

Chapter VII. Rotating Coordinate Systems

7.1. Frames of References

In order to really look at particle dynamics in the context of the atmosphere, we must now deal with the fact that we live and observe the weather in a non-inertial reference frame. Specially, we will look at a rotating coordinate system and introduce the Coriolis and centrifugal force.

Rotating and Non-rotating Frames of Reference

First, start by recognizing that presence of 2 coordinate systems when dealing with problems related to the earth:

- a) one fixed to the earth that rotates and is thus accelerating (non-inertial), our real life frame of reference
- b) one fixed with respect to the remote "star", i.e., an inertial frame where the Newton's laws are valid.

Apparent Force

In order to apply Newton's laws in our earth reference frame, we must take into account the acceleration of our coordinate system (earth). This leads to the apparent forces that get added to Newton's 2nd law of motion:

$$\frac{d\vec{V}}{dt} = \frac{\vec{F}_{real}}{m} \quad (\text{inertial}) \quad (7.1)$$

$$\frac{d\vec{V}}{dt} = \frac{\vec{F}_{real}}{m} + \frac{\vec{F}_{apparent}}{m} \quad (\text{non-inertial}) \quad (7.2)$$

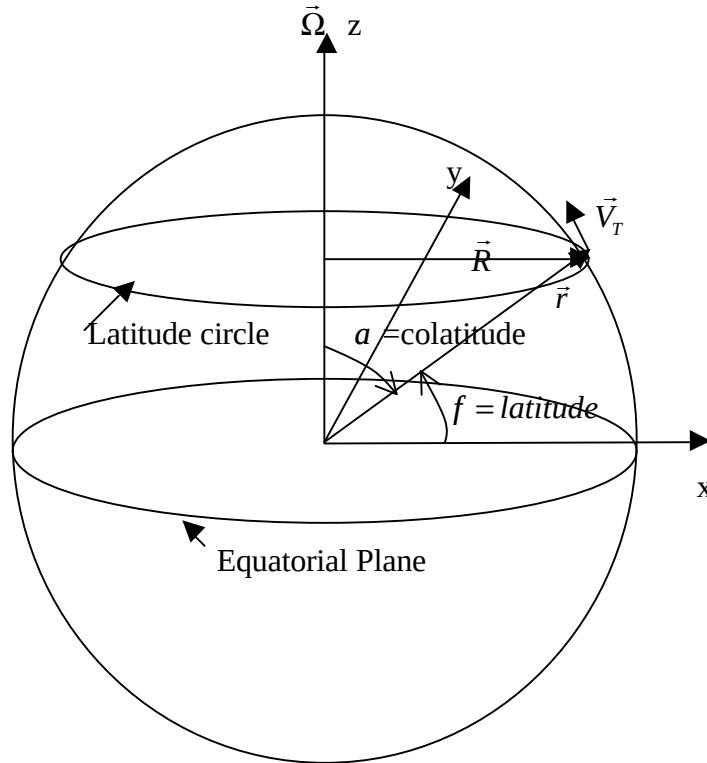
where $\frac{\vec{F}_{apparent}}{m}$ are due solely to the fact that we operate / observe in a non-inertial reference frame.

We need to relate $\left. \frac{d\vec{A}}{dt} \right|_{inertial}$ to $\left. \frac{d\vec{A}}{dt} \right|_{non-inertial}$.

7.2. Time Derivatives in Fixed and Rotating Coordinates

Reading Section 7.1 of Symon.

Using concepts developed in the last chapter, let's introduce some notion that will prove to be useful.



Let $\vec{r}$ = position vector from the origin (center of the earth). Then, we can find out that the instantaneous tangential velocity of a particle at any distance $\vec{r}$ ($\vec{a} + \vec{z}$ where a = mean radius of earth and z = altitude above ground) is given by

$$\boxed{\vec{V}_T = \vec{\Omega} \times \vec{r} .} \quad (7.1)$$

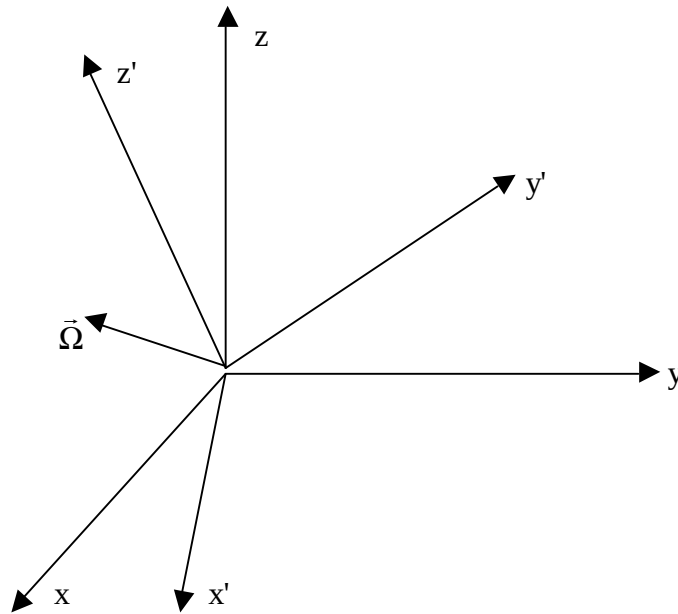
Here $\vec{\Omega}$ is the vector angular velocity of the earth rotation.

We can see this by first looking at the magnitude of $\vec{V}_T$. According to the above figure,

$$|\vec{V}_T| = |\vec{\Omega}| R = |\vec{\Omega}| |\vec{r}| \sin a = |\vec{\Omega} \times \vec{r}| \quad (7.2)$$

Now, let's try to relate $\frac{d\vec{A}}{dt}$ in 2 reference frames:

- a) Fixed (absolute) system $(\hat{i}, \hat{j}, \hat{k})$
- b) Rotating $(\vec{\Omega})$ system $(\hat{i}', \hat{j}', \hat{k}')$



Now, a vector $\vec{A}$ is the same vector no matter what coordinate system it is viewed from. This

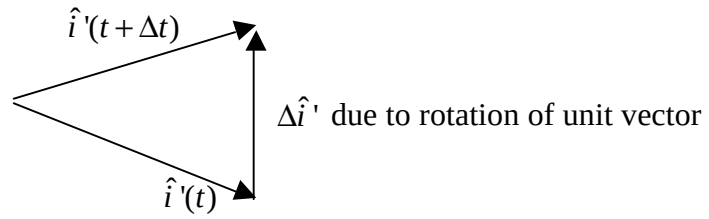
$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}'.$$

Given that the primed system is rotating, however, the time derivative of $\vec{A}$ will be different if viewed from the 2 systems. Mathematically, we have

'a' for absolute

$$\begin{aligned} \frac{d_a \vec{A}}{dt} &= \frac{dA_x}{dt} \hat{i} + \frac{dA_y}{dt} \hat{j} + \frac{dA_z}{dt} \hat{k} \\ &= \frac{dA'_x}{dt} \hat{i}' + \frac{dA'_y}{dt} \hat{j}' + \frac{dA'_z}{dt} \hat{k}' \\ &\quad + A'_x \frac{d\hat{i}'}{dt} + A'_y \frac{d\hat{j}'}{dt} + A'_z \frac{d\hat{k}'}{dt} \end{aligned} \quad (7.3)$$

Note that the primed unit vectors vary with time! (Remember our earlier derivation of the velocity in plane polar coordinates?)



Recall earlier that $\vec{V}_T = \vec{\Omega} \times \vec{r}$ for a particle rotating at angular velocity $\vec{\Omega}$. Since

$\vec{V}_T = \frac{d\vec{r}}{dt}$ therefore

$$\frac{d\vec{r}}{dt} = \vec{\Omega} \times \vec{r} \quad (7.4)$$

The above is correct only for a vector whose length does not change. The same is true for the unit vectors, therefore

$$\begin{aligned} \frac{d\hat{i}'}{dt} &= \vec{\Omega} \times \hat{i}' \\ \frac{d\hat{j}'}{dt} &= \vec{\Omega} \times \hat{j}' \\ \frac{d\hat{k}'}{dt} &= \vec{\Omega} \times \hat{k}' \end{aligned} \quad (7.5)$$

If we now look at $\frac{d\vec{A}}{dt}$ in the reference frame of the rotating (primed) system, then

$\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ appear to be fixed!

$$\frac{d_r \vec{A}}{dt} = \frac{dA'_x}{dt} \hat{i}' + \frac{dA'_y}{dt} \hat{j}' + \frac{dA'_z}{dt} \hat{k}' \quad (7.6)$$

which is a time rate of change in $\vec{A}$ seen from the rotating reference frame and is analogous to $\frac{d_a \vec{A}}{dt}$ earlier.

Thus, from (7.3), we can write

$$\begin{aligned}
\frac{d_a \vec{A}}{dt} &= \frac{d_r \vec{A}}{dt} + A'_x (\vec{\Omega} \times \hat{i}') + A'_y (\vec{\Omega} \times \hat{j}') + A'_z (\vec{\Omega} \times \hat{k}') \\
&= \frac{d_r \vec{A}}{dt} + \vec{\Omega} \times (A'_x \hat{i}' + A'_y \hat{j}' + A'_z \hat{k}') = \frac{d_r \vec{A}}{dt} + \vec{\Omega} \times \vec{A} \rightarrow
\end{aligned}$$

$$\frac{d_a \vec{A}}{dt} = \frac{d_r \vec{A}}{dt} + \vec{\Omega} \times \vec{A}$$

(7.7)

Let's look at a physical interpretation. If $\vec{A} = \vec{r}$, then we can write

$$\begin{array}{ccc}
\frac{d_a \vec{r}}{dt} & = & \frac{d_r \vec{r}}{dt} + \vec{\Omega} \times \vec{r} \\
\uparrow & \uparrow & \uparrow \\
\text{Absolute} & \text{Velocity relative} & \text{Velocity of the} \\
\text{velocity in} & \text{to moving} & \text{coordinate} \\
\text{the inertial} & \text{(rotating)} & \text{system itself} \\
\text{frame } V_a & \text{coordinates } (V_r) & \text{(remember} \\
& & \vec{V} = \vec{\Omega} \times \vec{r})
\end{array}$$
(7.8)

In another word, $\vec{V}_{abs} = \vec{V}_{rel} + \vec{V}_{coord}$.

Think of the earlier example of throwing a baseball towards a person on the ground from a moving railway car. If you can remember this simple analogy, you will be able to remember the formula in the box.

7.3 Equation of Motion in Absolute and Rotating Coordinates

Let's now apply concepts from the previous section to any arbitrary velocity vector of a particle assuming that the origins of the rotating and fixed (inertial) frames are the same. In this case, the position vector to any location is the same in both systems.

Our goal is to find the equations of motion in the absolute and rotating coordinates.

Now, we just showed that

$$\vec{V}_a = \vec{V} + \vec{\Omega} \times \vec{r} \quad (7.9)$$

where $\vec{V}$ is the relative velocity (we shall drop subscript r from now on).

