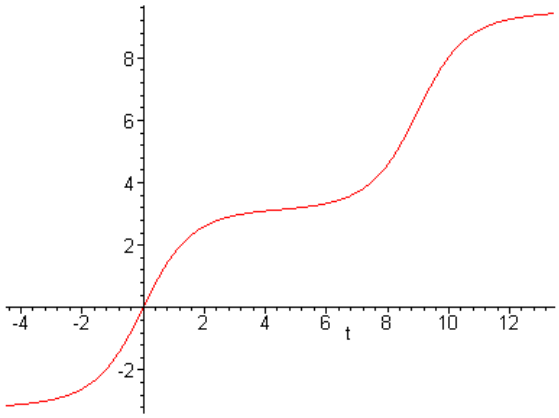
θ = 2*arcsin[sn(t; k= ω]

We know that K(k) is the quarter period of sn, and we have an extra factor of 2* out front, so here is a way to get Maple to plot this thing:

```
> k := .999:
> K := EllipticK(k);
```

```
                          K := 4.495596396
```

```
> f1 := 2*arcsin(JacobiSN(t,k)):
> f2 := 2*( arcsin(JacobiSN(t-2*K,k))+2*q):
> q := Pi/2:
> plot(piecewise(t>=-K and t<K,f1,t>K and t<=3*K, f2, 0),t=-K..3*K);
```



So finally we are getting the solution θ(t) for our "rotation problem". We have chosen k = .999 to emphasize slow motion at the top of the pendulum. It just barely reaches the top and then falls ahead to continue for another loop, so the angle θ as presented here just increases all the time. The thing never swings backwards of course (which would cause me more plotting problems).

So in reality, what we do is compute our solution in the range -K to K, then we just stamp that picture down again to the right and shifted up π. So I think this is a complete understanding of our rotation solution which is

$\theta(t) = 2*\arcsin[sn(vt/2; k=2\omega/v)]$

12. Phase Space.

I am happy now with the solution I have found. I would now like to generate that famous phase space picture I have seen in several places. From above we had

$$\tfrac{1}{2}\dot{\theta}^2 = \omega^2\cos\theta + C \qquad C = E/(m\ell^2) - \omega^2. \qquad \omega^2 = g/\ell$$

$$p_\theta = m\ell^2\dot{\theta}$$

From this I think I find

$$p_\theta^2 = 2m\ell^2E - (2m\ell^2\omega)^2 \sin^2(\theta/2) = [2m^2g\ell^3]\,(\ E' - 2\sin^2\theta/2\ )$$
$$= [2m^2g\ell^3]\,(\ \{E' - 1\}+\cos\theta\ )$$

where E is the pendulum energy. This of course has the form

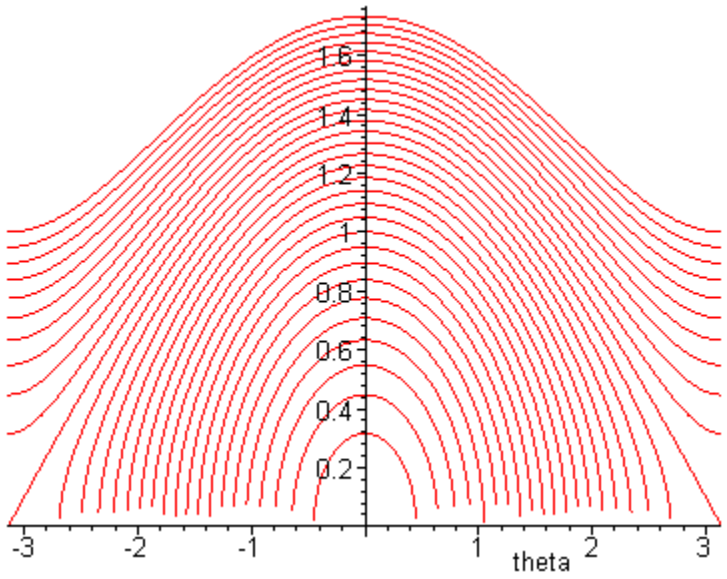
$$p_\theta^2 = E' - 2\sin^2(\theta/2)$$

This means that, horizontally, what you see in the range -π,π for θ will just repeat in windows to the left and right. Because there are ± square roots, the picture is reflected in the θ axis, so I will only plot it above this axis. If we just name E' to be E, and set A = 1, we get

$$p_\theta = +\sqrt{E'-2\sin^2(\theta/2)}$$

If E < 1, we get one kind of curve, and it E > 1 we get another. In this plot, I will run E in steps of 0.1 from .1 up to 3.0:

```
plot([seq((E/10 - 2*sin(theta/2)^2)^(1/2),E = 1..30)],theta = -Pi..Pi,color
=[seq(red,E = 1..30)], numpoints=1000);
```



So this is the famous picture! See Marion page 164 or Goldstein page 290. My 10th curve from the inside has E=1 and you see it is the critical curve as shown better in Marion.

