Active vs Passive in tensor doc PhL 11.23.16

This doc shows a tensor-doc model for the active/passive story using $\mathbf{e}_i = R^{-1}\mathbf{u}_i$ as the back-rotated vectors. Then Frame S = $\mathbf{u}_i$ and Frame S' = $\mathbf{e}_i$ and x'-space is not even involved.

For me, this is a rather difficult issue. In tensor doc when I write **V**' = R**V**, I always say that **V** is a vector in x-space and **V**' is a vector in x'-space. So I don't have the notion of **V**' as a new-named vector in x-space which is the active rotation in x-space of the vector **V**. So "active rotation" in my usual sense does not really appear in tensor doc. And "passive rotation" is also quite confusing to me there as well, but I hope to clear it up below. [all reasonable comments]

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1. Statement of a tensor doc Fact which I failed to realize was true

First, I note that I added Section 7.19 to tensor doc to clarify the meaning of things like $\mathbf{e}'_i$ by adding superscripts like $^{(u)}$. I first ran into this in wedge doc, and it is written up in the wedge folder in a slightly different superscript notation in "tensor doc and component types", but the Section 7.19 writeup is fine. I think I did this 7.19 after encountering the issue in wedge doc involving various paradoxes.

In that writeup and in 7.19 I note that

$(\mathbf{u}_n)^i$ really means $(\mathbf{u}_n)^{(u)\,i}$ or $(\mathbf{u}_n{}^{(u)})^i$ which is $\langle \mathbf{u}^i \mid \mathbf{u}_n \rangle$

$(\mathbf{e}_n)^i$ really means $(\mathbf{e}_n)^{(u)\,i}$ or $(\mathbf{e}_n{}^{(u)})^i$ which is $\langle \mathbf{u}^i \mid \mathbf{e}_n \rangle$

Since these two basis vectors are unprimed, they are implicitly expressed in the $\mathbf{u}_i$ basis.

Now it is a fact that $\mathbf{u}_n = R\mathbf{e}_n$ but I never write this equation anywhere in tensor doc. First here is a proof that this is valid, [a newly realized and correct fact, it does appear I think in wedge doc]

$(\mathbf{u}_n)^i = \delta_n{}^i$ // from tensor doc (7.18.3)

$(R\mathbf{e}_n)^i = R^i{}_j(\mathbf{e}_n)^j = R^i{}_j\, R_n{}^j$ // from tensor doc (7.18.1)

$= \delta^i{}_n$ // orthog rule #3 of (7.6.4)

I always say that $\mathbf{u}_n$ and $\mathbf{e}_n$ are vectors in x-space, and in that space $\mathbf{u}_n = R\mathbf{e}_n$ shows how they are related. I think I avoid writing this because I always want the thing on the left to be in x'-space, and so what appears in tensor doc is $\mathbf{e}'_n = R\mathbf{e}_n$ where $(\mathbf{e}'_n)^i = (\mathbf{u}_n)^i = \delta_n{}^i$. I like this form for vector fields, $\mathbf{V}'(\mathbf{x}') = R\mathbf{V}(\mathbf{x})$, but for the non-field basis vectors it is just a simple fact that $\mathbf{u}_n = R\mathbf{e}_n$. Therefore:

Fact: In x-space of tensor doc, we have $\mathbf{e_n} = R^{-1}\mathbf{u_n}$ so the $\mathbf{e_n}$ are *back-rotated* versions of the $\mathbf{u_n}$. [right]

Of course R in general is not a pure rotation, so "back-transformed" would be better in that case.

This is a simple fact that I just never realized was true. In that doc I did not use $\mathbf{u_n}$ until late in my development, perhaps that is why I did not see this. I was always more interested in transformations between x-space and x'-space.

2. A neutral review of active and passive rotations using c,c' vectors

BASIS VECTORS

We use the following notation:

Frame S basis	$\mathbf{c_i}$	
Frame S' basis:	$\mathbf{c'_i}$	(0)

I have picked a "neutral" basis vector name **c** to avoid any inadvertent contact with tensor doc.