Apply Eq.(7.7) to $\vec{V}_a$ in (7.9) $\rightarrow$

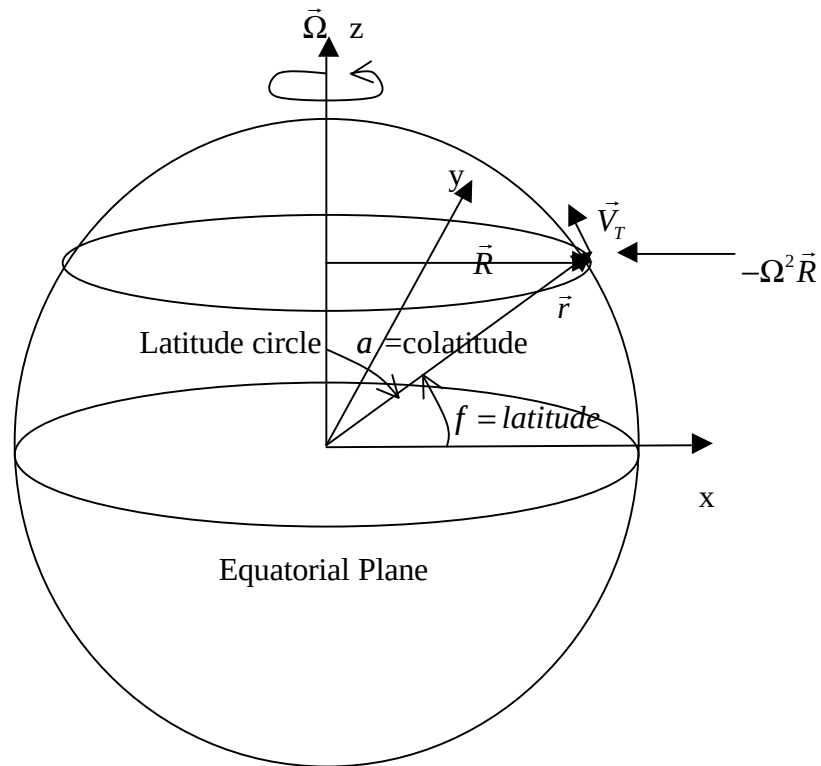
$$\begin{aligned} \frac{d\vec{V}_a}{dt} &= \frac{d}{dt} [\vec{V} + \vec{\Omega} \times \vec{r}] + \vec{\Omega} \times [\vec{V} + \vec{\Omega} \times \vec{r}] \\ &= \frac{d\vec{V}}{dt} + \frac{d}{dt} [\vec{\Omega} \times \vec{r}] + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times [\vec{\Omega} \times \vec{r}] \\ &= \frac{d\vec{V}}{dt} + \cancel{\frac{d\vec{\Omega}}{dt} \times \vec{r}} + \vec{\Omega} \times \frac{d\vec{r}}{dt} + \vec{\Omega} \times \vec{V} + \vec{\Omega} \times [\vec{\Omega} \times \vec{r}] \\ &= \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times [\vec{\Omega} \times \vec{r}] \end{aligned} \quad (7.10)$$

Apply the Newton's second law $\rightarrow$

$$\frac{d\vec{V}_a}{dt} = \boxed{\frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times [\vec{\Omega} \times \vec{r}]} = \frac{\sum_i \vec{F}_i}{m} \quad (7.11)$$

The part in the box is the **equation of motion** in the rotating coordinate system! It describes the change of (relative) velocity in time subjecting the net force. The forces on the right hand side are *real* forces, and the second and third term on the left arises because of the coordinate rotation, and there are *apparent* (not real) forces. We will discuss them in more details in the following.

We will look at the last term on the left with respect to an earth oriented coordinate system:



Note that $|\vec{R}| = |\vec{r}| \cos f$.

Look at the last term in (7.11):

$$\vec{\Omega} \times [\vec{\Omega} \times \vec{r}].$$

SHOW IT FOR YOURSELF that $\vec{\Omega} \times [\vec{\Omega} \times \vec{r}] = -\Omega^2 \vec{R}$! (7.12)

(Hint – use the right hand rule to find out the direction of this vector, then use the definition of cross product to determine its magnitude)

Using this we can rewrite equation (7.11) as

$$\frac{d\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R} = \frac{\sum \vec{F}_i}{m} \quad (7.13)$$

Abs. Accel Rel. Accel. Coriolis accel Centripetal accel Net real force

This is a VERY IMPORTANT EQUATION.

How do we make use of this equation?

Let's look at the forces on the right hand side of (7.13) in the real atmosphere. They are

Gravity = $\vec{g}$ (force per unit mass)

PGF (per unit mass) = $-\frac{1}{r} \nabla p$ (you have seen this before and you will derive this term in Dynamics I)

Friction = (neglected for now)

These are real forces (those seen and felt in an inertial / fixed reference frame), so let's put them into the equation:

$$\boxed{\frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R} = \vec{g} - \frac{1}{r} \nabla p} \quad (7.14)$$

this is the acceleration that we see and measure in our earth coordinate system – not an inertial frame, but the next 2 terms take that into account.

Rearranging (7.14) →

$$\frac{d\vec{V}}{dt} = -\frac{1}{r} \nabla p + \vec{g} - \frac{-2\vec{\Omega} \times \vec{V} + \Omega^2 \vec{R}}{\quad} \quad (7.15)$$

Coriolis force
Centrifugal force

Relative acceleration
Real forces
Apparent forces due solely to the rotation of the coordinate system

Equation (7.15) is "THE" EQUATION OF MOTION used in meteorology – it is the backbone of meteorology!

7.4 Forces in Equation of Motion

We are now going to spend more time looking in details at each of these terms.

7.4.1. Gravity

Let's look first at gravity. Recall that

$$\vec{g} = \frac{GM_e m}{a} \hat{r} \quad (7.16)$$

where M_e = mass of earth and a = mean radius of the earth. It turns out it is convenient to combine $\vec{g}$ and $\Omega^2 \vec{R}$, the **centrifugal** force.

Due to the presence of the centrifugal force that's directed away from the axis of rotation, the earth is actually elliptic, with the radius at the equator slightly larger than that at the poles. If one measures that force that's pulling a mass downward, it is the sum of true gravity (the one due to gravitational effect) and the centrifugal force, i.e.,

$$\vec{g}_{net} = \vec{g} + \Omega^2 \vec{R}. \quad (7.17)$$

Note that $\vec{g}$ points towards the center of the earth while $\vec{g}_{net}$ is slightly equatorward. Since $\vec{g}_{net}$ is what we really measure at the surface of the earth, from now on we drop the subscript net, and assume $\vec{g}$ includes both effects. $\vec{g}$ points in the direction of plumbline and is perpendicular to a static water surface – the horizontal plane.

Equation (7.15) becomes:

$$\frac{d\vec{V}}{dt} = -\frac{1}{r} \nabla p + \vec{g} - 2\vec{\Omega} \times \vec{V} \quad (7.17a)$$

here $\vec{g}$ is the $\vec{g}_{net}$ in (7.17).

See figure below.

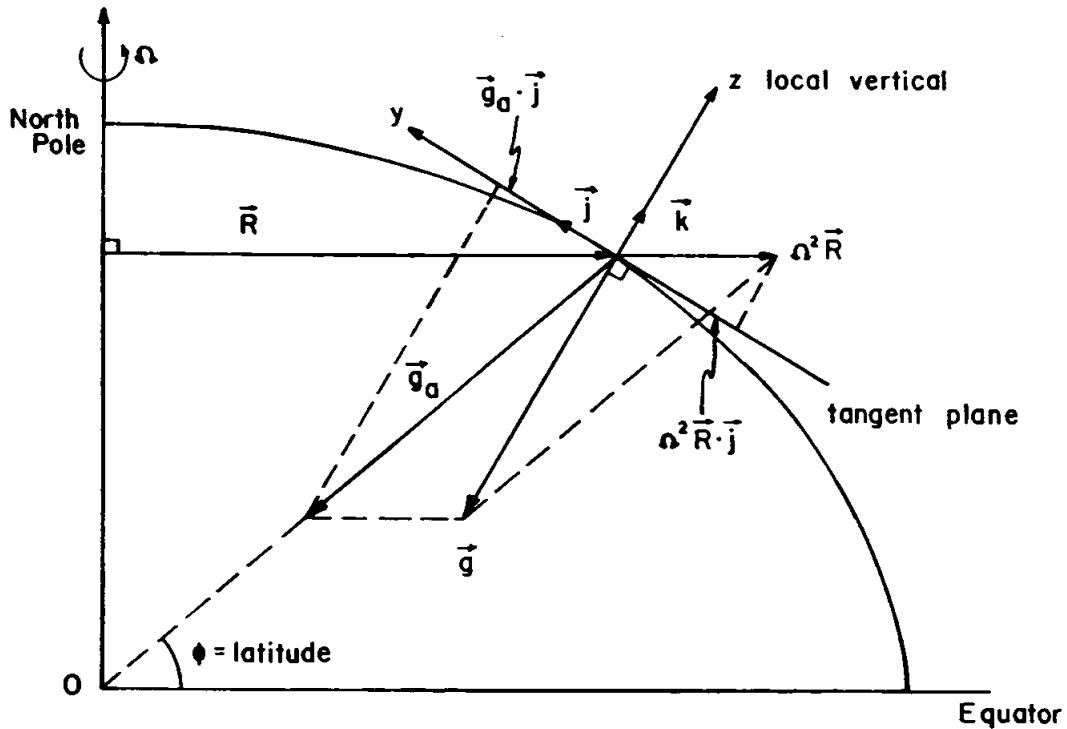


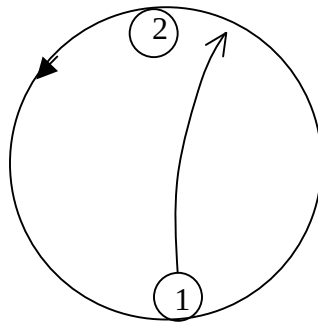
Fig. IV-6. The gravity force $\vec{g}$ is the resultant of the gravitation ($\vec{g}_a$) and centrifugal ($\Omega^2 \vec{R}$) forces.

Read Symon Sections 7.2 and 7.3, Riegal Pages 144-153!

7.4.2. Coriolis Force

Let's now look at the other apparent force – the Coriolis force. In contrast to the centrifugal force, **the Coriolis force is acting only when the object moves relative to the rotating / non-inertial coordinate system.**

Consider the case of a merry-go-around.



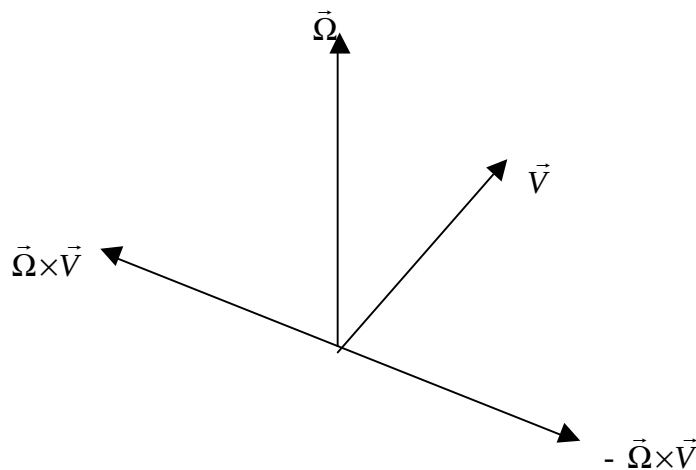
If person #1 rolls a ball toward 2 on a frictionless surface at a uniform speed in a straight line, it will appear to deflect to the right (in this case) – or in a direction opposite the rotation. To a person on the ground, the ball appears to travel in a straight line (non-

accelerating) – but from the coordinate system of the merry-go-around, the ball accelerates (velocity changes).

Let's look physically at the mathematical expression for the Coriolis force:

$$\vec{F}_{cor} = -2\vec{\Omega} \times \vec{V} \quad (7.18)$$

- (1) The minus sign is important, just like in advection. Don't forget it!
- (2) The Coriolis force acts perpendicular to the direction of the velocity vector and in direct proportion to its amplitude:



In the northern hemisphere, where $\vec{\Omega}$ is counterclockwise, the **Coriolis force acts to the right of the velocity vector**, i.e., it tends to deflect air parcel to the right of their direction of motion (see merry-go-around example).

- (3) The Coriolis force changes only the direction of the particle's motion, not its speed, because the force is always perpendicular to the direction of motion, just like the centripetal force in uniform circular motion.

Components of Coriolis Force

Now, because the Coriolis force is a vector, we need to look at its components. It turns out, as we will see, that it affects motion in all three directions:

$$\frac{du}{dt} = C_x + \text{other}$$

$$\frac{dv}{dt} = C_y + \text{other}$$

$$\frac{dw}{dt} = C_z + \text{other}$$

How do we obtain those three components? One is purely mathematical – we just need to find the components of $\vec{F}_{cor} = -2\vec{\Omega} \times \vec{V}$ in a chosen Cartesian coordinate system. We can also obtain them in a more physical way.

Let's do it in the physical way first.

Consider first the situation where we have a particle of unit mass moving freely on the frictionless surface of the rotating earth if the particle is initially at rest in the rotating frame, then only the centrifugal force is acting, along with the gravitational force. We combined the sum of these two into a new gravity that is perpendicular to the local horizontal.