As you run the pendulum with different E, you will loop on the different phase space orbits shown, or you will do rotation. The larger orbits have longer periods as we know already, and as we will probably find soon from Hamilton-Jacobi theory if we ever get to it with this pendulum which has consumed multiple days of time so far.

13. Hamilton-Jacobi theory applied to this problem.

From Section 2 above we know that

$$H(q_i, p_i) = H(\theta,p_\theta) = p_\theta^2/(2m\ell^2) + mg\ell(1-\cos\theta) \quad = \text{Hamiltonian.} \quad p_\theta = m\ell^2\dot{\theta}$$

so the HJE says this (using the fixed energy $E = \alpha_1$ situation)

$$H(\theta, p_\theta = \partial W/\partial\theta, p_\theta = \alpha_\theta) = \alpha_1 = E$$

which we write out as

$$(\partial W/\partial\theta)^2/(2m\ell^2) + mg\ell(1-\cos\theta) = E$$

On scratch paper I fiddle this to get

$dW = A\sqrt{(E'-1) + \cos\theta}\ d\theta \qquad A = \sqrt{2}\, m\ell^2\omega \qquad E' = E/mg\ell$

$W = A \int d\theta \sqrt{(E'-1) + \cos\theta}$

$\partial W/\partial\alpha_1 = (1/\sqrt{2}\,\omega) \int d\theta\ 1/\sqrt{(E'-1) + \cos\theta}$ " second integral"

To do the second integral we go to our same GR integrals on p 154

$\int_{}^{\theta} d\theta\ [1/\sqrt{\cos\theta + a}\ ] = \sqrt{2}\ F(\sin^{-1}\frac{\sin(\theta/2)}{\sin(\theta_0/2)}), \sin(\theta_0/2))$ // libration form

$= \sqrt{2}\ \csc(\theta_0/2)\ F[\ \theta/2, \csc(\theta_0/2)\ ]$ // rotation form

where for now θ_0 represents the unknown constant of integration.

Then according to page 287 we should have

$t + \beta_1 = \partial W/\partial\alpha_1 = (1/\sqrt{2}\,\omega)\sqrt{2}\ F(\sin^{-1}\frac{\sin(\theta/2)}{\sin(\theta_0/2)}), \sin(\theta_0/2))$

$= (1/\omega)\ F(\sin^{-1}\frac{\sin(\theta/2)}{\sin(\theta_0/2)}), \sin(\theta_0/2))$

$= (1/\omega)\ \csc(\theta_0/2)\ F[\ \theta/2, \csc(\theta_0/2)\ ] \qquad k = \csc(\theta_0/2) = 2\omega/v$

which we recognize as our libration solution from above, with $\beta_1 = 0$ for our initial conditions.

Just for fun, what is the W function itself?

$W = A \int d\theta \sqrt{(E'-1) + \cos\theta} \qquad A = \sqrt{2}\, m\ell^2\omega$

We can use GR page 156 with a = E'-1 and b = 1 to get and a+b = E'

$W(\theta, E) = A\ 2\sqrt{E'}\ \ E(\theta/2, r{=}\sqrt{2/E'} = \csc(\theta_0/2)\ )$ // see below on last part

$= 2\sqrt{2}\ \sqrt{m\ell^2}\ \sqrt{E}\ \ E(\theta/2, \csc(\theta_0/2)) = B\sqrt{E}\ E(\theta/2, \csc(\theta_0/2)) \qquad B = 2\sqrt{2}\ \sqrt{m\ell^2}$

which is a second-kind elliptic integral function.

Now we should be able to differentiate this to get our previous result. In this set of equations, we shall use $k = \csc(\theta_0/2)$:

$E = \frac{1}{2}\, m\ell^2v^2 \qquad$ where $v = 2\omega \sin(\theta_0/2) \Rightarrow \qquad E = 2mg\ell \sin^2(\theta_0/2)$

so that $\sin^2(\theta_0/2) = E/(2mg\ell) = E'/2 = k^{-2}$ => $2mg\ell = Ek^2$ and $E = 2mg\ell k^{-2}$

=> $0 = E2kdk + dEk^2$ => $dk/dE = -k/2E$

and $k = \csc(\theta_0/2) = \sqrt{2/E'}$

Then we get

$\partial W(\theta, E)/\partial E = B[\sqrt{E}\ \partial E(\theta/2, k)/\partial k * \partial k/\partial E + (1/(2\sqrt{E}))\, E(\theta/2,k)\ ]$ $k = \csc(\theta_0/2)$

