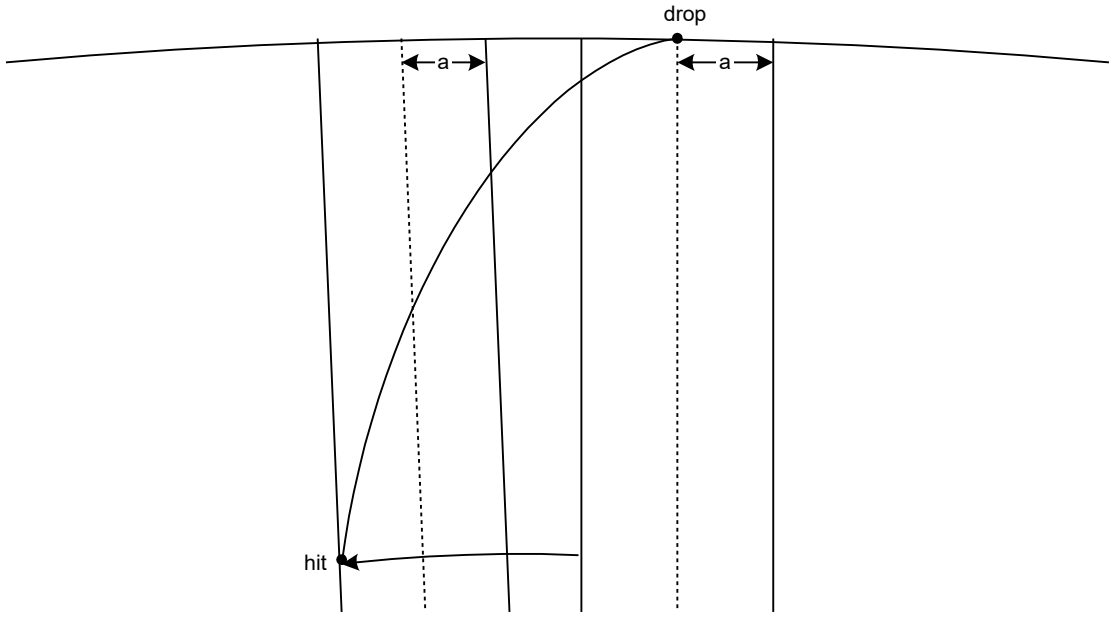
The Problem of Dropping a Rock in a Well (Amended)

Here is the picture of interest. We assume that the rock hits the wall at time $t = t_1$ and that the hit spot is a distance r_1 from earth center. The well radius is a. The velocity of a point on the well wall that is r from the center of the earth is $v(r) = 2\pi r/T$ and R = radius of earth. Gravity g is assumed constant.



We now hop onto the rock and solve the problem from the frame of reference of a gnat riding down with the rock. At t=0 the gnat sees the left wall moving to the left at zero velocity (relative to the gnat), but by the time the rock hits the wall, the relative velocity is $v(R) - v(r_1)$. The claim is that we compute the horizontal distance the rock travels in time t_1 and set that equal to a, the radius of the well. So

$$\int_0^{t_1} dt\, [\, v(R) - v(r(t))] = a \qquad (1)$$

Since $v(r) = 2\pi r/T$, this becomes

$$(2\pi/T) \int_0^{t_1} dt\, [R - r(t)] = a \qquad \Rightarrow \qquad \int_0^{t_1} dt\, [R - r(t)] = aT/(2\pi)$$

But we know that $R - r(t) = (1/2)gt^2$ because that is what falling rocks do. So

$$aT/(2\pi) = \int_0^{t_1} dt\, [R - r(t)] = \int_0^{t_1} dt\, [(1/2)gt^2] = (1/2)g\,(1/3)t_1^3$$

We then solve for t_1 to get

$$t_1^3 = 3aT/(\pi g) \qquad (2)$$

This is the answer accepted by the website, but it is the wrong answer. The reason it is wrong is that the gnat is in an accelerating frame of reference, and the naive laws of physics don't apply in such reference frames. The fact that the gnat is accelerating downward is not the problem. The problem is that the gnat's frame of reference is *rotating* as the earth rotates, more on this below.

So let's solve the problem in a fixed frame of reference. In this frame of reference, the rock has velocity v(R) to the left at t = 0 and in fact at all subsequent times t, right up to $t = t_1$ when it hits the wall. The rock starts with this velocity at time t=0 and nothing causes its horizontal velocity to change (no air friction for example). Meanwhile, the spot where it will hit the wall is moving to the left at velocity $v(r_1)$, if we assume the hit occurs distance r_1 from earth center. This hit-spot on the well wall moves to the left at this same speed all the while from t = 0 to $t = t_1$. [The earth is so large that we ignore the fact that this point on the wall is actually moving down in our picture slightly as it moves to the left. In this problem, that small vertical distance is about 4 feet which we can compare to the hit distance down which is more than 1000 feet.]

So we can write expressions for the horizontal position of rock and the hit-spot:

$$x_{rock}(t) = - v(R)t \qquad x_{spot}(t) = -v(r_1)t - a$$

The x-position of our coordinate system's origin is chosen at the rock drop point. That is why $x_{rock}(0) = 0$ and why $x_{spot}(0) = -a$. (The hit spot does lie on the left wall).

The rock hits the wall when $x_{rock}(t) = x_{spot}(t)$, which occurs at $t = t_1$, so we then get

$$[V(R) - v(r_1)] t_1 = a \qquad (3)$$

This has a simple interpretation. If you rode on a (non-rotating) reference frame moving v(R) to the left along with the top of the well, you would say that the hit spot was moving to the right at $[V(R) - v(r_1)]$ and in time t_1 it travels distance a and hits the rock. Notice how equation (3) is very different from (and simpler than) equation (1) above. We rewrite (3) using $v(r) = 2\pi r/T$ to get

$$(R - r_1) t_1 = aT/(2\pi)$$

But we know that $R - r_1 = (1/2)g t_1^2$, the distance the rock drops , so we have

$$[(1/2)g t_1^2] \ t_1 = aT/(2\pi)$$

which we can solve for t_1 to get

$$t_1^3 = aT/(\pi g) \qquad (4)$$

This is similar to answer (2), but does not have the 3 in the numerator. That 3 came from the integration of t^2 in the first solution method.

From a horizontal perspective, the gnat's rotating frame of reference is the same as a frame glued to the surface of the earth (on the equator) at the rock drop point. To correct for the fact that the frame is rotating, one has to add a "fictitious" Coriolis force which appears to push projectile objects to the east. One can think of shooting a long range mortar shell from the equator toward the north pole. The shell has a horizontal velocity to the east when launched, but the target has a lesser velocity to the east because it is closer to the pole (where there is no horizontal velocity). Thus, the fired shall lands to the east of where it was aimed, and this "problem" is explained by adding a fictitious force pushing the projectile to the east. In our rock problem, we are (so to speak) firing a projectile not to the north, but toward the center of the earth. The landing spot still has a lesser velocity than the launching spot, so the projectile (our rock) lands to the east of where we naively compute. So there is a mysterious force pushing it to the east that we have not accounted for in our "gnat" analysis, and this causes the rock to hit the east wall of the well sooner than we otherwise think. Sooner by $(3)^{1/3} = 1.44$. This causes the rock to drop much less further than it does without including the Coriolis force.

In the northern hemisphere, mortar shells veer to the right (east), so you have to aim to the left. If we shoot from the equator toward the south pole, shells veer to the left (east), so we have to aim to the right. I think I have this right now....