Suppose now that the particle is suddenly set in motion toward the east. It is now rotating faster than the earth, and thus the centrifugal force will be stronger. If Ω = magnitude of angular velocity of the earth, $\vec{R}$ the position vector from the axis of rotation, as shown earlier, and u the eastward speed relative to the ground (i.e., the speed relative to the rotating coordinate system), then the total centrifugal force is that from earth's rotation plus that due to the eastward motion, i.e.,

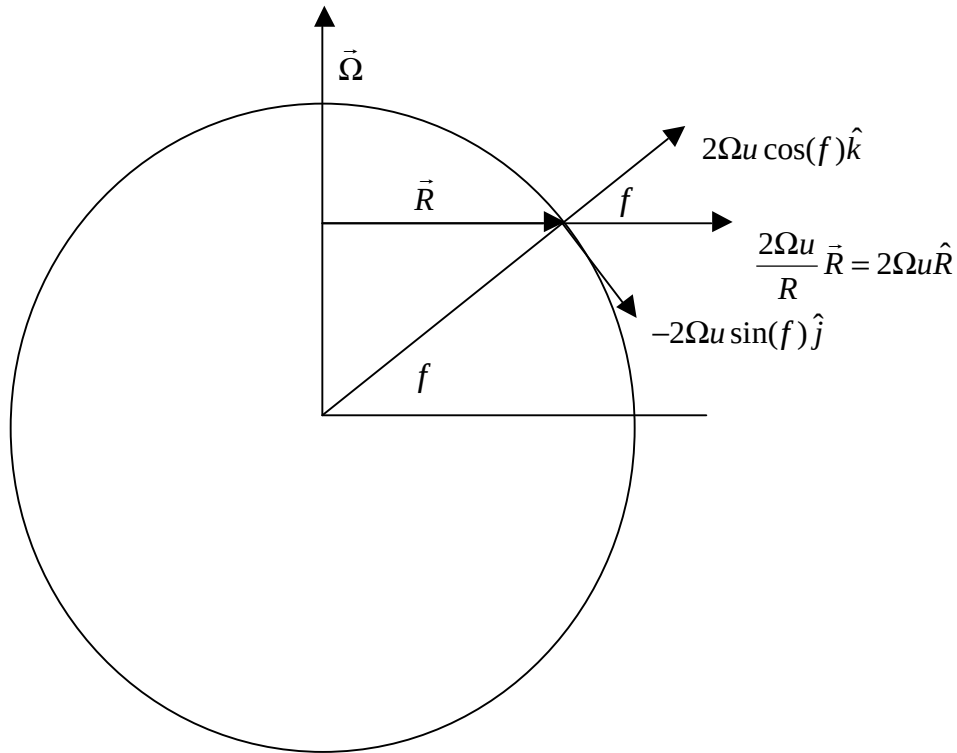
$$\left(\underset{\substack{\uparrow \\ \text{Original} \\ \text{rotation}}}{\Omega} + \underset{\substack{\uparrow \\ \text{Due to} \\ \text{relative} \\ \text{motion}}}{\frac{u}{R}} \right)^2 \vec{R} = \underset{\substack{\uparrow \\ \text{Centrifugal force} \\ \text{due to earth} \\ \text{rotation as before.} \\ \text{It is added to } \vec{g}}}{\Omega^2 \vec{R}} + \underset{\substack{\nearrow \\ \text{Deflecting force that} \\ \text{acts outward, in the} \\ \text{direction of } \vec{R}}}{\frac{2\Omega u}{R} \vec{R} + \frac{u^2}{R^2} \vec{R}} \quad (7.19)$$

For synoptic-scale motion, $|u| \ll \Omega R$ or $\frac{|u|}{R} \ll \Omega$, the last term can be neglected to a first approximation. How do we show this?

$$\frac{2\Omega u}{R} \vec{R} + \frac{u^2}{R^2} \vec{R} = \frac{2\Omega u}{R} \vec{R} \left(1 + \frac{u}{2\Omega R} \right) \approx \frac{2\Omega u}{R} \vec{R}$$

Thus, the only term left is the second one, which is the Coriolis force (recall that it's due to the relative motion only) due to motion along a latitude circle.

Let's look at the component of this vector!

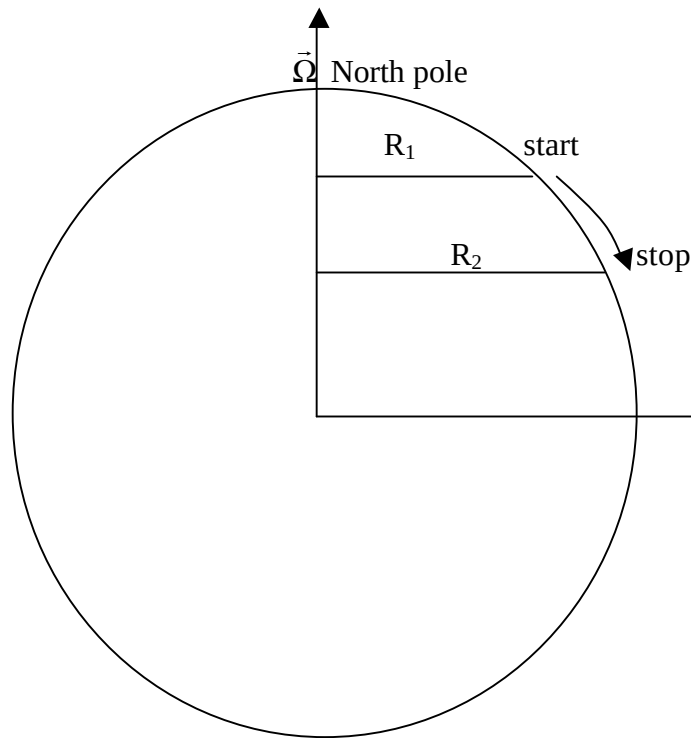


We see that, due to the Coriolis force alone,

$$\begin{aligned} \left(\frac{dv}{dt} \right)_{cor} &= -2\Omega u \sin(f) \\ \left(\frac{dw}{dt} \right)_{cor} &= 2\Omega u \cos(f) \end{aligned} \quad (7.20)$$

Thus, if a particle is moving eastward in the horizontal plane in the Northern Hemisphere, it is deflecting Southward and Upward! The opposite is true for a particle moving from east to west. If you plug in numbers, you will find that the vertical component of the Coriolis force is much smaller than gravity, and thus it usually does not cause big change in the vertical position (height) of an object, but the horizontal components can be significant, compared to other horizontal forces (horizontal pressure gradient force is the other dominant force).

Now, let's repeat this exercise for a particle displaced toward the equator. What's different here? Now, R , the distance from the particle from the axis of earth rotation changes (increases).



Because the forces acting on the particle are central, the torque is zero – there is no torque in the east-west direction, therefore, the particle conserves angular momentum. So, if R increases, what will happen to the velocity?

Initially, it has no motion tangential to the direction of rotation. The angular momentum is ΩR^2 , so as R increases, the Ω of the particle has to decrease $\rightarrow$ the absolute eastward velocity decreases, the particle starts to deflect toward west. So, the particle moving south is deflected to the west.

Let's try to derive mathematically the speed at which the deflection occurs.

Let δR be the change in R and δu the change in zonal wind speed. Note that $\delta \phi < 0$ for an equator-ward displacement but $\delta R > 0$. δu is to be determined.

$$\left(\frac{dq}{dt} R^2 \right)_{start} = \left(\frac{dq}{dt} R^2 \right)_{stop}$$

At the start, the particle has no speed in the east-west direction, so it's A.M. is

$$\Omega R^2$$

where Ω is earth's angular velocity.

At the ending point, it has acquired a relative angular velocity of $\frac{du}{R+dR}$, so the final A.M. is

$$\left(\Omega + \frac{du}{R+dR} \right) (R+dR)^2 = \Omega(R+dR)^2 + du(R+dR).$$

Equating the A.M at start and ending point $\rightarrow$

$$\Omega R^2 + 2\Omega R dR + \Omega(dR)^2 + du(R+dR) = \Omega R^2$$

Consider small displacements therefore small δu and δR , the 2 δ terms can be neglected (since they are even smaller), and solve for δu , we get

$$du = -2\Omega dR \quad (7.21)$$

Now, we can rewrite δR in terms of $\delta\phi$ as follows:

$$R = a \cos(\phi) \rightarrow \delta R = -a \sin(\phi) \delta\phi$$

Thus, $\delta u = 2\Omega a \delta\phi \sin(\phi) \rightarrow$

$$\left(\frac{du}{dt} \right)_{cor} = 2\Omega \frac{adf}{dt} \sin(f)$$

Furthermore, $a d\phi = dy$, therefore

$$\boxed{\left(\frac{du}{dt} \right)_{cor} = 2\Omega v \sin(f)} \quad (7.22)$$

(7.22) gives the acceleration (deflection) in east-west direction due to the earth-relative motion in the north-south direction.

For northward motion, $v > 0 \rightarrow du/dt > 0 \rightarrow$ eastward deflection, deflection to the right.

If you repeat the same analysis for a particle launched vertically, then the conservation of absolute angular momentum shows that you will see a

$$\left(\frac{du}{dt} \right)_{cor} = -2\Omega w \cos(f) \quad (7.23)$$

i.e., the particle will accelerate toward the east. Adding this to the result just obtained gives the component of the Coriolis force in the zonal direction!

$$\left(\frac{du}{dt} \right)_{cor} = 2\Omega v \sin(f) - 2\Omega w \cos(f) \quad (7.24)$$

Having derived the components of the Coriolis force, $-2\vec{\Omega} \times \vec{V}$, the equation of motion, (7.15) can now be written as (neglecting friction):

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial x} + 2\Omega v \sin(f) - 2\Omega w \cos(f) \\ \frac{dv}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial y} - 2\Omega u \sin(f) \\ \frac{dw}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial z} - g + 2\Omega u \cos(f) \end{aligned} \quad (7.25)$$

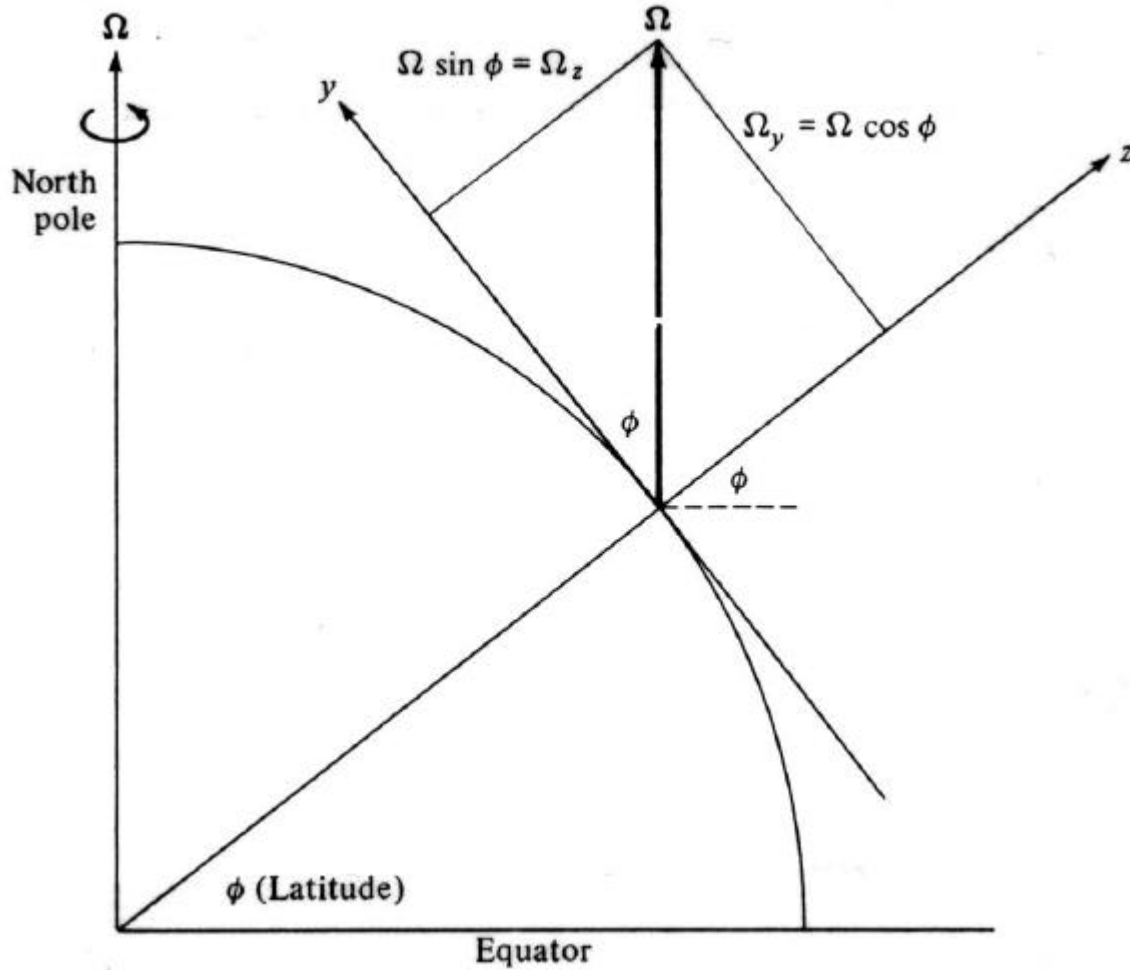
If we let $f = 2\Omega \sin(f)$ and $\tilde{f} = 2\Omega \cos(f)$ (they are called Coriolis parameters) then

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial x} + f v - \tilde{f} w \\ \frac{dv}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial y} - f u \\ \frac{dw}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial z} - g + \tilde{f} u \end{aligned} \quad (7.26)$$

The above equations are EXTREMELY important for meteorology!