For the derivative, we consult GR p age 907 8.123.3 which says $\partial E/\partial k = (E-F)/k$ so we get

$\partial W(\theta, E)/\partial E = B[\sqrt{E}\ \partial E(\theta/2, k)/\partial k * \partial k/\partial E + (1/(2\sqrt{E}))\, E(\theta/2,k)\ ]$

$= B[\sqrt{E}\ [(E(\theta/2,k)-F(\theta/2,k))/k] * [-k/2E] + (1/(2\sqrt{E}))\, E(\theta/2,k)\ ]$

$= B[-(1/(2\sqrt{E}))[(E(\theta/2,k)-F(\theta/2,k))] + (1/(2\sqrt{E}))\, E(\theta/2,k)]$

$= B(1/(2\sqrt{E}))\, F(\theta/2,k) = (k/\omega)\, F(\theta/2, k = \csc(\theta_0/2))$

which agrees exactly with our previous result. Just more exercise in use of elliptic formulas.

14. Action-angle variables applied to this problem.

Goldstein 9-32 page 290 gives the left equation here, and from above I keep going:

$p_\theta = \partial W/\partial\theta = A\sqrt{(E'-1) + \cos\theta}$ $A = \sqrt{2}\, m\ell^2\omega$ $E' = E/mg\ell$

But inside the square root we have

$E' - (1-\cos\theta) = E' - 2\sin^2(\theta/2)$ => $p_\theta = \partial W/\partial\theta = A\sqrt{E' - 2\sin^2(\theta/2)}$

We have seen in Section 12 that this provides the characteristic phase space graphic. In our current notation for A and E' (different from what I used earlier), the critical energy is E' = 2.

Notice that $A^2E' = 2m\ell^2E$ and reproduces our first term in our previous p_θ^2 expression. Similarly, notice that $2A^2 = (2m\ell^2\omega)^2$ which reproduces our second coefficient, so our result here is exactly the same as the previous result in Section 12 above.

Now let's compute the "action" variable J_θ [remember: this is the area inside the phase space loop]

$J_\theta = \oint d\theta\, (\partial W/\partial\theta) = \oint d\theta\, [A\sqrt{(E'-1) + \cos\theta}\ ]$ $A = \sqrt{2}\, m\ell^2\omega$ $E' = E/mg\ell$

We know the curve is symmetric, so the area is 2 times the integral 0 to π (see G p 290)

$$J_\theta = 2A \int_0^\pi d\theta \sqrt{(E'-1) + \cos\theta} \;\; = 4A \sqrt{E'}\, E(\pi/2, k) \; = 4A\sqrt{E'}\;\; \mathbf{E}(k) \qquad k = \sqrt{2/E'} \; = \csc(\theta_0/2)$$

where **E** is the complete second kind elliptic function.

Next we want to find the "angle" variable which is

$$w \equiv \partial W/\partial J_\theta \; = \partial W/\partial E' * \partial E'/\partial J_\theta \; = \; \partial W/\partial E' \,/\, \partial J_\theta \,/\partial E'$$

Here is what we know so far:

$$W(\theta, E) = 2A\sqrt{E'}\;\; E(\theta/2,k) \qquad k = \sqrt{2/E'} \; = \csc(\theta_0/2) \qquad A = \sqrt{2}\; m\ell^2\omega \qquad E' = E/mg\ell$$

$$J_\theta \qquad = 4A\sqrt{E'}\;\; \mathbf{E}(k) \;\; = 2W(\pi/2, E)$$

So we can compute the required derivatives as follows:

$$\partial W/\partial E' \; = 2A\, [\sqrt{E'}\;\; \partial E(\theta/2,k)/\partial k * \partial k/dE' \; + (1/2\sqrt{E'}\,)\, E(\theta/2,k)\,]$$

But we know that $k = \sqrt{2/E'}$ so $k^2E' = 2$ so $\partial k/dE' = -\,k/(2E')$. Also we look up GR page 197 which tells us that $\partial E(\theta/2,k)/\partial k = [(E-F)/k]$ so we get

$$\partial W/\partial E' \; = 2A\, [\sqrt{E'}\, [(E-F)/k] * -k/(2E') + (1/2\sqrt{E'}\,)\, E(\theta/2,k)\;\;]$$

$$= \; 2A(1/2\sqrt{E'}\,)\, [[-\,(E-F)] \; + \; E(\theta/2,k)\;\;] \; = (A/\sqrt{E'}\,)\, F(\theta/2,k)$$

$$= (A/\sqrt{E'}\,)\, F(\theta/2,k)$$

The other derivative is the same as what we just did except it has $\theta = \pi$ and factor 2, so get

$$\partial J_\theta \,/\partial E' = \; (2A/\sqrt{E'}\,)\, F(\pi/2,k) \; = (2A/\sqrt{E'}\,)\; \mathbf{K}(k)$$

