Foucault PhL 8.20.12

This is a big flailing mess all done in a rush that got nowhere.

I have not looked that the texts on this recently, I want to try my own simple method first.
I need to do a flying-ant problem where ant has V(t) instead of a constant V.

Realization: All of problem 3 is just the same if we replace V → V(t) so I should make that change! The reason is that I never differentiate anything. What about my Hard Way section? Flying ant in Frame S' feels certain fict forces yes. But I want to specify V(t), how do I do that? It implies a certain a(t) in Frame S. Namely, $\mathbf{A}(t) = \mathbf{d}V(t)/dt$. Then we have

$$\mathbf{F'}_{\mathtt{eff}} = m\mathbf{a'}$$

$$\mathbf{F'}_{\mathtt{eff}} = \mathbf{F} + \mathbf{F'}_{\mathtt{fict}}$$

$$\mathbf{F'}_{\mathtt{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{Special Case \#1} \qquad (8.12)$$

$$\mathbf{F} = m\mathbf{A}$$

So we then would have (assume $\dot{\omega} = 0$)

$$m\dot{\mathbf{v}}' = m\dot{\mathbf{V}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

(d) Problem 4: Inverse Problem: Ant flies in Frame S at variable velocity V(t)

$$\mathbf{v} = V(t)\hat{\mathbf{n}}$$
$$\mathbf{r} = V(t)t\,\hat{\mathbf{n}} + \mathbf{r}_0$$
$$\mathbf{a} = 0 \ . \qquad (15.37)$$

$$\mathbf{r}' = V(t)t\,R_z(\theta-\varphi)\,\mathbf{e'}_1 + R_z(-\varphi)\,[(r_0)_i\,\mathbf{e'}_i] + b\,\mathbf{e'}_2$$

(15.41)

$$\mathbf{r}'(t) = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$$

where (15.42)

$$x' = V(t)t\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi$$
$$y' = V(t)t\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi + b$$
where $\varphi = \varphi_0 + \omega t$

This then is the trajectory of the flying ant as seen in Frame S'.

What is V(t) for a pendulum?

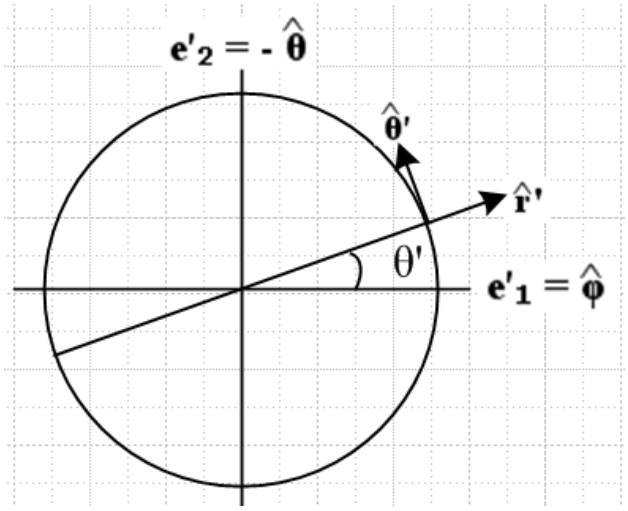
This is written up in my pendulum doc. For small angles it is this:

$v(t) = v_0 \cos(\Omega t)$ $V_0 = L\theta_0\Omega$ $\Omega^2 = g/L$ L = length of string

Other facts are these

$v = L\dot{\theta}$ $\theta(t) = \theta_0 \sin(\Omega t)$ $\dot{\theta}(t) = \theta_0\ \Omega\cos(\Omega t)$ $r' = L\theta = L\ \theta_0 \sin(\Omega t) = R_0 \sin(\Omega t)$

Now in our Earth-fixed if the pendulum is momentarily at some angle α, here is what you see looking down on the floor



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What are the forces on the mass at the end of the string?

real = **T**, m**g** fict = $-2m\,\boldsymbol{\omega} \times \mathbf{v}'$

where this is the local **g** not $\mathbf{g_0}$. Can we ignore the distinction in this problem?? That is to say, can we ignore the centrifugal fictitious force? I think yes, so let's try it that way.