In the above, we derived the components of the Coriolis force using physical methods. Let's now derive these components using a purely **mathematical method**.

We choose a Cartesian coordinate system whose x-y coordinate surface is tangent to the surface of the surface (see figure below). The coordinate origin is at the earth surface of a given latitude and longitude.



The Coriolis force per unit mass is given by $-2\vec{\Omega} \times \vec{V}$. We can expand this mathematically for the Cartesian-coordinate as follows:

$$-2\vec{\Omega} \times \vec{V} = -2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

Of course, we need the components of the earth's rotation vector $(\Omega_x, \Omega_y, \Omega_z)$, and we can most easily obtain them by defining a coordinate system with its origin at a point on

the earth's surface with the z-axis parallel to the radial vector ($\vec{r}$) with z increasing away from the earth's surface (local vertical -- see figure above). The x and y axes together form a plane that is tangent to the earth, with x increasing toward the east and y increasing toward the north. The figure shows the earth's rotation vector at the defined point on the earth, along with its components (projected onto the three Cartesian axes). Note that the x-component is zero (x-axis into the page) because the rotation vector is perpendicular to the x-axis. Thus, based upon the geometry shown, we find that

$$\Omega_x = 0, \quad \Omega_y = \Omega \cos f, \quad \Omega_z = \Omega \sin f.$$

Using these in the above determinant, we have

$$-2\vec{\Omega} \times \vec{V} = \hat{i}(2\Omega v \sin f - 2\Omega w \cos f) + \hat{j}(-2\Omega u \sin f) + \hat{k}(2\Omega u \cos f) \quad (7.27)$$

Plug (7.27) into (7.17a), we obtain exactly the same equations for velocity components as given in (7.25)!

The mathematical way is apparently simpler and more elegant. The drawback is that the physical meaning of each individual term is lost in the derivation.

Comments:

You can see from the equations that the acceleration due to Coriolis effect is proportional to the velocity – and deflection it causes has to be considered when the force acts on an object for a long time, i.e., when the object travels over a long distance, such as the upper-level air stream does! Therefore for large-scale atmospheric flows, Coriolis force can not be neglected.

Example 1:

Suppose a ballistic missile is first due eastward at 43 degrees N latitude. If the missile travels 1000km at a speed of 500 m/s, by how much is it deflected due to the Coriolis force?

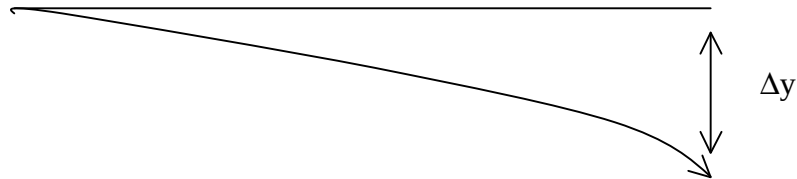
Given: $\phi = 43^\circ\text{N}$,
 u of missile = 500m/s
 distance traveled = 1000km = 10^6 m

To find: lateral deflection,

The equations to use is (how do we know)?

$$\frac{dv}{dt} = -2\Omega u \sin(f) .$$

Note. Missile starts with $u \neq 0$ and $v = 0$, but soon is deflected to the right (south) $\rightarrow u \neq 0$ and $v < 0$. So we should get $v < 0$ and $\Delta y < 0$, i.e., a southward deflection.



Δy is what we ultimately need.

To be Δy , we first have to find v .

$$\begin{aligned} \frac{dv}{dt} &= -2\Omega u \sin(f) = -2(7.292 \times 10^{-5} \text{ s}^{-1})(500 \text{ m/s}) \sin(43^\circ) \\ &= -0.0497 \text{ s}^{-1} \end{aligned}$$

$$\int_{v_0=0}^{v(t)} dv = \int_0^t -0.0497 \text{ s}^{-1} dt$$

$$v(t) = -0.0497t = \frac{dy}{dt}$$

$$\int_0^{y(T)} dy = \int_0^t (-0.0497t) dt$$

$$y(T) = -0.0497 \frac{T^2}{2}$$

What is T ? $T = \frac{1000000 \text{ m}}{500 \text{ m/s}} = 2000 \text{ s} .$

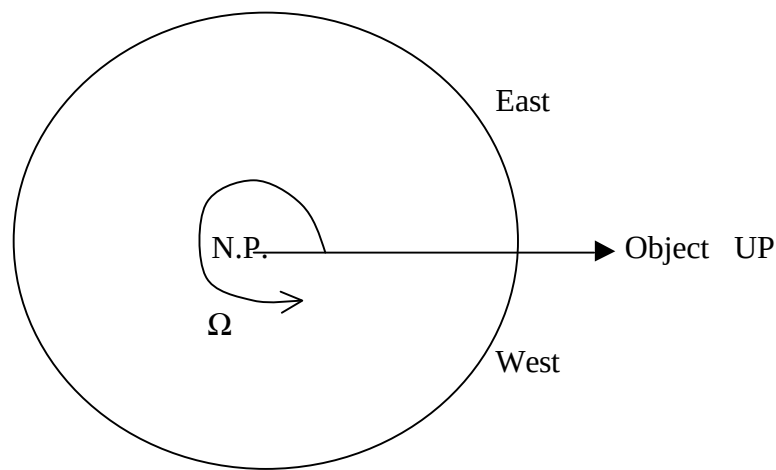
Therefore $\Delta y = -0.0497 \frac{2000^2}{2} = -99.4 \text{ km} .$

Example 2:

Suppose we drop an object from a tower of height h at the equator. When will it land relative to a "plumb line" at the release point?

Let's first try to guess the answer – will it fall to the east or west?

In this problem, gravitational and centrifugal forces are in the radial direction, i.e., no projection onto the longitudinal circle (c.f. Figure in page 7-13). So, our picture is relatively simple: Looking from above:



The Coriolis force is in the equatorial plane (no y deflection). What equation governs the motion? The object falls at a speed w (we use info from chapter 4 on falling body), and we expect the deflection to be east-west. So, we need an equation that, given w , predicts the evolution of u . Thus, we use equation (7.24)

$$\frac{du}{dt} = 2\Omega v \sin(f) - 2\Omega w \cos(f).$$

Initially $v = 0$, so the first term is zero $\rightarrow \frac{du}{dt} = -2\Omega w \cos(0) = -2\Omega w$.

Now, $m \frac{dw}{dt} = -mg \rightarrow$

$$\int_{w_0}^{w(t)} dw = \int_0^t (-g) dt$$

$$w(t) = -gt$$

Plug $w(t)$ into du/dt equation and integrating $\rightarrow$

$$\int_0^{u(t)} du = 2\Omega g \int_0^t t dt$$

$$u(t) = \Omega g t^2$$

The deflection is found from $dx/dt = u(t) = \Omega g t^2$

or

$$\Delta x = \Omega g t^3 / 3.$$

Since $\Delta x > 0$, the object falls to the EAST of the drop point.

If the height to fall is h , then

$$\frac{dz}{dt} = w = -gt$$

$$\int_h^z dz = -g \int_0^t t dt$$

$$z = h - gt^2 / 2$$

For $z = 0$, the time T is

$$T = \sqrt{\frac{2g}{h}} \rightarrow$$

$$\Delta x = \frac{1}{3} g \Omega \left(\frac{2g}{h} \right)^{3/2}$$

For a 500 m tower, $\Delta x \sim 0.77$ cm at the equator. This is no different from the merry-go-around example!

7.5. Foucault Pendulum

We live on the rotating earth. How do we know that the earth is rotating? It's easy to tell if you observe the earth from the space, but not so if you stand at the surface of the earth.

There is actually a simple way to demonstrate that the earth is rotating, even inside a closed building. It is called Foucault pendulum. It was invented by French physics, Foucault*, by accident.

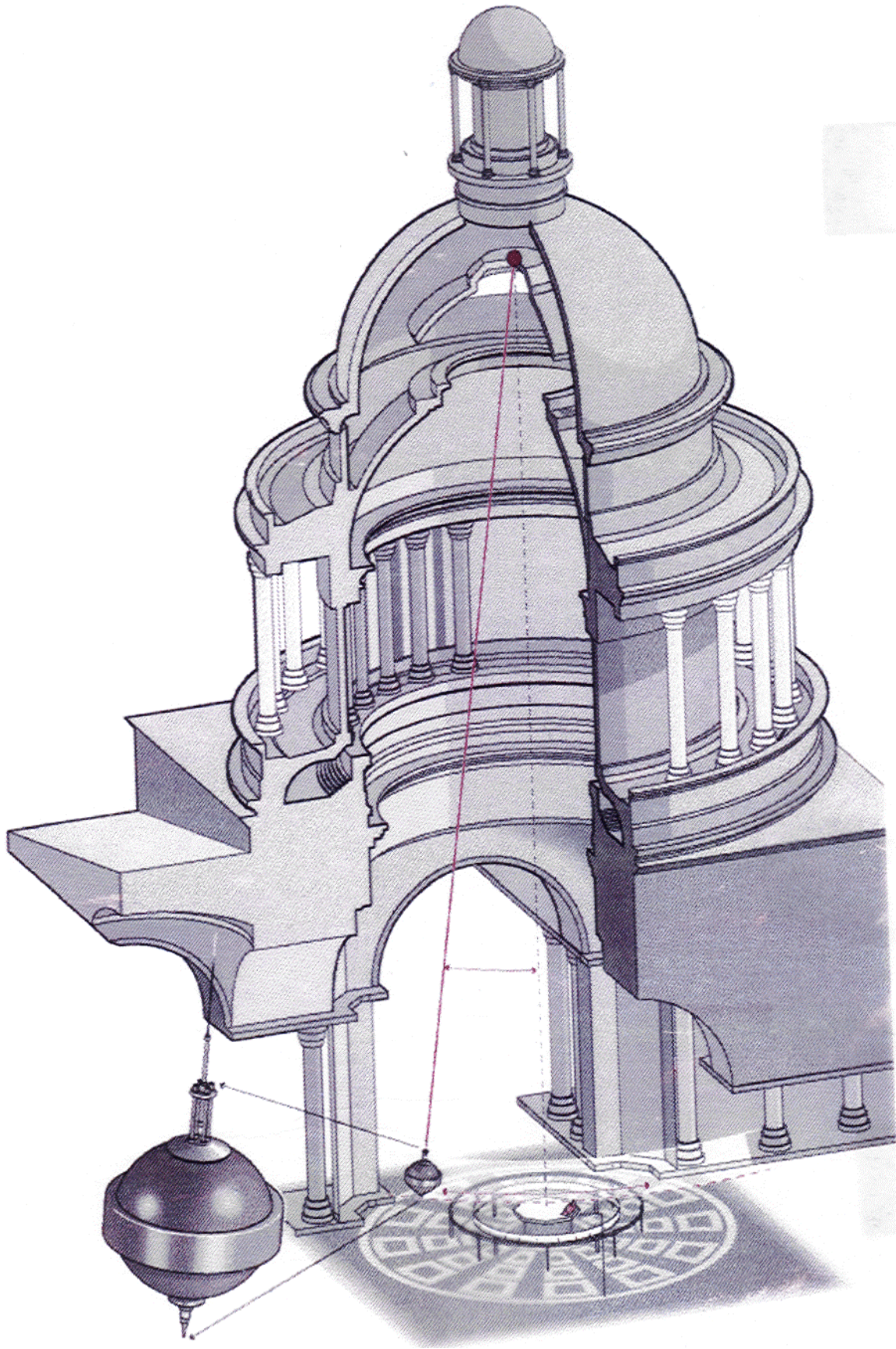
***Foucault, Jean Bernard Leon.** 1819-1868. French physicist who estimated the speed of light and determined that it travels more slowly in water than in air (1850). He also employed a pendulum to prove the rotation of the earth (1851) and invented the gyroscope (1852).

