Flywheels PhL 11.4.04

Questions: Is it practical to store energy in a rotating flywheel? If so, what is the best geometry and material for such a flywheel? Could a flywheel be used to store energy from a windmill at an isolated home location (like Torrey)? What is the cost of such a flywheel? Can these things be used in electric cars? How does a flywheel compare to normal batteries, or to fuel-cell technology? None of these questions are answered here, but the basic facts are stated for the reader's possible interest.

The Physics.

(1) The kinetic energy of a *linearly moving* object is $1/2\ mv^2$ where m = mass, v = velocity.

The kinetic energy of a *rotating* object is $1/2\ I\ \omega^2$ where I = "moment of inertia" and ω = "angular velocity". The two formulas are analogous, with $m \rightarrow I$ and $v \rightarrow \omega$. You can regard the kinetic energy of an object as energy being "stored" in that object.

If an object rotates at a frequency F, then $\omega = 2\pi F$. For example, if F = 100 times/second, then ω = 628 radians per second. A circle has 2π radians.

(2) The moment of inertia I is something you have to calculate for a rotating object of general shape. It is given by this integral:

$$I = \int \mathbf{r}^2 [\ \rho(\mathbf{r})\ dV\ ]$$

where $\mathbf{r}$ = some location in the object, $\mathbf{r}^2$ is the square of the distance of this location *from the rotation axis*, and dV is a small differential volume element around the point $\mathbf{r}$, and the integral is over the entire 3D object. The thing $\rho(\mathbf{r})$ [rho] is the density at the point $\mathbf{r}$ perhaps in kg/m^3.

It is easy to compute this integral for symmetrical shapes. The most popular shape is a hollow cylinder of uniform (ρ = constant) material. If the inside radius is a, and the outside radius is R, one calculates I as follows:

$dV = 2\pi r\ dr\ h$ // ring at radius r, thickness dr, height h

$$I = \int \mathbf{r}^2\ \rho(\mathbf{r})\ dV = \rho \int \mathbf{r}^2 [\ 2\pi r\ dr\ h\ ] = 2\pi\rho h \int_a^R r^3\ dr = 2\pi\rho h[\ R^4 - a^4]/4$$

$$= (\pi\rho h/2)\ [\ R^4 - a^4] = (\pi\rho h/2)\ [\ R^2 - a^2][\ R^2 + a^2]$$

The volume of material in this hollow cylinder is

$$V = \int dV = \int [\ 2\pi r\ dr\ h\ ] = 2\pi\ h \int_a^R r\ dr = 2\pi\ h\ [\ R^2 - a^3]/2 = \pi\ h\ [\ R^2 - a^2] = V$$

so the conclusion is that

$$I = (\rho V/2)\,[\,R^2 + a^2\,] = (M/2)\,[\,R^2 + a^2\,] = MR^2\,[\,1 + a^2/R^2\,]/2$$

Two limits of this result are interesting. If a = R, we are talking about a thin rim of radius R, and the result is $I = MR^2$. On the other hand, if a = 0, we are talking about a solid cylinder, and $I = (1/2)MR^2$. The above formula can be restated as:

$$I = f\,MR^2 \qquad f = \text{"factor"} = [\,1 + a^2/R^2\,]/2 \qquad 1/2 \le f \le 1$$

(3) So we know that $E = 1/2\ I\ \omega^2$ and $I = f\,MR^2$ for a flywheel, where f is some fraction that might be 1/2 or 3/4 or something like that, so we get $E = 1/2\ f\,MR^2\omega^2$ as the stored energy in the flywheel. We can insert $M = \rho V$ to get

$$E = 1/2\ f\,\rho V\,R^2\omega^2$$ // energy stored in flywheel of volume V, density ρ
// radius R, rotation rate ω.

You can see that if you double the rotation rate ω, the stored energy (perhaps in kilowatt-hours) is increased by a factor of 4. Obviously, you want to spin this thing as fast as possible to get the maximal stored energy. It also seems (at this point) advantageous to make the density ρ be large and the radius R also be large, and of course a large volume V also helps (but these parameters are linear, not quadratic as with ω).

(4) However, if ω gets too large, the flywheel will rip apart (explode), and that is the topic of this little section. You cannot have a flywheel that is just a "rim". We want to calculate the force per unit area that centrifugal force puts on the material on the inner surface of the hollow cylinder.