The bob has an acceleration along the line shown in the above drawing with size $a = mg\alpha$ where α is the pendulum angle not shown,

$\tan\alpha = a/mg$ where a is the restoring acceleration in the usual pendulum picture, but α is small so

$m\mathbf{a}' \bullet \hat{\mathbf{r}}' = -mg\alpha = -(mg/L)\mathbf{r}'$

$\mathbf{a}' \bullet \hat{\mathbf{r}}' = -g\alpha = -(g/L)\mathbf{r}' = -\Omega^2\,\mathbf{r}'$ $= a'_{r'}\,\mathbf{r}'$

But the total acceleration we shall assume is more like

$$\mathbf{a}' = a'_{r'}\,\hat{\mathbf{r}}' + a'_{\theta'}\,\hat{\boldsymbol{\theta}}'$$

What is the equation of motion please. Hemming and hawing for an hour here. I think the **T** and m**g** forces cause just the component you see above. So

effect if **T** and m**g** only is this $\mathbf{F} = a'_{r'}\,\hat{\mathbf{r}}'$ $a'_{r'} = -\Omega^2 r'$

Then our full equation is

$$\mathbf{F}'_{\mathbf{eff}} = m\mathbf{a}' = ma'_{r'}\,\hat{\mathbf{r}}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

or

$$m\mathbf{a}' = -\Omega^2 r'\,\hat{\mathbf{r}}' - 2m\,\boldsymbol{\omega} \times \mathbf{v}'$$

and finally we have an equation. Let's try this expansion for **v**'

$$\mathbf{v}' = v'_{r'}\,\hat{\mathbf{r}}' + v'_{\theta'}\,\hat{\boldsymbol{\theta}}'$$

Now we know that

$$\hat{\mathbf{r}}' = \cos\theta'\,\mathbf{e}'_1 + \sin\theta'\,\mathbf{e}'_2$$
$$\hat{\boldsymbol{\theta}}' = -\sin\theta'\,\mathbf{e}'_1 + \cos\theta'\,\mathbf{e}'_2$$

or

$$\hat{\mathbf{r}}' = \cos\theta'\,\hat{\boldsymbol{\varphi}} - \sin\theta'\,\hat{\boldsymbol{\theta}}$$
$$\hat{\boldsymbol{\theta}}' = -\sin\theta'\,\hat{\boldsymbol{\varphi}} - \cos\theta'\,\hat{\boldsymbol{\theta}}$$

Invert the above

$$\mathbf{e}'_1 = \cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}'$$
$$\mathbf{e}'_2 = +\sin\theta'\,\hat{\mathbf{r}}' + \cos\theta'\hat{\boldsymbol{\theta}}'$$

or

$$\hat{\boldsymbol{\varphi}} = \cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}'$$
$$\hat{\boldsymbol{\theta}} = -+\sin\theta'\,\hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}'$$

So then

$$\mathbf{v}' = v'_{r'}\,\hat{\mathbf{r}}' + v'_{\theta'}\,\hat{\boldsymbol{\theta}}'$$

$$= v'_{r'}\,[\cos\theta'\,\hat{\boldsymbol{\varphi}} - \sin\theta'\,\hat{\boldsymbol{\theta}}] + v'_{\theta'}\,[-\sin\theta'\,\hat{\boldsymbol{\varphi}} - \cos\theta'\,\hat{\boldsymbol{\theta}}]$$

$$= (v'_r\, \cos\theta' - v'_\theta\, \sin\theta')\, \hat{\boldsymbol{\varphi}} - (v'_r\, \sin\theta' + v'_\theta\, \cos\theta')\hat{\boldsymbol{\theta}}$$

But we also know that

$$\hat{\boldsymbol{e}}_i = R(\xi)\, \mathbf{e}_i. \tag{A.6}$$

$$R(\xi) = \mathcal{R}\, R_2 = \begin{pmatrix} \cos\varphi\cos\theta & -\sin\varphi & \cos\varphi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi & \sin\varphi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \tag{A.9}$$

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{A.12a}$$

$$\hat{\mathbf{r}} = \cos\varphi\, \sin\theta\, \hat{\mathbf{x}} + \sin\varphi\, \sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\varphi\, \cos\theta\, \hat{\mathbf{x}} + \sin\varphi\, \cos\theta\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \tag{A.12c}$$