An interesting article on foucault pendulum can be found at <http://www.abc.net.au/surf/pendulum/default.htm>.

See illustration on the next page and a photo of a real thing.



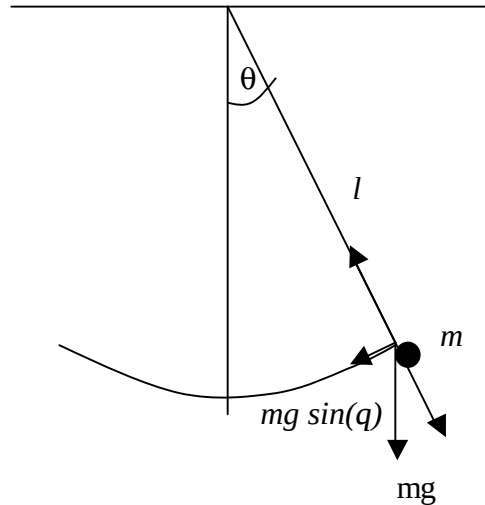
A picture of Foucault pendulum at Smithsonian Institution's (<http://www.si.edu>) National Museum of American History (<http://americanhistory.si.edu>) in Washington, D.C. Taken in 1996. Courtesy of April Strout. As the pendulum moves, it knocks over the tiny red things around the circumference of the circle.



It is basically a large pendulum hanging from a tall building / church hall that is fixed to the rotating earth.

Let's look at the simple pendulum first.

We know for a regular pendulum made up of a mass m hanging from a massless string of length l ,



the equation describing the motion is

$$m \frac{d}{dt} \left[\frac{d(lq)}{dt} \right] = -mg \sin(q)$$

i.e.,

$$l\ddot{q} = -g \sin(q).$$

Since we assume that the swing is mild, i.e., q remains relatively small, we can approximate $\sin(q)$ with q , i.e., $\sin(q) \approx q$ (because $\sin(q) = q + q^2/2 + \dots$) $\rightarrow$

$$\ddot{q} + \frac{g}{l}q = 0. \quad (7.28)$$

This is exactly the SHO equation we solved earlier in the semester. It has a general solution of

$$q(t) = A \cos(\omega t) + B \sin(\omega t)$$

where $w = \sqrt{\frac{g}{l}}$ (you can show it yourself). It has the unit of s^{-1} and is the frequency of oscillation. If the pendulum starts from rest at angle θ_0 ($d\theta/dt = 0$ at $t = 0$... from rest), then $A = \theta_0$ and $B=0 \rightarrow$

$$q(t) = q_0 \cos(wt) . \quad (7.29)$$

In the absence of friction, the motion is perpetual and periodic. The period is

$$T = \frac{2\pi}{w} = 2\pi \sqrt{\frac{l}{g}} . \quad (7.30)$$

Note that the period is *independent* of the amplitude θ_0 , and this is why pendulum are good for use in clocks! They are very regular!

Now let's go back to the Foucault pendulum. The pendulum is set in motion, if we neglect the effect of friction and observe it from a fixed star, what will we see?

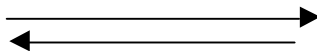
We should see that the pendulum behaves like a simple pendulum, doing its periodic motion perpetually, and within a fixed vertical plane whose angle does not change relative to the star, but because the earth is rotating underneath it, the angle of the plane changes relative to the earth. This is easier to understand if the pendulum is located at the north pole. The earth rotates underneath the pendulum while the hanging point of the pendulum does not change. As a result, we see relative motion between the pendulum and the earth.

What if we observe the pendulum standing on the earth? Will we still see the simple periodic motion of a simple pendulum? Does the earth rotation have any effect? Yes – remember the Coriolis force in a rotating frame?

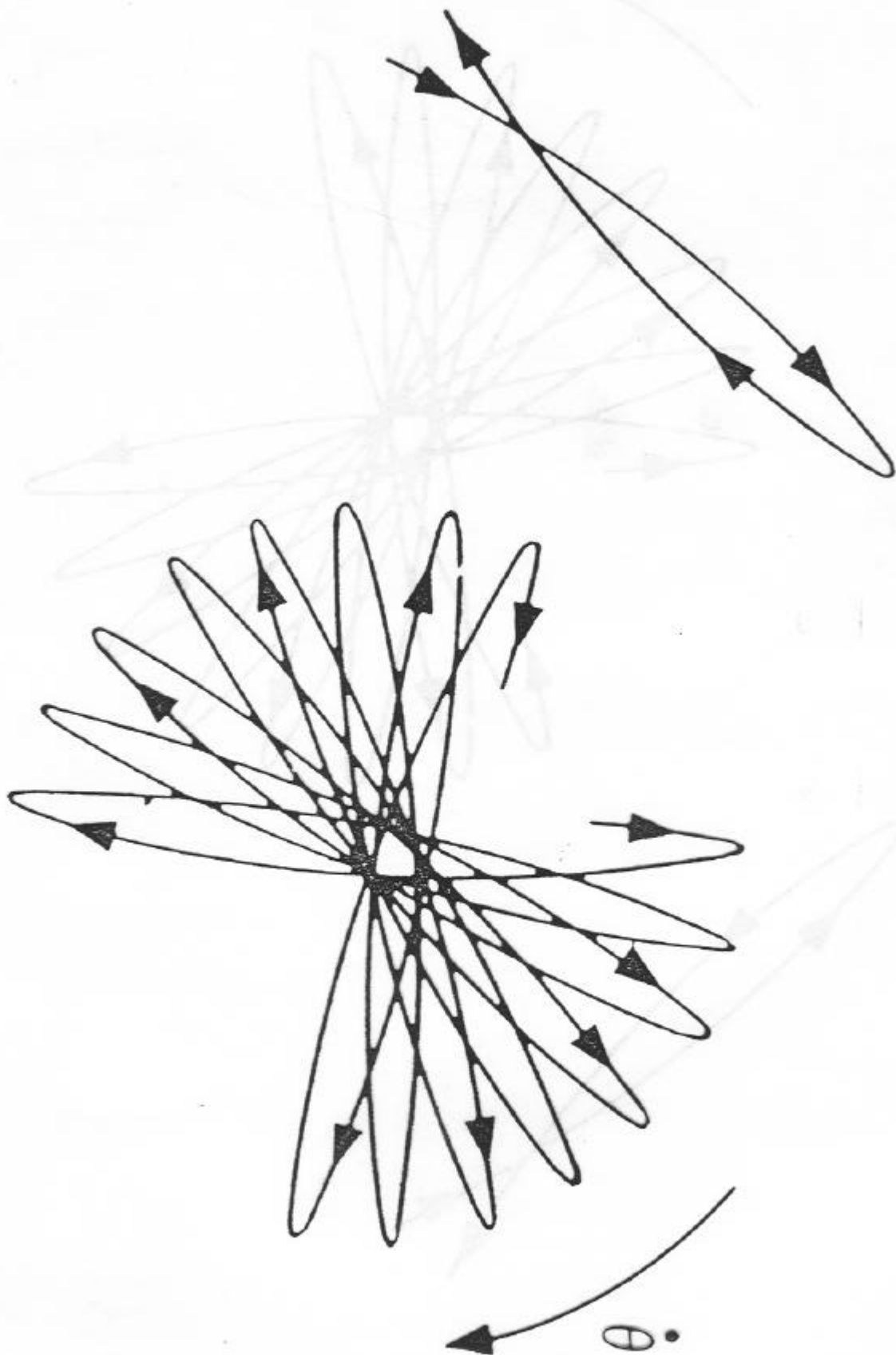
Because the plane in which the pendulum swings is fixed in the absolute coordinate while the earth we stand on rotates counterclockwise relative to that plane, we (who rotates together with the earth) observe that the pendulum plane rotates clockwise relative to the us and the earth!

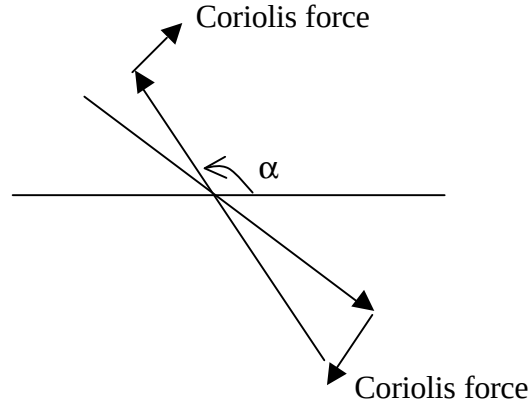
In the earth coordinate, such 'mysterious' rotation of the pendulum plane is due to the Coriolis force, which we know always acts perpendicular to the direction of motion.

If we view the pendulum from the top, the trajectory is a fixed straight line if there is no Coriolis force:



When Coriolis force is present, it deflects the 'bob' of the pendulum to its right, the trajectory becomes something like





Applying the equation of motion in the direction normal to the direction of bob movement, i.e., $F_{\text{cor}} = ma$, we can find (details not show) that the change in the angle α of pendulum plane relative to the earth is governed by equation

$$\frac{da}{dt} = -\Omega \sin(f). \quad (7.31)$$

The right hand side is actually the (local) vertical component of the angular velocity of the earth, except for the negative sign. Since the earth rotates counterclockwise, the pendulum plane rotates clockwise (indicated by the negative sign). So we observe a trajectory depicted in the previous page.

The rate of rotation of the pendulum plane is equal to Ω at the north pole ($-\Omega$ at the south pole) and zero at the equator (the Foucault pendulum will not work if placed at the equator).

The period of rotation of the pendulum plane is

$$T = \frac{2p}{a} = \frac{2p}{\Omega \sin(f)}. \quad (7.32)$$

At the north pole, it is exactly 24 hours, and at 45 degree latitude, it is about 34 hours, and at the equator, there is not rotation and the period is infinitely large. At the equator, we will not see the effect of earth rotation – the Coriolis force is zero there.

It is the projection of earth's angular velocity to the local vertical direction [$\Omega_z = \Omega \sin(f)$] that matters!

7.6. Coriolis Force and Weather

The Coriolis force is extremely important for large-scale atmospheric flow. It is actually one of the largest force acting on air parcels in the horizontal direction. The balance between the Coriolis force and horizontal pressure gradient force give rise to the so-called geostrophic wind that is the dominant component of winds at the large scales.