The $\mathbf{c_i}$ are assumed to be axis-aligned in Frame S so we know that

$$(\mathbf{c_i})^j = \delta_i{}^j \qquad (1)$$

The $\mathbf{c_i}'$ are **back-rotated** versions of $\mathbf{c_i}$ so that

$$\mathbf{c'_i} = R^{-1}\mathbf{c_i} \qquad (2)$$

which one can of course also write as

$$\mathbf{c_i} = R\,\mathbf{c'_i}\ . \qquad (3)$$

In components equation (2) says

$$(\mathbf{c'_i})^k = (R^{-1})^k{}_n\,(\mathbf{c_i})^n \;=\; (R^{-1})^k{}_n\;\delta_i{}^n \;=\; (R^{-1})^k{}_i \;= (R^T)^k{}_i = R_i{}^k$$

and this is true even if R is not a rotation (used tensor doc Section 7.9). So we have this simple fact for the components of the back-rotated basis vectors,

$$(\mathbf{c'_i})^k = R_i{}^k \qquad (4)$$

Fact: Equation (3) is equivalent to

$$\mathbf{c'_i} = R_i{}^j\mathbf{c_j}\ . \qquad \text{// right side is sum of vectors} \qquad (5)$$

Proof:

$[R_i{}^j \mathbf{c}_j]^k = R_i{}^j (\mathbf{c}_j)^k$ // all is in the c-basis here

$= R_i{}^j \delta_j{}^k$ // from (1)

$= R_i{}^k$

$= (\mathbf{c}'_i)^k$ // from (4)

[I do the above proof also in frames doc Section 1 (a)]

VECTORS

We can expand the vector **V** in either basis,

$\mathbf{V} = (\mathbf{V}^{(c)})^i \mathbf{c}_i$ $\quad \mathbf{V} \bullet \mathbf{c}^i = (\mathbf{V}^{(c)})^i$ Frame S expansion (6)

$\mathbf{V} = (\mathbf{V}^{(c')})^i \mathbf{c}'_i$ $\quad \mathbf{V} \bullet \mathbf{c}'^i = (\mathbf{V}^{(c')})^i$ Frame S' expansion (7)

Now using just the $\mathbf{c}_i$ basis, I can create a new vector in Frame S called **V'** where

$\mathbf{V}' = R\mathbf{V}$ or $(\mathbf{V}'^{(c)})^i = R^i{}_j (\mathbf{V}^{(c)})^j$ // active rot in Frame S (8)

where $R^i{}_j$ is the rotation matrix in the $\mathbf{c}_i$ basis. We can expand this new vector **V'** in two ways

$\mathbf{V}' = (\mathbf{V}'^{(c)})^i \mathbf{c}_i$ $\quad \mathbf{V}' \bullet \mathbf{c}^i = (\mathbf{V}'^{(c)})^i$ Frame S expansion (9)

$\mathbf{V}' = (\mathbf{V}'^{(c')})^i \mathbf{c}'_i$ $\quad \mathbf{V}' \bullet \mathbf{c}'^i = (\mathbf{V}'^{(c')})^i$ Frame S' expansion (10)

At this point we can say [ok]

1. The components of **V'** in Frame S are: $(\mathbf{V}'^{(c)})^i$
2. The components of **V** in Frame S' are: $(\mathbf{V}^{(c')})^i$ (11)

Fact: The active/passive claim is that these two components are exactly the same.

Proof:

$(\mathbf{V}'^{(c)})^i = R^i{}_j (\mathbf{V}^{(c)})^j$ // from (8)

$= R^i{}_j (\mathbf{V} \bullet \mathbf{c}^j)$ // from (6)

$= \mathbf{V} \bullet (R^i{}_j \mathbf{c}^j)$ // a rewrite

$= \mathbf{V} \bullet \mathbf{c'}^{i}$ // from (5) tilted

$= (\mathbf{V}^{(c')})^{i}$ // from (7)

3. A connection to tensor doc

I will cut and paste the entire Section 2 above showing what I think is the connection between Section 2 above and tensor doc.