The inverse of the above is then

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix}$$

$$\hat{\mathbf{x}} = \cos\varphi\sin\theta\, \hat{\mathbf{r}} + \cos\varphi\cos\theta\, \hat{\boldsymbol{\theta}} - \sin\varphi\, \hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{y}} = \sin\varphi\sin\theta\, \hat{\mathbf{r}} + \sin\varphi\cos\theta\, \hat{\boldsymbol{\theta}} + \cos\varphi\, \hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}}$$

Now we know by staring at the picture that

$$\boldsymbol{\omega} \times \mathbf{e}'_2 = \sin\theta_L \mathbf{e}'_1 = \cos\theta\, \mathbf{e}'_1$$
$$\boldsymbol{\omega} \times \mathbf{e}'_1 = -\hat{\boldsymbol{\rho}}$$

$$\hat{\boldsymbol{\rho}} = \sin\theta\, \mathbf{e}'_3 - \cos\theta\, \mathbf{e}'_2 = \sin\theta\, \hat{\mathbf{r}} + \cos\theta\, \hat{\boldsymbol{\theta}}$$

so then

$$\boldsymbol{\omega} \times \mathbf{e}'_2 = \cos\theta\, \mathbf{e}'_1$$
$$\boldsymbol{\omega} \times \mathbf{e}'_1 = -\sin\theta\, \mathbf{e}'_3 + \cos\theta\, \mathbf{e}'_2$$

$$\boldsymbol{\omega} \times \hat{\boldsymbol{\theta}} = -\cos\theta\, \hat{\boldsymbol{\varphi}}$$
$$\boldsymbol{\omega} \times \hat{\boldsymbol{\varphi}} = -\sin\theta\, \hat{\mathbf{r}} - \cos\theta\, \hat{\boldsymbol{\theta}}$$

Rewrite

$$\boldsymbol{\omega} \times \hat{\boldsymbol{\theta}} = \sin\theta_L \hat{\boldsymbol{\varphi}}$$
$$\boldsymbol{\omega} \times \hat{\boldsymbol{\varphi}} = -\sin\varphi\sin\theta\, \hat{\mathbf{r}} - \sin\varphi\cos\theta\, \hat{\boldsymbol{\theta}} - \cos\varphi\, \hat{\boldsymbol{\varphi}}$$

Now recall from above that

$$\mathbf{v}' = (v'_r \cos\theta' - v'_\theta \sin\theta')\, \hat{\boldsymbol{\varphi}} - (v'_r \sin\theta' + v'_\theta \cos\theta')\hat{\boldsymbol{\theta}}$$

Then

$$\boldsymbol{\omega} \times \mathbf{v}' = \boldsymbol{\omega} \times \{ (v'_r \cos\theta' - v'_\theta \sin\theta')\, \hat{\boldsymbol{\varphi}} - (v'_r \sin\theta' + v'_\theta \cos\theta')\hat{\boldsymbol{\theta}}\}$$

$$= (v'_r \cos\theta' - v'_\theta \sin\theta')\, \boldsymbol{\omega} \times \hat{\boldsymbol{\varphi}} - (v'_r \sin\theta' + v'_\theta \cos\theta')\, \boldsymbol{\omega} \times \hat{\boldsymbol{\theta}}$$

$$= (v'_r \cos\theta' - v'_\theta \sin\theta')\, [-\sin\varphi\sin\theta\, \hat{\mathbf{r}} - \sin\varphi\cos\theta\, \hat{\boldsymbol{\theta}} - \cos\varphi\, \hat{\boldsymbol{\varphi}}] - (v'_r \sin\theta' + v'_\theta \cos\theta')\sin\theta_L \hat{\boldsymbol{\varphi}}$$

$$= -(v'_r \cos\theta' - v'_\theta \sin\theta')\sin\varphi\sin\theta\, \hat{\mathbf{r}}$$
$$+ (v'_r \cos\theta' - v'_\theta \sin\theta')(-\sin\varphi\cos\theta)\, \hat{\boldsymbol{\theta}}$$
$$+ [\, (v'_r \cos\theta' - v'_\theta \sin\theta')(-\cos\varphi) - (v'_r \sin\theta' + v'_\theta \cos\theta')\sin\theta_L\,]\, \hat{\boldsymbol{\varphi}}$$