Earlier, we presented the equations of motion

$$\begin{aligned}\frac{du}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial x} + fv - \tilde{f}w \\ \frac{dv}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial y} - fu \\ \frac{dw}{dt} &= -\frac{1}{r} \frac{\partial p}{\partial z} - g + \tilde{f}u\end{aligned}\tag{7.33}$$

where $f = 2\Omega \sin(\phi)$ and $\tilde{f} = 2\Omega \cos(\phi)$. At the large scales (scales on the order of 1000 km), the vertical wind speed is much smaller than the horizontal speed. Vertical motion becomes large only in small-scale systems such as the thunderstorm, hurricanes and tornadoes. Therefore we focus our attention on horizontal motion (u,v). We also neglect the Coriolis force related to w as appearing in u equation. We therefore have

$$\frac{du}{dt} = -\frac{1}{r} \frac{\partial p}{\partial x} + fv\tag{7.34a}$$

$$\frac{dv}{dt} = -\frac{1}{r} \frac{\partial p}{\partial y} - fu.\tag{7.34b}$$

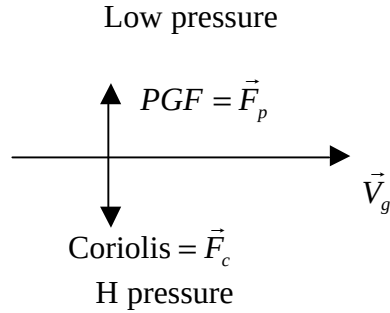
7.6.1. Geostrophic Flow

The geostrophic winds exist when there is a perfect balance between the pressure gradient force and the Coriolis force. This balance is called geostrophic balance. As a result, the acceleration is zero (assuming friction can be neglected). Scale analysis (you will learn it next semester) shows that this is nearly true for large-scale flows, hence

$$-\frac{1}{r} \frac{\partial p}{\partial x} + fv_g = 0\tag{7.35a}$$

$$-\frac{1}{r} \frac{\partial p}{\partial y} - fu_g = 0\tag{7.35b}$$

We denote winds that satisfy geostrophic balance, $\vec{V}_g = (u_g, v_g) = \hat{i}u_g + \hat{j}v_g$.



Equation (7.35) can be written in a vector form:

$$-\frac{1}{r} \nabla p - f \hat{k} \times \vec{V}_g = 0 \quad (7.36)$$

(because $\hat{k} \times \vec{V}_g = \hat{k} \times (\hat{i}u_g + \hat{j}v_g) = \hat{j}u_g - \hat{i}v_g$), or

$$-\frac{1}{r} \nabla p = f \hat{k} \times \vec{V}_g \quad (7.37)$$

$\hat{k} \times (7.37) \rightarrow$

$$-\frac{1}{r} \hat{k} \times \nabla p = f \hat{k} \times (\hat{k} \times \vec{V}_g) = -f \vec{V}_g$$

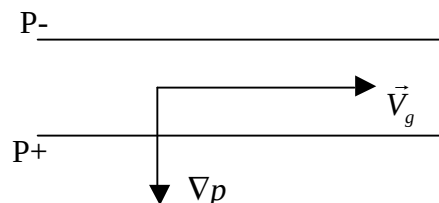
therefore

$$\vec{V}_g = \frac{1}{f r} \hat{k} \times \nabla p$$

(7.38)

(7.38) is the vector form of (7.35) (it can also be obtained directly from 7.35). It defines the geostrophic wind.

It says that, in the northern (southern) hemisphere, if you walk in the direction of geostrophic winds, high pressure is always on your right (left). The southern hemisphere is opposite because the Coriolis parameter f is negative there.



The large-scale (synoptic scale and above) upper-level (where friction can be neglected) atmosphere flow is nearly geostrophic. Therefore the wind is essentially parallel to the geopotential height (a representation of pressure field on a constant pressure surface) contours or perpendicular to the pressure gradient force. See figure below.

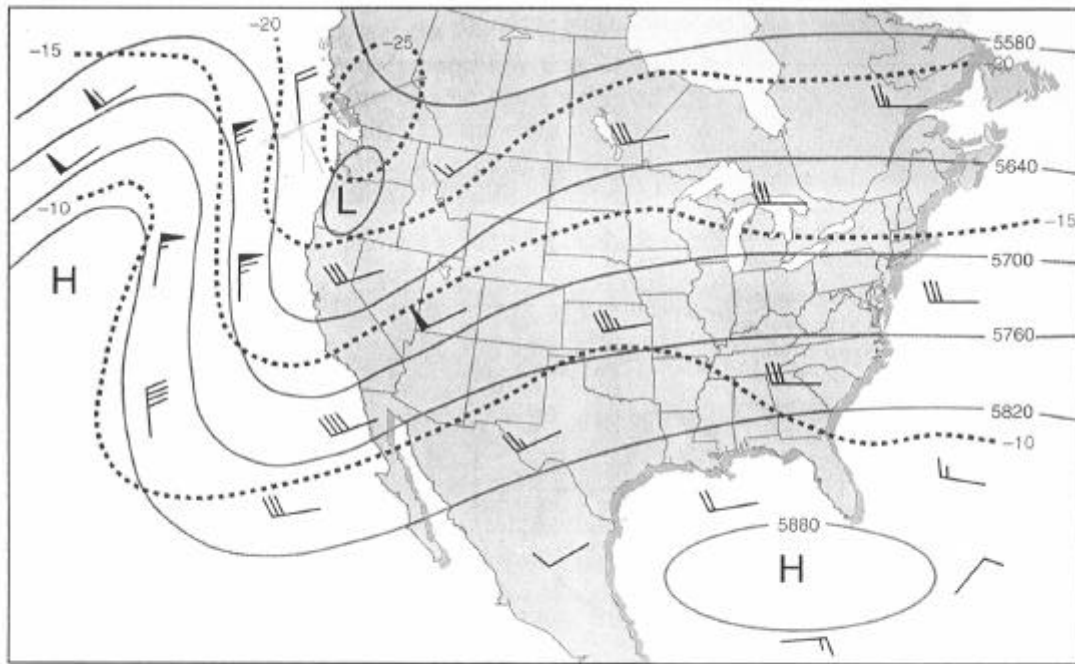


FIGURE 9.26

A 500-mb chart. Solid lines are contours (meters above sea level). Dashed lines are isotherms in °C.

Let's look closer at the geostrophic wind equation

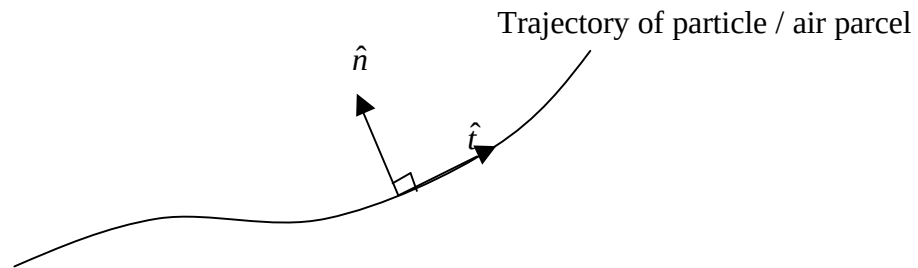
$$u_g = -\frac{1}{f} \frac{\partial p}{\partial y}.$$

- 1) If $\frac{\partial p}{\partial y} < 0$ (pressure decreases to the north), then $u > 0 \rightarrow$ high pressure on the right.
- 2) At the equator, $f = 0$, the geostrophic wind is undefined – i.e., there is no geostrophic balance to talk about at the equator.
- 3) $f = 2W\sin(\phi)$ is largest at the poles. Given the same pressure gradient force, the velocity is smallest at the poles, there one needs a smaller u_g to obtain the same Coriolis force to balance the pressure gradient force.

- 4) If we know that the wind is in a geostrophic balance, we only need to observe the pressure field in order to know the wind field. The geostrophic balance can be used to quality control wind and pressure observations.
- 5) Near the surface, the geostrophic balance is usually not as well satisfied because of friction, one more force that needs to be considered in the force balance. Also for flows with large curvature, centrifugal force (arising from centripetal acceleration, not from earth rotation) also needs to be considered.

7.6.2. Curved Flow, Natural Coordinates

For straight airflow along straight parallel isobaric contours, when a geostrophic balanced is reached, acceleration due to either speed and direction change is zero. In reality, airflow is rarely perfectly straight, its **trajectory** is often **curved**. To be study curved flows, it is convenient to define and use the **natural coordinates**.



Nature coordinates consist of 2 unit vectors, one along the direction of motion ($\hat{t}$) and one normal to and directed to the left of the velocity vector ($\hat{n}$). The coordinate directions are s and n , respectively.

<u>Cartesian</u>	$x, \hat{i}$	$y, \hat{j}$
------------------	--------------	--------------

<u>Natural</u>	$s, \hat{t}$	$n, \hat{n}$
----------------	--------------	--------------

Here we assume we only deal with 2D motion.

In the natural coordinates, because $\hat{t}$ is always in the direction of velocity,

$$\vec{V} = V\hat{t} . \quad (7.39)$$

If s is length along the trajectory, $V = \frac{ds}{dt}$.
 V is always positive.

The acceleration in the natural coordinates is

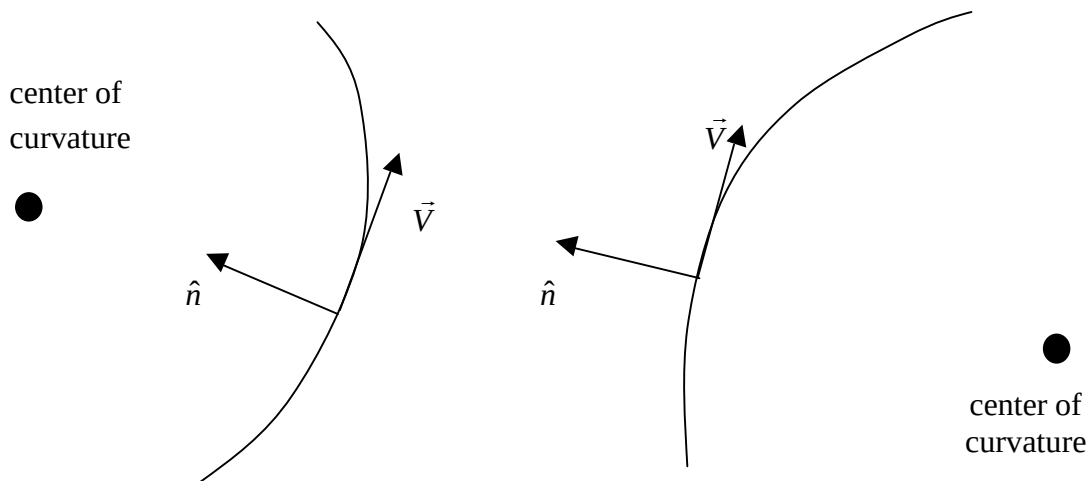
$$\frac{d\vec{V}}{dt} = \frac{d(V\hat{t})}{dt} = \frac{dV}{dt}\hat{t} + V\frac{d\hat{t}}{dt} = \frac{dV}{dt}\hat{t} + \frac{V^2}{R}\hat{n} \quad (7.40)$$

In obtaining (7.40), we used

$$\frac{d\hat{t}}{dt} = \frac{V}{R}\hat{n} \quad (7.41)$$

where $R = \text{radius of curvature}$ following the parcel's motion. It is analogous to the time derivative of the unit vector in radial direction of the plane polar coordinate and we will not derive it rigorously here.

Unlike the radius of a circle, the radius of curvature R can have both positive and negative signs. It is defined as



The center of curvature on the left side
 ($+\hat{n}$ side) of trajectory $R > 0$

The center of curvature on the right side
 ($-\hat{n}$ side) of trajectory, $R < 0$

Note that R is locally calculated and can change long the trajectory.

In (7.40), we have

$$\begin{array}{ccc} \frac{d\vec{V}}{dt} & = \frac{dV}{dt} \hat{t} & + \frac{V^2}{R} \hat{n} \\ \text{Total} & \text{Acceleration} & \text{Acceleration due to} \\ \text{acceleration} & \text{due to speed} & \text{direction change –} \\ & \text{change} & \text{centripetal acceleration} \end{array} \quad (7.42)$$

In the equation of motion, $\frac{d\vec{V}}{dt} = -\frac{1}{r} \nabla p - f\hat{k} \times \vec{V}$, the pressure gradient force can be written as

$$-\frac{1}{r} \nabla p = -\frac{1}{r} \frac{\partial p}{\partial s} \hat{t} - \frac{1}{r} \frac{\partial p}{\partial n} \hat{n}. \quad (7.43)$$

What about the Coriolis force $-f\hat{k} \times \vec{V}$? We know it always points to the right of trajectory (assuming northern hemisphere where $f > 0$). The result we obtain will still be valid when $f < 0$, therefore

$$-f\hat{k} \times \vec{V} = -fV\hat{n} \quad (7.44)$$

Putting all three expressions together $\rightarrow$

$$\frac{dV}{dt} \hat{t} + \frac{V^2}{R} \hat{n} = -\frac{1}{r} \frac{\partial p}{\partial s} \hat{t} - \frac{1}{r} \frac{\partial p}{\partial n} \hat{n} - fV\hat{n}$$

Equating the $\hat{n}$ and $\hat{t}$ components, we obtain the equations of motion for the respective components:

$$\frac{dV}{dt} = -\frac{1}{r} \frac{\partial p}{\partial s} \quad (7.45a)$$

$$\frac{V^2}{R} = -\frac{1}{r} \frac{\partial p}{\partial n} - fV \quad (7.45b)$$

Notes:

- (1) The equations are coupled through V and p.
- (2) For motion parallel to the isobars, $\frac{\partial p}{\partial s} = 0 \rightarrow V = \text{constant}$ following the motion.
- (3) For purely straight flow, $R = \infty \rightarrow$ centripetal acceleration and the corresponding centrifugal force is zero. We get the geostrophic relation $-\frac{1}{r} \frac{\partial p}{\partial n} - fV = 0 \rightarrow$

$$V_g = -\frac{1}{f r} \frac{\partial p}{\partial n}.$$

- (4) We know that if we choose a coordinate system that follows the air parcel undergoing a curved motion, we experience a centrifugal force that is related to the centripetal acceleration seen in a fixed coordinate. (7.45b) can be rewritten as

$$0 = -\frac{1}{r} \frac{\partial p}{\partial n} - fV - \frac{V^2}{R} \quad (7.46)$$

where $-\frac{V^2}{R}$ is the centrifugal force.