Thus it appears that our angle variable is this: [all this time $k = \sqrt{2/E'} \; = \csc(\theta_0/2)$]

$$w_\theta = \; (A/\sqrt{E'}\,)\, F(\theta/2,k) \,/\, (2A/\sqrt{E'}\,)\; \mathbf{K}(k) \; = \; ½ \; F(\theta/2,k)/\; \mathbf{K}(k) \; = \; ½ \; F(\theta/2,k) \,/\, F(\pi/2,k)$$

which is mildly interesting. But now it is supposed to be true that

$$\nu_\theta = \dot{w}_\theta$$

In our *rotation* solution above, we found that

$$\omega t/k = (v/2)t \; = F(\theta/2,k) \qquad\qquad k = (2\omega/v) = \csc(\theta_0/2)$$

so let's now use this in our last result

$$w_\theta = \tfrac{1}{2}\ F(\theta/2,k)/\ \mathbf{K}(k)\ = \tfrac{1}{2}\ (v/2)t\ /\mathbf{K}(k)\ = (v/4)\ t\ /\mathbf{K}(k)$$

Then we find that

$$\nu_\theta = \dot{w}_\theta\ = (v/4)/\mathbf{K}(k)\ = \nu_{rot}$$

so we are consistent with our previous calculation in the rotation section above.

Summary of this little section:

- The W function for this problem is a second-kind elliptic function,

$$W(\theta,E)\ = A\int_0^\theta d\theta\ \sqrt{(E'-1) + \cos\theta}\ = 2A\sqrt{E'}\ E(\theta/2,k\ ) \qquad \text{where}$$

$$A = \sqrt{2}\ m\ell^2\omega \qquad E' = E/mg\ell \qquad k = k = \sqrt{2/E'}\ = \csc(\theta_0/2)$$

The k value here is appropriate for the "rotation" regime of the pendulum where E' > 2 so the integrand is always real and k is always < 1. You could express this in a form appropriate for libration by using this identity which I derived from the GR page 908 transformation table,

$$E(\varphi,k) = k\ F(k\sin\varphi,\ 1/k) + (1-k^2)/k\ *\ F(k\sin\varphi,\ 1/k) \qquad \text{with } \varphi = \theta/2$$

but the other form is much simpler so we stick with it.

- We can compute the action and the angle variables and we find that

$$J_\theta = 2A\int_0^\pi \sqrt{(E'-1) + \cos\theta}\ \ = 4A\ \sqrt{E'}\ E(\pi/2,\ k\ )\ = 4A\sqrt{E'}\ \ \mathbf{E}(k) \qquad k = \sqrt{2/E'}$$

$$w_\theta = \ \tfrac{1}{2}\ F(\theta/2,k)/\ \mathbf{K}(k)\ =\ \tfrac{1}{2}\ F(\theta/2,k)\ /\ F(\pi/2,k)$$

Remember that we did a canonical transformation from θ, p_θ to w_θ, J_θ which is a particular case of doing the HJ transformation with a W. Notice that J_θ is a constant since E' is a constant.

- Instead of what we did above, at this point we could *use the HJ knowledge that* $v_\theta = \dot{w}_\theta$ = *constant*, and then we would say [this is going to lead us to a solution of the problem!]

$$\nu_\theta = \dot{w}_\theta = (1/2\mathbf{K}(k)\ )\ \ \partial F(\varphi,k)/\partial\varphi\ |^{\varphi=\theta/2}\ *\ \partial\varphi/\partial t|^{\varphi=\theta/2}$$

But from the definition of the integral F, we know that

$$\partial F(\varphi,k)/\partial\varphi\,|^{\varphi=\theta/2} = 1/\Delta\varphi|^{\varphi=\theta/2} = 1/(\sqrt{1 - k^2\sin^2(\theta/2)})$$

$$\partial\varphi/\partial t|^{\varphi=\theta/2} = \tfrac{1}{2}\,\dot{\theta}(t)$$

to get that

$$\nu_\theta = \dot{w}_\theta = (1/2\mathbf{K}(k))\;\; 1/\sqrt{1 - k^2\sin^2(\theta/2)} \;\; * \tfrac{1}{2}\,\dot{\theta}(t)$$

which we could then solve for $\dot{\theta}(t)$ to get

$$\dot{\theta}(t) = 4\,\nu_\theta\,\mathbf{K}(k)\sqrt{1 - k^2\sin^2(\theta/2)}$$

$$\Rightarrow \int_0^\theta d\theta\; 1/\sqrt{1 - k^2\sin^2(\theta/2)} = 4\,\nu_\theta\,\mathbf{K}(k)\,t$$