Now we are supposed to shove this into our equation of motion

$$m\mathbf{a}' = -m\Omega^2 r'\, \hat{\mathbf{r}}' - 2m\, \boldsymbol{\omega} \times \mathbf{v}'$$

I think I will just ignore the $\hat{\mathbf{r}}$ term in $\boldsymbol{\omega} \times \mathbf{v}'$ since it is an upward force on the bob away from the earth. And I have already ignored radial v' etc etc. Just try it

$$\mathbf{a}' = a'_r\, \hat{\mathbf{r}}' + a'_\theta\, \hat{\boldsymbol{\theta}}'$$

Then

$$ma'_r\, \hat{\mathbf{r}}' + ma'_\theta\, \hat{\boldsymbol{\theta}}' = -m\Omega^2 r'\, \hat{\mathbf{r}}' - 2m\, \boldsymbol{\omega} \times \mathbf{v}'$$

and we have from above

$$\hat{\boldsymbol{\varphi}} = \cos\theta'\, \hat{\mathbf{r}}' - \sin\theta'\, \hat{\boldsymbol{\theta}}'$$
$$\hat{\boldsymbol{\theta}} = -\sin\theta'\, \hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}'$$

What a mess indeed! Then

$$ma'_{r'}\,\hat{\mathbf{r}}' + ma'_{\theta'}\,\hat{\boldsymbol{\theta}}' = -m\Omega^2 r'\,\hat{\mathbf{r}}'$$

$$-2m\{+(v'_{r'}\cos\theta' - v'_{\theta'}\sin\theta')(-\sin\varphi\cos\theta)[-\sin\theta'\,\hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}']$$

$$+[(v'_{r'}\cos\theta' - v'_{\theta'}\sin\theta')(-\cos\varphi) - (v'_{r'}\sin\theta' + v'_{\theta'}\cos\theta')\sin\theta_L][\cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}']\}$$

I think we are going to end up with coupled ODEs.

I guess I am done flailing now. I tried to roll my own and made a giant mess. I thought I could get some rough idea of a V(t) swing and use that to see how much rotation there was per swing. What happened to that plan in the above mess? I got completely lost in the trees.

To a good approximation, I think we do have this velocity in Frame S'

$$\mathbf{r}'(t) \approx r(t)\,\hat{\mathbf{r}}' \qquad r(t) = r_0\sin(\Omega t)$$

$$\mathbf{v}'(t) \approx v(t)\,\hat{\mathbf{r}}' \qquad v(t) = v_0\cos(\Omega t)$$

That is probably enough for the Coriolis calculation. So I need $\boldsymbol{\omega} \times \hat{\mathbf{r}}'$, but all I know is this

$$\boldsymbol{\omega} \times \hat{\boldsymbol{\theta}} = -\cos\theta\,\hat{\boldsymbol{\varphi}}$$

$$\boldsymbol{\omega} \times \hat{\boldsymbol{\varphi}} = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}$$

So OK, write from way back above

$$\hat{\mathbf{r}}' = \cos\theta'\,\hat{\boldsymbol{\varphi}} - \sin\theta'\,\hat{\boldsymbol{\theta}}$$

Then

$$\mathbf{v}'(t) \approx v(t)\cos\theta'\,\hat{\boldsymbol{\varphi}} - v(t)\sin\theta'\,\hat{\boldsymbol{\theta}}$$

$$\boldsymbol{\omega} \times \mathbf{v}'(t) \approx v(t)\cos\theta'[-\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}] - v(t)\sin\theta'[-\cos\theta\,\hat{\boldsymbol{\varphi}}]$$

$$= -v(t)\cos\theta'\sin\theta\hat{\mathbf{r}} - v(t)\cos\theta'\cos\theta\,\hat{\boldsymbol{\theta}} + v(t)\sin\theta'\cos\theta\,\hat{\boldsymbol{\varphi}}$$