We will look at two additional types of flow in the following sections.

7.6.3. Cyclostrophic Flow

(Read 4-page handout of today. From Holton – Introduction to Dynamic Meteorology).

For flows with large curvature (small R. e.g., tornadoes, dust devils, water sprouts), the centrifugal force $\gg$ Coriolis force, therefore the latter can be neglected. We then have the following equation

$$0 = -\frac{1}{r} \frac{\partial p}{\partial n} - \frac{V^2}{R}$$

which is a balance between the pressure gradient force and the centrifugal force. The velocity given is:

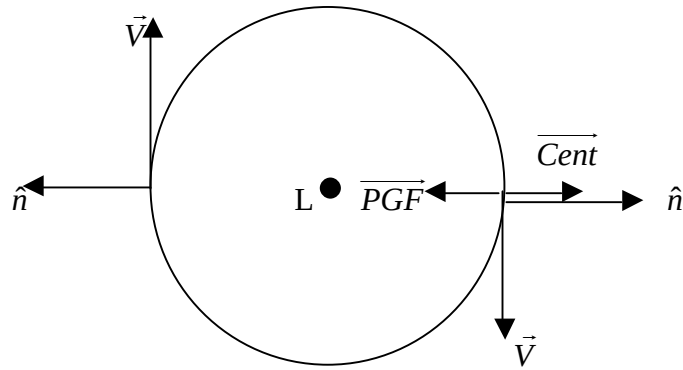
$$V = \sqrt{-\frac{R}{r} \frac{\partial p}{\partial n}}. \quad (7.47)$$

What kind of pressure pattern is possible? Physically, we know that the centrifugal force always directs away from the center of rotation (center of curvature in general), therefore to balance it, the pressure gradient force has to point to the center of curvature. This

means for a closed circulation, the circulation center always has to be a center of low pressure!

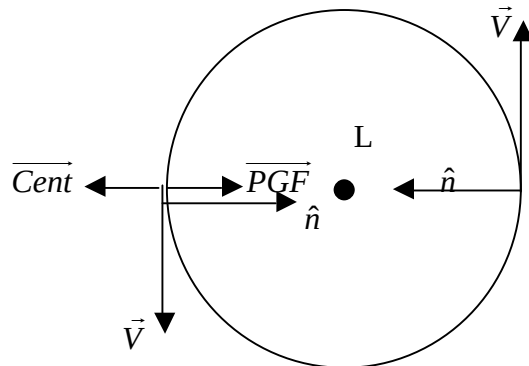
Does (7.47) fit our physical reasoning above? There are two possible cases.

CASE I: $R < 0$ and $\frac{\partial p}{\partial n} > 0$, V is real.



$R < 0$ and $\frac{\partial p}{\partial n} > 0$, V is real

CASE II: $R > 0$ and $\frac{\partial p}{\partial n} < 0$, V is real.



$R > 0$ and $\frac{\partial p}{\partial n} < 0$, V is real

Therefore, for cyclostrophic circulation, the pressure is always lower at the center and the circulation can be either cyclonic (counterclockwise) and anticyclonic (clockwise).

Anticyclonic low-pressure systems rarely occur at the large scales, however, as the Coriolis comes into play. The balance between pressure gradient force and the Coriolis force becomes dominant and the centrifugal force is small relative to those two (partly due to large R).

7.6.4. Gradient Wind

Gradient flow/wind is a three-way balance between the Coriolis force, pressure gradient force and the centrifugal force. Because of the balance, the acceleration in the direction of trajectory $dV/dt = 0$. The equation of force balance is (7.46)

$$0 = -\frac{1}{r} \frac{\partial p}{\partial n} - fV - \frac{V^2}{R}$$

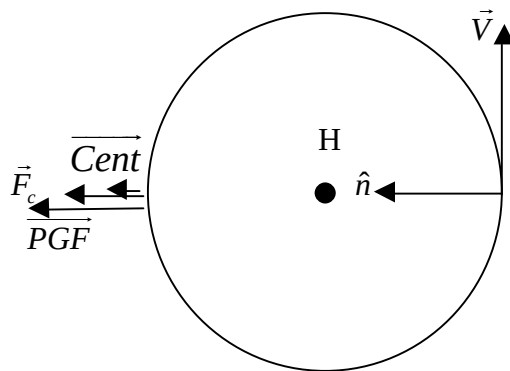
solving for $V \rightarrow$

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2}. \quad (7.48)$$

The physically meaningful solution (V is real and positive) exists only in certain situations.

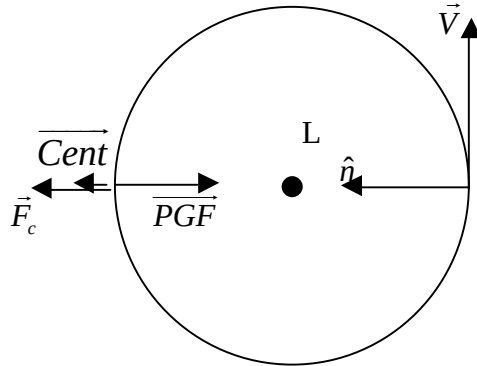
The four possible scenarios are illustrated in the following figures.

CASE I: $R > 0$ and $\frac{\partial p}{\partial n} > 0$, then V will be either complex or negative – neither is permitted.



All three forces point in the same direction – impossible to have a balance.

CASE II: $R > 0$ and $\frac{\partial p}{\partial n} < 0$, V is real and positive (with + sign in front of square root)

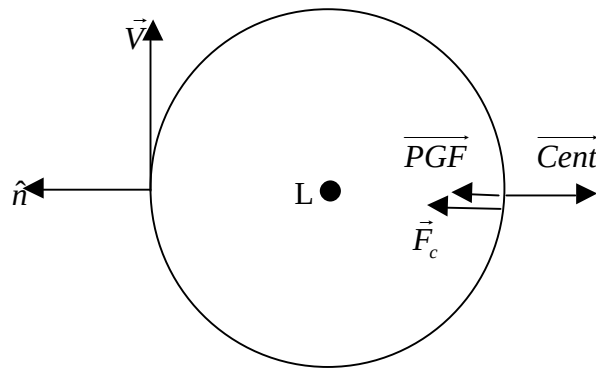


This is a case of **Normal or Regular Low** with **baric flow**.

A flow is called **baric** when Coriolis force and PGF are opposite in directions.

Such a flow and pressure pattern is typical of the low-pressure systems, mid-latitude cyclones observed in the atmosphere.

CASE III: $R < 0$ and $\frac{\partial p}{\partial n} > 0$.



In this case, the PGF and F_c are in the same direction. Such flow is called **antibaric**. **PGF and F_c are balanced by the centrifugal force.** Obviously the centrifugal force has to be large enough – which requires large speed, large curvature (small radius of curvature) or both.

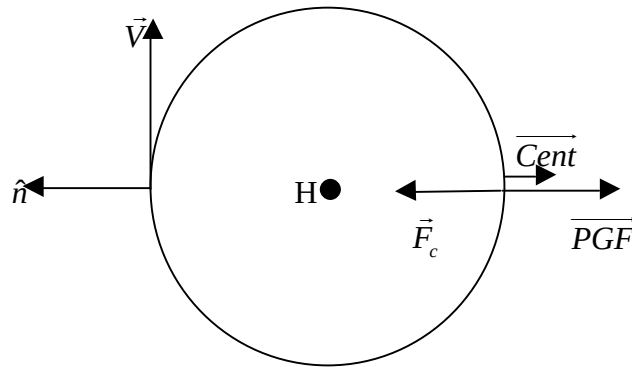
This type of circulation rarely occurs at the synoptic or large scales in the atmosphere where centrifugal force is typically not large enough to balance the other two forces.

Another reason is that such a circulation is hard to get established. If we have a pressure pattern with low center, an air parcel starting from rest will be driven towards low pressure center and turned to the right by Coriolis force – resulting in a cyclonic rather than anti-cyclonic circulation.

Such force balance is possible at the small scales where F_c plays a much smaller role. That circulation is then similar to the cyclostrophic anticyclonic circulation we discussed in the previous section.

This is the case of **Anomalous low**.

CASE IV: $R < 0$ and $\frac{\partial p}{\partial n} < 0$.

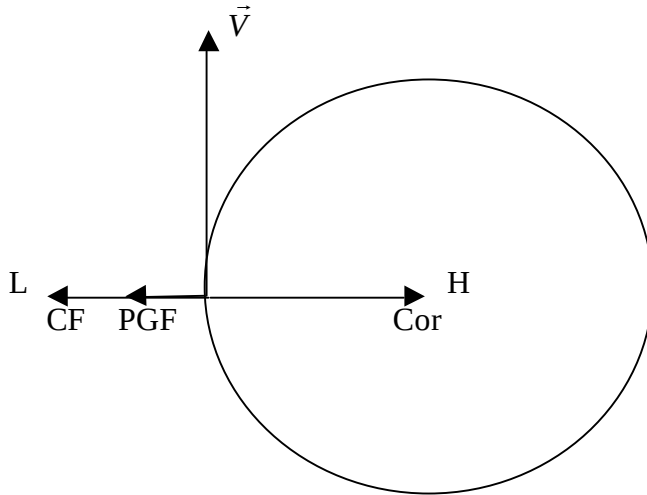


In this case, the radical could be negative, therefore we require that

$$\left| \frac{R}{r} \frac{\partial p}{\partial n} \right| < \frac{f^2 R^2}{4} \rightarrow$$

get two positive real roots for V .

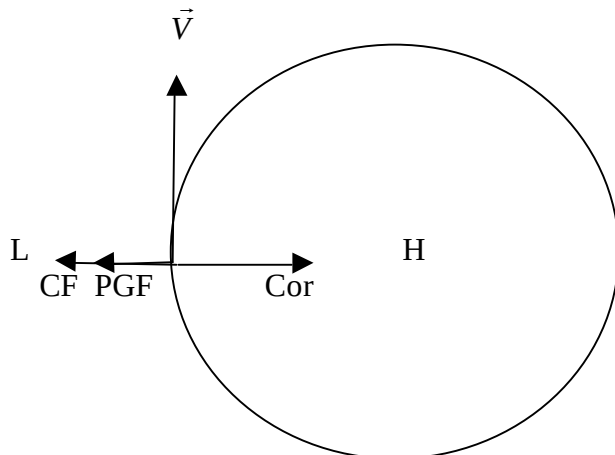
Positive sign case, $V = -\frac{fR}{2} + \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2} > -\frac{fR}{2} > 0$, V is larger than the negative sign case.



Since Cor (Coriolis force) is proportional to V and CF (centrifugal force) proportional to V^2/R (large R), the PGF (which is in the same direction as CF) needs to be only relatively small for the balance to occur.