Change variables to $\varphi = \theta/2$ in the integral so it becomes

$$2\int_0^{\theta/2} d\varphi\; 1/\sqrt{1 - k^2\sin^2\varphi} = 2\,F(\theta/2,k) \qquad \text{GR page 904}$$

Then we have as our problem's solution

$$[2\,\nu_\theta\,\mathbf{K}(k)]\;t = F(\theta/2,k)$$

Looking up 5 equations, we could look at t=0 where we have θ=0 so

$$\nu_\theta = (1/2\mathbf{K}(k))\;*\;1\;*\;\tfrac{1}{2}\,\dot{\theta}(0) = v/(4\mathbf{K}(k))$$

which tells us that

$$[2\,\nu_\theta\,\mathbf{K}(k)] = v/2$$

so our problem solution is then

$$vt/2 = F(\theta/2,k)$$

which we can compare to our known rotational solution which is

$$\omega t/k = (v/2)t = F(\theta/2,k) \qquad k = (2\omega/v) = \csc(\theta_0/2)$$

which is the same! So in this way, we have used the action-angle variables version of the Hamilton-Jacobi method to find the solution to our pendulum problem. We never had a non-linear differential equation which we had to solve. This is really a different method of solution!

Comment on the rotational case.

At the top of the rotation we have $\theta = \pi$. Our motion equation says

$$\dot{\theta}(t) = 4\,\nu_\theta\,\mathbf{K}(k)\sqrt{1 - k^2\sin^2(\theta/2)} \quad = v\sqrt{1 - k^2\sin^2(\theta/2)}$$

Velocity at the top is then $\dot{\theta}_{top} = v\,k'$.

$T_{top} = \tfrac{1}{2}\,m\,\ell^2\,\dot{\theta}^2 = \tfrac{1}{2}\,m\,\ell^2\,v^2\,(1-k^2)$ $\qquad k = 2\omega/v$
$V_{top} = 2mg\ell$

$T_{bottom} = \tfrac{1}{2}\,m\,\ell^2\,v^2$
$V_{top} = 0$

$T_{top}/V_{top} = (1-k^2)/k^2 = \cot^2\alpha$ $\qquad$ if $k = \sin\alpha$

Here is a possible interpretation:

$$\dot{\theta}_{top}/\dot{\theta}_{bot} = k' = \cos\alpha$$

Conclusion: during rotation mode, I can find no "angle" in the picture which has any correspondence with k. I was wondering about my idea of $k = \csc(\theta_0/2)$. This is just not something useful.

15. What are the classical waves doing in this problem?

The action S is following surfaces such that $W(\theta,z,E) = Et$ as t increases. So we need to solve this for φ to see what the surfaces look like.

$$W(\theta,E) = 2A\sqrt{E'}\;E(\theta/2,k)$$

But what is the inverse of the E function? This is not obvious from a quick look, it is not sn or cn or dn, but of course it is "well defined" and here then are our surfaces:

$$2A\sqrt{E'}\;E(\theta/2,k) = Et$$

As t increases, we get some $\theta = f(t,E)$. This is a flat surface at angle $\theta(t)$ which stretches flat into the z direction. As t increases, these surfaces move sort of perpendicular to the pendulum mass, just as you would expect. Show them with your right hand. We could compute the wave velocity u, and we know the

wavelength is $\lambda = h/mv$. The waves don't keep up with the mass. Easier to study this for a free particle as done in Goldstein, I just thought I would mention it here.

16. What is the solution to the quantum plane pendulum problem? It would have bound states for low E. Not clear how you would tie a string onto a non-localized mass, so I don't know how you would even set this problem up. I guess the string follows the same ψ as the mass.

Setup. By setting up in terms only of angle θ as in the classical case, we are saying "there is a string". The TI SE Says this: (Laplacian in cylindricals in effect has that $1/r^2$ out frong.

$$(1/\ell^2)\,\partial^2\psi(\theta)/\partial\theta^2 + A^{-1}(E-V)\psi(\theta) = 0 \qquad A^{-1} = 2m/\hbar^2 \qquad V(\theta) = mg\ell(1-\cos\theta)$$

$$(A/\ell^2)\,\partial^2\psi(\theta)/\partial\theta^2 + (E-V)\psi(\theta) = 0$$

Let $\theta = x$ for the moment and write this as

$$-(A/\ell^2)\psi_{xx} + mg\ell(1-\cos x)\psi = E\psi \qquad E' = E/mg\ell$$

Define $B = A/mg\ell^3 = \hbar^2/(2m^2g\ell^3)$ and use same $E' = E/mg\ell$ as in the classical stuff above, so then

$$-B\psi_{xx} + (1-\cos x)\psi = E'\psi$$

Notice that B and E' are dimensionless. The angular momentum of the pendulum going at its basic ω rate would be $L = rp = rmv = \ell m\ell\omega = \ell^2 m\sqrt{g/\ell}$ so that $L^2 = m^2\ell^3 g$ which is what appears in the denominator of B, and of course $\hbar$ also has dimensions of L.