[in this stuff below u_n and e_n are exactly the tensor doc basis vectors]

BASIS VECTORS

We use the following notation:

Frame S basis	$\mathbf{u}_i$	
Frame S' basis:	$\mathbf{e}_i$	(0)

The $\mathbf{u}_i$ are axis-aligned in Frame S so we know that

$$(\mathbf{u}_i)^j = \delta_i{}^j \qquad (1)$$

The $\mathbf{e}_i$ are back-rotated versions of $\mathbf{u}_i$ so that

$$\mathbf{e}_i = R^{-1}\mathbf{u}_i \quad \text{// as in Section 1 above} \qquad (2)$$

which one can of course also write as

$$\mathbf{u}_i = R\,\mathbf{e}_i \ . \qquad (3)$$

In components equation (2) says

$$(\mathbf{e}_i)^k = (R^{-1})^k{}_n\,(\mathbf{u}_i)^n = (R^{-1})^k{}_n\,\delta_i{}^n = (R^{-1})^k{}_i = (R^T)^k{}_i = R_i{}^k$$

and this is true even if R is not a rotation (used tensor doc Section 7.9). So we have this simple fact for the components of the back-rotated basis vectors,

$$(\mathbf{e})^k = R_i{}^k \qquad (4)$$

which agrees with tensor doc (7.18.1)

Fact: Equation (3) is equivalent to

$$\mathbf{e}_i = R_i{}^j\mathbf{u}_j \ . \quad \text{// right side is sum of vectors} \qquad (5)$$

Proof:

$[R_i{}^j \mathbf{u}_j]^k = R_i{}^j (\mathbf{u}_j)^k$ // all is in the c-basis here

$= R_i{}^j \delta_j{}^k$ // from (1)

$= R_i{}^k$

$= (\mathbf{e}_i)^k$ // from (4)

Note that equation (5) appears in wedge doc (2.4.4) second line!

VECTORS

We can expand the vector **V** in either basis,

$\mathbf{V} = (\mathbf{V}^{(u)})^i \mathbf{u}_i$ $\quad \mathbf{V} \bullet \mathbf{u}^i = (\mathbf{V}^{(u)})^i$ Frame S expansion (6)

$\mathbf{V} = (\mathbf{V}^{(e)})^i \mathbf{e}_i$ $\quad \mathbf{V} \bullet \mathbf{e}^i = (\mathbf{V}^{(e)})^i$ Frame S' expansion (7)

Now using just the $\mathbf{u}_i$ basis, I can create a new vector in Frame S called **V'** where

$\mathbf{V}' = R\mathbf{V}$ or $(\mathbf{V}'^{(u)})^i = R^i{}_j (\mathbf{V}^{(u)})^j$ // active rot in Frame S (8)

where $R^i{}_j$ is the rotation matrix in the $\mathbf{u}_i$ basis. We can expand this new vector **V'** in two ways

$\mathbf{V}' = (\mathbf{V}'^{(u)})^i \mathbf{u}_i$ $\quad \mathbf{V}' \bullet \mathbf{u}^i = (\mathbf{V}'^{(u)})^i$ Frame S expansion (9)

$\mathbf{V}' = (\mathbf{V}'^{(e)})^i \mathbf{e}_i$ $\quad \mathbf{V}' \bullet \mathbf{e}^i = (\mathbf{V}'^{(e)})^i$ Frame S' expansion (10)

At this point we can say

1. The components of **V'** in Frame S are: $(\mathbf{V}'^{(u)})^i$
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Fact: The active/passive claim is that these two components are exactly the same.