Consider a thin annular ring of height h, thickness dr at some radius r lying between a and R:

$$dV = 2\pi r\,dr\,h \qquad dM = \rho dV = 2\pi\rho h\,r\,dr$$

where dM is the mass of the thing rim. Centrifugal force is $F = mv^2/r$ for a mass m spinning in "orbit" at radius r, and velocity v. But $v = r\omega$ as one can easily show, so $F = m\omega^2 r$. In our application we then have

$$dF = dM\omega^2 r = 2\pi\rho h\,\omega^2 r^2\,dr$$

If we add up the centrifugal force of all the thin rings between r =a and r = R we integrate to get

$$F = 2\pi\rho h\,\omega^2 \int_a^R r^2\,dr = (2\pi\rho h\,\omega^2/3)\,[R^3 - a^3\,]$$

This is the total force on the inner surface of the cylinder tending to pull it outward. The area of this surface is $A = 2\pi ah$, so the tension on this surface is

$$T = F/A = (\rho\,\omega^2/3a)\,[R^3 - a^3\,]$$

TBC

It is simple calculation to find out the force/area ("tension") that is needed to hold the spoke in at the axis. The answer is this:

$T = (1/2)\ \rho\ \omega^2\ R^2$

To compute this, you consider a small mass $dM = \rho\ d^3V$ at distance r from the axis, you figure the centrifugal force pulling on that mass is $(dM)\ v^2/r$, divide by cross sectional area at r, then integrate over the spoke of material from r out to the outer edge R. You then set r = 0 in the answer to find the force at the axis, where it is greatest. That is to say, the spoke from r = r out to r = R is what is pulling out on the little cross sectional area of the spoke, and of course the worst case is when r = 0.

If this force F exceeds the "tensile strength" of the spoke material, the flywheel explodes. The best we can do for energy storage is to spin the thing faster until $T = T_{max}$ and hold at that point. So

$T_{max} = (1/2)\ \rho\ \omega^2\ R^2$

WRONG. (1) The spoke model is no good. The mass supported by a spoke is much more than just the spoke itself, so this calculation is no good. It is OK for a disk of material. (2) Real applications use a cylinder of material and there is no spoke at all. The cylinder is somehow constrained by magnetic bearings. Nothing has to physically attach it to the axis.

(5) Combine $E = 1/2\ f\ \rho V\ R^2\omega^2$ with this last result to get

$E = f\ V\ T_{max}$ // max energy you can store in your flywheel

This is a fascinating result. Remember that maybe f = .75 or something like that, V = volume of the flywheel material (perhaps mostly at the rim), and T_{max} is a property of the material that makes the spokes. Surprisingly, the result does not depend on the mass of the flywheel, as you think it might. Nor does it depend on the radius R. To see why this is so, you have to review the above derivation.

So now to maximize the energy you can store in a flywheel, you make the geometry as "rim like" as possible to maximize the fraction f (at most, f = 1), you maximize the volume of the rotating material, and you select a material that has a large T_{max} . If you make the flywheel be a cylinder of solid material so there are then no distinct "spokes", T_{max} then applies to that material itself, and f = 1/2.

Here is a table showing T_{max} for a selection of materials:

Material	Tensile Strength (GPa)
cement	.004
cast iron	0.17
steel alloy	0.76
stainless steel	0.86
titanium	0.90
spider silk	1.3
SiC	3.5
Kevlar	3-4
Bucky nanotubes	63
C-C bonds	200

In the mks system of units, a "tension" is force/area which is newtons/square meter or nt/m^2. One nt/m^2 is called a "Pascal" or Pa. The units shown above are in billions of Pascals, or gigaPascals or GPa. (force per area is also called pressure, and a tension is sort of a negative pressure. One GPa =145,038 PSI.)

The above table shows why "modern" flywheels are made of high-performance fiber materials like Kevlar, and are not made of steel or other metals. These flywheels can be very lightweight and can spin very fast. They are usually put in a can with the air removed, and they rotate on good quality bearings. Magnets are spaced along the outer surface of the flywheel, and these are used as part of a generator to spin the thing up, and as a motor when power is extracted from the wheel.

Here are numbers for a "practical" modern flywheel half the size of a hot water heater.

A flywheel rim having: Height = 2.5 feet OD = 2.5 feet f ~ .75
Max Tensile Stress = 500,000 psi = 3.4 GPa = T_{max} (something like Kevlar)
Specific Gravity = 1.1 = ρ
will store **50-kwh**, weigh **370 pounds**, spin up to 44,000 rpm.

Lead-acid batteries, to store 50-kwh, weigh over **4000 pounds**.

These numbers are from site http://rpm2.8k.com/basics.htm which is a commercial site for a company that makes flywheels for "intermediate UPS" storage. This thing could run a typical household using 2 kilowatts (20 100W light bulbs) for 24 hours. Unlike the batteries, the flywheel storage unit presumably never needs maintenance or replacement (until the bearings wear out if it does not use magnetic bearings).

The materials list contains interesting numbers. Nature has made spider silk stronger than titanium or stainless steel in terms of tensile strength. (Remember, it is force per cross sectional area). The strongest possible "material" is to somehow use idealized carbon strands, and someone has estimated the tensile strength at 200 GPa, and someone else has actually measured 63 GPa for BuckyTubes (made of carbon in a very regular arrangement). This last material could make the above flywheel store almost 1000 kilowatt hours. This thing would be rotating pretty fast, you might want it to be underground for safety reasons.