This is linear in ψ, so a linear ODE. Compare to $\theta_{tt} = -\omega^2\sin\theta$ we had before which was "auto". Change from θ to ψ and t to x to get $\psi_{xx} = -\omega^2\sin\psi$ which is "autonomous". So now we have a completely different equation. Write our new equation as

$$-B\psi_{\theta\theta} = [(E'-1) + \cos\theta]\,\psi$$

Maybe this helps to write it using x = t:

$$-B\ddot{\psi} = [(E'-1) + \cos(t)]\,\psi(t) = f(t)\psi$$

So how do you go about solving a "new" ODE that you are not familiar with? Could always do power series then try to find the series. Then end up with mysterious recursion relation. How about our old friend Maple! Well, Maple cannot do it. You can turn on the info level and see it try. Here is the Maple output:

```
> restart;
> with(DEtools):

> infolevel[dsolve]:= 3;
> diff_eq1 := B*D(D(y))(x) + (C + cos(x))*y(x) = 0;
```

$$diff_eq1 := B\,(\mathrm{D}^{(2)})(y)(x) + (C + \cos(x))\,y(x) = 0$$

```
> init_con := y(0)=1, D(y)(0)=1;
```

$$init_con := y(0) = 1, \mathrm{D}(y)(0) = 1$$

```
> dsolve( {diff_eq1, init_con} , {y(x)} );

Methods for second order ODEs:
trying to isolate the derivative d^2y/dx^2
successful isolation of d^2y/dx^2
 -> trying classification methods
trying a quadrature
trying a symmetry of the form [xi=0, eta=F(x)]
dsolve/diffeq/plus_min_expsols:   checking sin-cos solver
dsolve/diffeq/linearODE:   checking special function equation
dsolve/diffeq/linearODE:   checking special function equations: II
dsolve/diffeq/plus_min_expsols:   checking sin-cos solver
dsolve/diffeq/polylinearODE:   checking Euler equation
dsolve/diffeq/secorder:   checking for special function equations
dsolve/diffeq/secorder:   checking special function equations: II
dsolve/diffeq/secorder:   checking hypergeometric equation
DEtools[expsols]   trying exponential solutions
dsolve/diffeq/secorder:   checking Riemann-Papperitz equation
DEtools[expsols]   trying exponential solutions
trying 2nd order linear exact
trying symmetries linear in x and y(x)
trying reduction of order to Riccati
trying Riccati sub-methods
trying 2nd order reducible2
 -> trying 2nd set of classification methods
 -> trying Lie symmetry methods
```

So here we are again with

$$B\,\ddot{\psi} + [\,(E'-1) + \cos(t)]\,\psi(t) = 0 \qquad B = A/mg\ell^3 = \hbar^2/(2m^2g\ell^3) \qquad E' = E/mg\ell$$

I have no idea what to do! I look up "SE" and pendulum web, was led to this fact:

Mathieu equation

The canonical form for **Mathieu's differential equation** is

$$\frac{d^2u}{dx^2} + [a_u - 2q_u\cos(2x)]u = 0$$

That was a good lead, GR has a whole section on this guy starting page 991. Page 993 shows "negative q" which is our case. In fact, here is what GR show

$\psi_{tt} + (a + 2q\cos(2t))\psi = 0$

If I define t' = 2t so that $\psi_t = 2\psi_{t'}$ etc, then we have

$4\,\psi_{t't'}(t') + (a + 2q\cos(t'))\psi(t') = 0$

Define a' and q' as a/4 and q/4 so we then have

$\psi_{t't'}(t') + (a' + 2q'\cos(t'))\psi(t') = 0$ then set t' back to t, arbitrary

$\psi_{tt}(t) + (a' + 2q'\cos(t))\psi(t) = 0$

But this is our pendulum SE where we have a' = (E' - 1)/B and 2q' = 1/B. Therefore, here is our claim.