Proof:

$(\mathbf{V}'^{(u)})^i = R^i{}_j (\mathbf{V}^{(u)})^j$ // from (8)

$= R^i{}_j (\mathbf{V} \bullet \mathbf{u}^j)$ // from (6)

$= \mathbf{V} \bullet (R^{i}{}_{j}\mathbf{u}^{j})$ // a rewrite

$= \mathbf{V} \bullet \mathbf{e}^{i}$ // from (5) tilted

$= (\mathbf{V}^{(e)})^{i}$ // from (7)

In tensor doc we write $(\mathbf{V}'^{(u)})^{i} = (V')^{i}$ and also $(\mathbf{V}^{(e)})^{i} = (V')^{i}$ so we can rewrite the expansions above

$\mathbf{V} = \mathbf{V}^{i}\mathbf{u}_{i}$	$\mathbf{V} \bullet \mathbf{u}^{i} = \mathbf{V}^{i}$	Frame S expansion	(6)'
$\mathbf{V} = \mathbf{V}'^{i}\mathbf{e}_{i}$	$\mathbf{V} \bullet \mathbf{e}^{i} = \mathbf{V}'^{i}$	Frame S' expansion	(7)'
$\mathbf{V}' = \mathbf{V}'^{i}\mathbf{u}_{i}$	$\mathbf{V}' \bullet \mathbf{u}^{i} = \mathbf{V}'^{i}$	Frame S expansion	(9)'
$\mathbf{V}' = (\mathbf{V}'^{(e)})^{i}\mathbf{e}_{i}$	$\mathbf{V}' \bullet \mathbf{e}^{i} = (\mathbf{V}'^{(e)})^{i}$	Frame S' expansion	(10)

Lines 1 and 2 above agree with lines 1 and 3 from tensor doc,

$\mathbf{V} = V^{1}\,\mathbf{u}_{1} + V^{2}\,\mathbf{u}_{2} + \ldots$	$= \Sigma_{n} V^{n}\,\mathbf{u}_{n}$	where $\mathbf{u}^{n} \bullet \mathbf{V} = V^{n}$	$\mathbf{u}^{n} = g^{ni}\,\mathbf{u}_{i}$
$\mathbf{V} = V_{1}\,\mathbf{u}^{1} + V_{2}\,\mathbf{u}^{2} + \ldots$	$= \Sigma_{n} V_{n}\,\mathbf{u}^{n}$	where $\mathbf{u}_{n} \bullet \mathbf{V} = V_{n}$	
$\mathbf{V} = V'^{1}\mathbf{e}_{1} + V'^{2}\mathbf{e}_{2} + \ldots$	$= \Sigma_{n} V'^{n}\,\mathbf{e}_{n}$	where $\mathbf{e}^{n} \bullet \mathbf{V} = V'^{n}$	$\mathbf{e}^{n} = g'^{ni}\,\mathbf{e}_{i}$
$\mathbf{V} = V'_{1}\mathbf{e}^{1} + V'_{2}\,\mathbf{e}^{2} + \ldots$	$= \Sigma_{n} V'_{n}\,\mathbf{e}^{n}$	where $\mathbf{e}_{n} \bullet \mathbf{V} = V'_{n}$	(7.13.10)

The other two expansions do not appear in tensor doc but they are valid. I think I could show that the last line above (10') is equivalent to this

$\mathbf{V}' = \mathbf{V}^{i}\mathbf{e}_{i}$	$\mathbf{V}' \bullet \mathbf{e}^{i} = \mathbf{V}'^{i}$	Frame S' expansion	(10)

but I will leave that for now as just a conjecture.

The main fact above then is this association

Frame S basis $\mathbf{u}_i$
Frame S' basis: $\mathbf{e}_i$

These two frames are associated only with x-space in tensor doc. Remember that these two basis vectors exist in x-space, as I say many times in tensor doc.

My Big Confusion I think was that I was making this wrong association

Frame S = x-space
Frame S' = x'-space // wrong

The two spaces are not "frames", they are "spaces". Within each space you could construct whatever "frames" you like, and above I have constructed two "frames" in x-space.;

Two comments:

(1) I have above realized a version of the active passive story in relation to tensor doc by using $\mathbf{e}_i = R^{-1}\mathbf{u}_i$ as the back-rotation starting point, and then I just say Frame S = $\mathbf{u}_n$ and Frame S' = $\mathbf{e}_i$ and yes both the Frames are in x-space and having nothing to do with x'-space.

(2) Thus for the active/passive story, I give up the notion that Frame S' is x'-space and Frame S is x-space.