Even though the above case (with positive sign) is theoretically possible, it almost never occurs in the atmosphere. At about 43°N ($f = 10^{-4} \text{ s}^{-1}$), with density $\rho = 1 \text{ kg m}^{-3}$, a radius of curvature $R = -1000 \text{ km}$, and a pressure gradient of $\partial p / \partial n = -1 \text{ mb} / 100 \text{ km}$, $V = 11 \text{ m/s}$ with negative sign and about 89 m/s with positive sign. Strong winds are rarely observed in high-pressure systems. Therefore, this is the case of **Anomalous High**.

The negative sign case, $V = -\frac{fR}{2} - \left(\frac{f^2 R^2}{4} - \frac{R}{r} \frac{\partial p}{\partial n} \right)^{1/2} < -\frac{fR}{2} > 0$, V is smaller than the positive sign case $\rightarrow$ Cor and CF smaller $\rightarrow$ PGF larger for balance



According to earlier discussions, this is the case of **Normal or Regular High**.

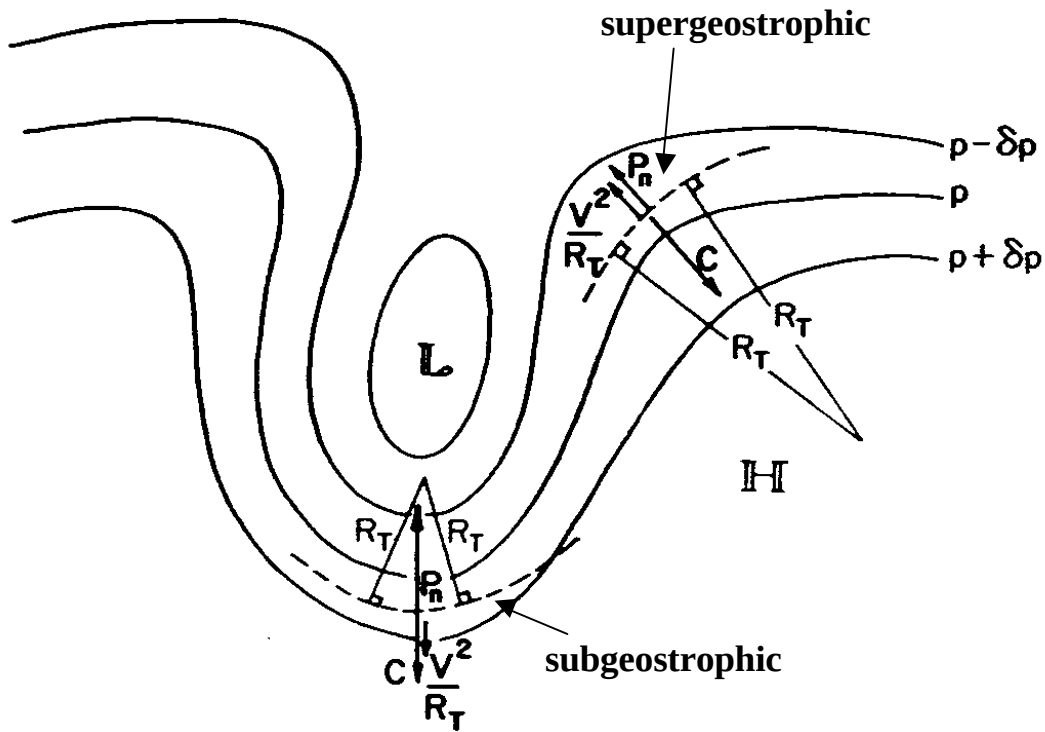
Subgeostrophic and Supergeostrophic Flows

Within sharp troughs in the middle-latitude westerlies the **observed velocities are often subgeostrophic** by as much as 50%, even though the streamlines still tend to be oriented parallel to the isobars. These large departures from geostrophic balance are a consequence of the large centripetal acceleration associated with the sharply curved flow within such regions. In order to illustrate how these accelerations alter the balance between the Coriolis force and the pressure gradient force it is convenient to represent them in terms of an apparent centrifugal force. The three-way balance between pressure gradient and Coriolis and centrifugal forces is shown in the following figure for cyclonic and anticyclonically curved trajectories. In both cases, the centrifugal force is directed outward from the center of curvature of the air trajectories (denoted by the dashed lines) and has a magnitude given by V^2 / R where R , is the local radius of curvature. In the cyclonic case, the centrifugal force reinforces the Coriolis force so that, in effect, a balance of forces can be achieved with a wind velocity smaller than would be required if the Coriolis force were acting alone. Thus, in this case, it is possible to maintain a sub-geostrophic flow parallel to the isobars. **For the anticyclonically curved trajectory** the situation is just the opposite: the centrifugal force opposes the Coriolis force and, in effect, necessitates a **super-geostrophic wind velocity** in order to bring about a three-way balance of forces.

For the combination of sharp anticyclonic curvature and a strong pressure gradient it is not possible to achieve a three-way balance of forces; that is to say, for all possible values of V ,

$$P_n > fV - \frac{V^2}{R}$$

It is no coincidence, then, that the combination of sharp troughs and tight pressure gradients is not uncommon, whereas sharp ridges are rarely if ever observed in conjunction with tight pressure gradients



The three-way balance between the horizontal pressure gradient force, the Coriolis force and the centrifugal force, in flow along curved trajectories (- -) in the Northern Hemisphere.

We can also look at the above issues by comparing the gradient wind speed with the geostrophic wind speed corresponding to the same pressure gradient. Since

$fV_g = -\frac{1}{r} \frac{\partial p}{\partial n}$ where V_g is the geostrophic wind speed, equation (7.46) can be rewritten as

$$0 = fV_g - fV - \frac{V^2}{R} \quad (7.49)$$

and V is the gradient wind speed. Therefore

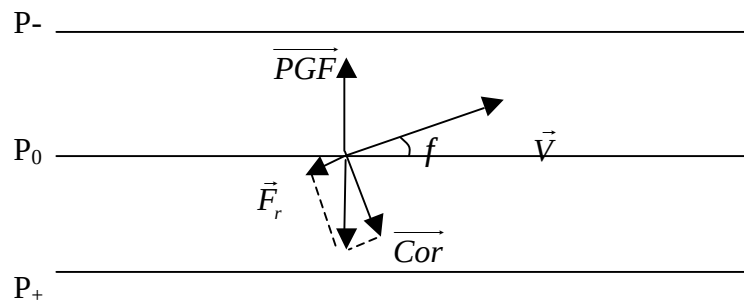
$$\frac{V_g}{V} = 1 + \frac{V^2}{fR} \quad (7.50)$$

For cyclonic circulation in northern hemisphere, $fR > 0$, therefore $V < V_g$, i.e., in flows with cyclonic curvature, gradient wind speed (which is a better approximation to the real wind) is smaller than the geostrophic wind speed. One gets subgeostrophic flow. Similarly, $V > V_g$ with anticyclonic curvature.

7.6.5. Frictional Effect

Up to this point, we have neglected the influence of friction. Friction actually plays a key role in setting the structure of large-scale atmosphere for the development of deep convection. For now, let's take a more simplistic view of the quantitative effect of friction.

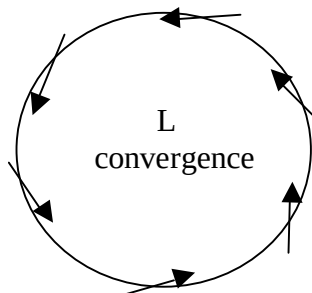
In a geostrophic flow, the PGF and the Coriolis force are in balance. When friction is present, friction acts in the opposite direction as the velocity vector and is therefore also perpendicular to the Coriolis force. The balance between the three forces are illustrated as follows:

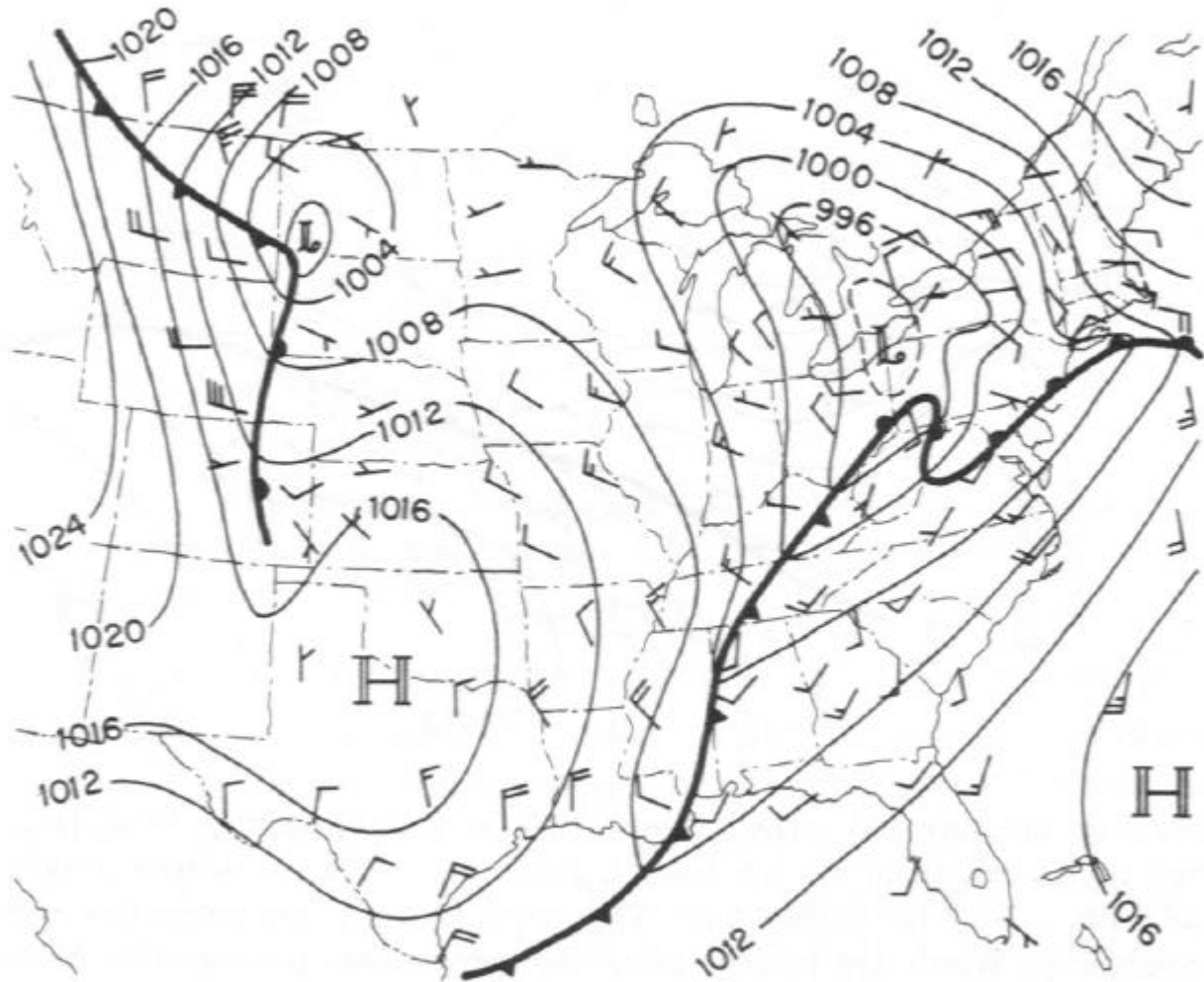


As can be seen above, the wind vector is no longer parallel to the pressure contours, the flow is crossing the isobars to the lower pressure side! Based on this balance, it can be shown that this velocity is less than the geostrophic velocity.

ϕ , the angle of departure from geostrophy, can be calculated from the force balance.

Friction is usually non-negligible near the ground (in the boundary layer), where the air is more turbulent and is subject to surface drag. The crossing of air to the low-pressure side at the lower levels contributes significantly to the low-level convergence in low-pressure systems and the convergence induces (through mass continuity – remember the kinematic method for calculating vertical motion from horizontal divergence?) ascending motion that destabilizes the atmosphere so that convection to occur (bad weather). Similarly, friction induces low-level divergence in high pressure systems → descending motion → good weather.





Mean-sea level pressure and surface winds, showing the surface winds mostly follow the isobaric contours but the cross-contour (from high to low pressure) component is significant. At a result, low-pressure centers are associated with convergence and high-pressure centers associated with divergence. The convergence forces air in low-pressure system to rise, producing weather.