"The pendulum SE

$B\,\psi_{tt} + [\,(E'-1) + \cos(t)]\,\psi(t) = 0$

is really the Mathieu equation,

$\psi_{tt} + (a + 2q\cos(2t))\psi = 0$

with these parameters

$a = (E' - 1)/B$ and $2q = 1/B$

and

$B = A/mg\ell^3 = \hbar^2/(2m^2g\ell^3)$ $E' = E/mg\ell$

so we have dimensionless q and a as follows,

$q = m^2g\ell^3/\hbar^2 = 1/(2B)$ $a = 2q(E' - 1)$

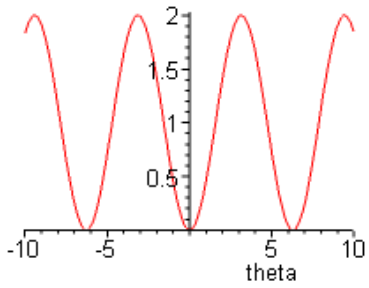
end of claim. "

Then on GR page 993 we see solution functions called ce and se. It is noted that the periodic solutions exist for certain eigenvalues of a. Each solution is presented as a Fourier sum of cosines or sines, hence the names ce and se. There is a label 2n and 2n+1 which is the usual kind of thing.

Since I have the TISE with t really θ, periodic would mean periodic in θ, not time.

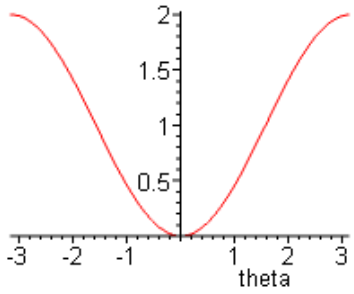
Meanwhile, here is what the potential V(θ) looks like:

```
plot(1-cos(theta),theta = -10..10);
```



I am not used to this kind of potential. Since it is periodic, I guess you would expect the Mathieu solution to be periodic. But in our physical system, we really have this

```
plot(1-cos(theta),theta = -Pi..Pi);
```



But we can think of θ going up forever in the rotation solution, and then the first does apply! The second applies for the libration solutions of lower energy. That would be bound states in this well. The rotation solution I guess would be a wave packet perhaps moving to the right through the wells! Very good.

A&S also have a section on Mathieu functions page 722. The parameters are a and q, and page 724 shows a plot of the eigenvalues a_r as functions of q for the various solutions "r".

" The French mathematician E.L. Mathieu investigated this equation in 1868 while seeking a description of the vibrations of an elliptical membrane (drumheads)."

I finally found something on the web that at least addresses this problem, a PDF, from which I quote

An interesting one-dimensional example where the AMF occur is the quantum pendulum.[14–16] Consider a plane pendulum of length L and mass M oscillating under the action of gravity. The time-independent Schrödinger equation corresponding to this problem is

$$-\frac{\hbar^2}{2ML^2}\frac{d^2\Psi}{d\theta^2}+[V(\theta)-E]\Psi=0, \tag{7}$$

where θ is the angular displacement from the vertical, $V(\theta)=-MgL\cos\theta$ is the potential energy, and $\Psi(\theta)$ is the wave function associated with the energy E. The boundary condition to be imposed on Ψ is that it be single valued in θ, that is, $\Psi(\theta)$ has period 2π: $\Psi(\theta+2\pi)=\Psi(\theta)$. Equation (7) can be rewritten in the form of an angular Mathieu equation (3), by defining

$$\theta=2\eta, \tag{8a}$$

In the above, AMF refers to "angular Mathieu functions", and the text continues:

$$a=\frac{4E}{(\hbar^2/2ML^2)}, \tag{8b}$$

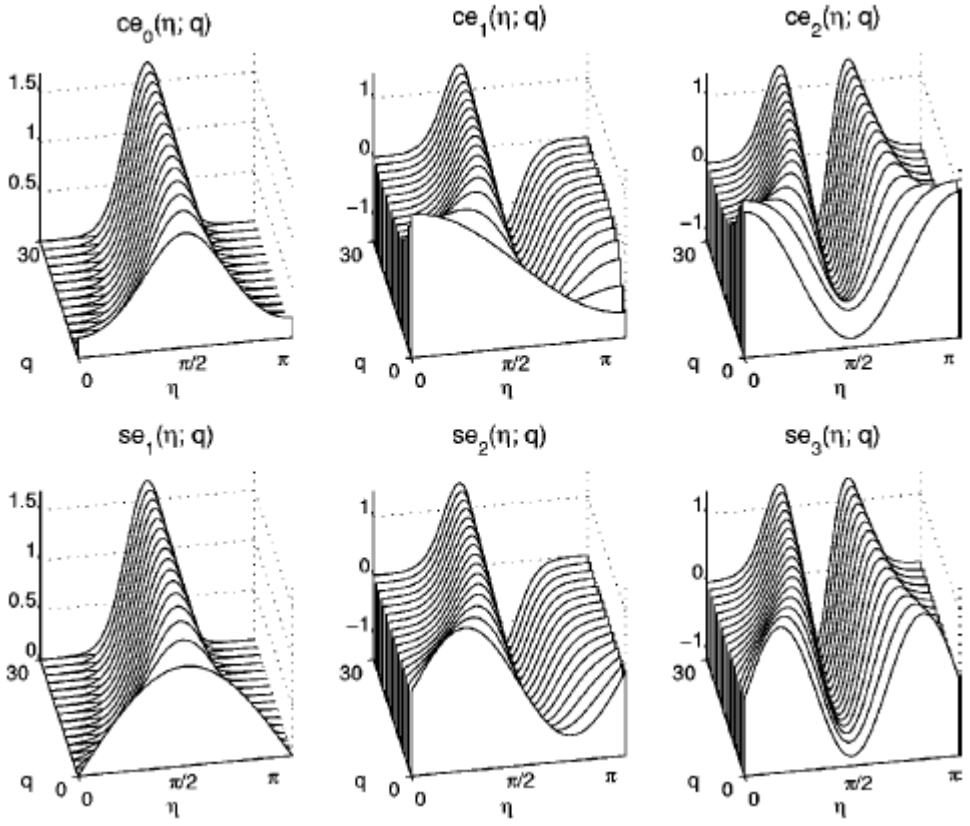
$$q=-\frac{2MgL}{(\hbar^2/2ML^2)}. \tag{8c}$$