I don't care about the $\hat{\mathbf{r}}$ term so get

$$\boldsymbol{\omega} \times \mathbf{v}'(t) \approx -v(t)\cos\theta'\cos\theta\,\hat{\boldsymbol{\theta}} + v(t)\sin\theta'\cos\theta\,\hat{\boldsymbol{\varphi}}$$

But now recall that

$$\hat{\boldsymbol{\varphi}} = \cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}'$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta'\,\hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}'$$

so get

$$\boldsymbol{\omega} \times \mathbf{v}'(t) \approx -v(t)\cos\theta'\cos\theta\,[-\sin\theta'\,\hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}'] + v(t)\sin\theta'\cos\theta\,[\cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}']$$

$$\boldsymbol{\omega} \times \mathbf{v}'(t) / v(t) = -\cos\theta'\cos\theta\,[-\sin\theta'\,\hat{\mathbf{r}}' - \cos\theta'\hat{\boldsymbol{\theta}}'] + \sin\theta'\cos\theta\,[\cos\theta'\,\hat{\mathbf{r}}' - \sin\theta'\,\hat{\boldsymbol{\theta}}']$$

$$= [\cos\theta'\cos\theta\sin\theta' + \sin\theta'\cos\theta\cos\theta']\,\hat{\mathbf{r}}'$$

$$+ [\cos\theta'\cos\theta\cos\theta' - \sin\theta'\cos\theta\sin\theta']\hat{\boldsymbol{\theta}}'$$

I expected first term to be 0, sign error somewhere. Then we at least have something for our result

$$\boldsymbol{\omega} \times \mathbf{v}'(t) \approx v(t)\cos\theta\,[\cos^2\theta' - \sin^2\theta']\,\hat{\boldsymbol{\theta}}'$$

At least this says no Coriolis at the equator. If $\theta' = 0$, then I expect $\boldsymbol{\omega} \times \mathbf{v}'(t)$ = large.
If $\theta' = \pi/2$ I expect a simpler result. Maybe this is the answer when I fix my sign error

$$\boldsymbol{\omega} \times \mathbf{v}'(t) \approx -v(t)\cos\theta\,\hat{\boldsymbol{\theta}}'$$

Then at least it deflects to the left. So finally after 2 hours I have something for this Coriolis term which should have taken 5 minutes. Typical. What do I do next? What is the size of the deflection for one swing due to this Cor force

$$-2m\,\boldsymbol{\omega} \times \mathbf{v}'(t) \approx 2mv(t)\cos\theta\,\hat{\boldsymbol{\theta}}' \qquad v(t) = v_0\cos(\Omega t)$$

I need to get the position. Maybe we have

$$m\mathbf{a}' = 2mv(t)\cos\theta\,\hat{\boldsymbol{\theta}}'$$

$$\mathbf{a}' = a'_{r'}\,\hat{\mathbf{r}}' + a'_{\theta'}\,\hat{\boldsymbol{\theta}}'$$

$$a'_{\theta'} = 2m\cos\theta\,v(t)$$

$$(dv'_{\theta'}/dt) = 2m\cos\theta\,v(t)$$

$$v'_{\theta'} = 2m\cos\theta\,v_0\int_0^t \cos(\Omega t)dt = (2m\cos\theta\,v_0/\Omega)\sin(\Omega t)$$

$$r'_{\theta'} = (2m\cos\theta\,v_0/\Omega^2)\cos(\Omega t) + K$$

So then in one half period maybe it moves something on this order

$(2m \cos\theta \, v_0/\Omega^2)$

STOP. I give up on this shoot from the hip approach after 8 pages of mess. I see the idea of how you solve the problem, but my kinematics setup stinks, the primes stink, the approximations are unclear. So I have no "quick fix" to throw into frames doc. This is not in fact a flying-ant problem where you are given something in one frame and want it in the other frame. This is really an F = ma problem in a non-inertial frame with a messy geometry. You need polar coordinates on your "turntable" and you also have sphericals going at the same time. Probably his wants to be done with angle θ' as the variable itself. Eventually I will read what my favorite authors have to say about this problem. I probably have notes on Goldstein, but don't want to start into this now at 6 PM before Cod.