In view of Eq. (8a), the boundary condition is $\Psi[2(\eta+\pi)]=\Psi[2\eta]$, that is, as a function of η, the wave function has to be periodic with period π. From Table I, such solutions are AMF of even order: $\mathrm{ce}_{2r}(\eta;q)$ and $\mathrm{se}_{2r+2}(\eta;q)$ for $r=0,1,2,\ldots$. All other AMF are excluded by the periodicity condition. We note further that characteristic values of the energy E_m are defined by the eigenvalues a_m of the Mathieu equation; from Eq. (8b) we find $E_m(\hbar^2/8ML^2)a_m$.

Also, we can see from Eq. (8c) that, for the quantum pendulum, the parameter q depends only on the given physical constants of the problem, and that Eq. (7) really corresponds to a Mathieu equation with negative q. To satisfy the usual definition of the Mathieu functions, we must perform the change of variable $\eta\rightarrow(\pi/2-\eta)$ according to Table I.

When visualizing mathematical functions, a suitable plot depends on the purpose. For example, the surfaces shown in Fig. 2 were done following a mathematical point of view. These diagrams allow us to appreciate at a glance the evolution of ce and se as functions of η or q. In Fig. 3 we adopt a physical point of view to show the probability distributions Ψ^2 of the quantum pendulum. As stated, the wave functions are written in terms of AMF of even order, $\mathrm{ce}_{2r}(\eta;q)$ and $\mathrm{se}_{2r+2}(\eta;q)$. We might plot these solutions against the polar coordinate θ in a rectangular system (θ,Ψ^2), but a more meaningful picture is usually a polar diagram. In Fig. 3, we can visualize how the probability distributions of a quantum pendulum vary with θ.

I think you need to have eigenvalues in order for the solutions to have the right periodicity to match the π periodicity of the potential in this problem. That sets the eigenenergies, and then you can look at the lowest states, here are some graphs from the same paper



Each graph is shown for several values of parameter q. The angle $\eta = \theta/2$ is zero at the TOP of the pendulum (my classical analysis had it at the bottom.). So the bottom is $\theta = \pi$ or $\eta = \pi/2$. The top left graph shows the ground state ce_0 which has probability peaking at the bottom position. Then the next state up has none at the bottom, and some on the two sides. As usual, the number of nodes increases with the index. The se functions are also solutions, and I don't understand why things are doubled. The se are odd if you go to the - angle region and the ce are even, but not sure we care about that region. I suppose all these solutions exist and are different. The AS graph shows a_0 and b_1 are very close.

"for ce_m the eigenvalues are usually denoted as $a_m(q)$, whereas for se_m they are represented as $b_m(q)$."

Summary: The quantum plane pendulum can be studied using the TISE with ∇^2 in cylindrical coordinates which just means 1D with an extra $1/\ell^2$. The differential equation which results is completely different from that which arises in the classical plane pendulum problem:

$\theta_{tt} = -\omega^2 \sin\theta$ // classical solutions are Elliptic functions

$\psi_{tt} + (a' + 2q' \cos(t))\psi = 0$ // quantum solutions are Mathieu functions

As expected, the TDSE has eigenstates with the "usual shapes" in the θ space. We would then construct the classical solutions by taking wave packets formed from these states. At low energy, these packets would bounce back and forth, emulating the classical librating pendulum. At higher energy, these packets

would skid along the potential to ever-increasing θ, emulating the rotational solutions. So yes, this is a pretty good problem to compare classical and quantum solutions. It corresponds to a real physical situation, unlike particles in boxes.