

Chapter 7

Rotating Frames

7.1 Angular Velocity

A rotating body always has an (instantaneous) axis of rotation.

Definition: a frame of reference S' is said to have *angular velocity* $\boldsymbol{\omega}$ with respect to some fixed frame S if, in an infinitesimal time δt , all vectors which are fixed in S' rotate through an angle $\delta\theta = \omega \delta t$ about an axis $\mathbf{n} = \boldsymbol{\omega}/\omega$ through the origin, where $\omega = |\boldsymbol{\omega}|$.

Consider a vector $\mathbf{u}$ which is *fixed* in the rotating frame. In a time δt it rotates through an angle $\delta\theta$ about $\mathbf{n}$; i.e., it moves to

$$\begin{aligned}\mathbf{u} + \delta\mathbf{u} &= \mathbf{u} \cos \delta\theta + (\mathbf{u} \cdot \mathbf{n})\mathbf{n}(1 - \cos \delta\theta) - \mathbf{u} \times \mathbf{n} \sin \delta\theta \\ &= \mathbf{u} + \mathbf{n} \times \mathbf{u} \delta\theta + O(\delta\theta^2) \\ \implies \delta\mathbf{u} &= \mathbf{n} \times \mathbf{u} \omega \delta t + O(\delta t^2)\end{aligned}$$

$$\implies \boxed{\dot{\mathbf{u}} = \omega \mathbf{n} \times \mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}.}$$

(This fact is often regarded as “obvious” and can be quoted; it is sometimes taken as the definition of $\boldsymbol{\omega}$.)

Now let S be an inertial frame and S' be a frame rotating with angular velocity $\boldsymbol{\omega}$ with respect to S . Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis for S and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis for S' . Let $\mathbf{a}$ be any vector — not necessarily fixed in either S or S' — with components a_i and a'_i respectively, i.e.,

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = a'_1 \mathbf{e}'_1 + a'_2 \mathbf{e}'_2 + a'_3 \mathbf{e}'_3.$$

Then

$$\dot{\mathbf{a}} = \frac{d}{dt}(a_i \mathbf{e}_i) = \dot{a}_i \mathbf{e}_i$$

since the $\{\mathbf{e}_i\}$ are fixed. Also,

$$\begin{aligned}
\dot{\mathbf{a}} &= \frac{d}{dt}(a'_i \mathbf{e}'_i) \\
&= \dot{a}'_i \mathbf{e}'_i + a'_i \dot{\mathbf{e}}'_i \\
&= \dot{a}'_i \mathbf{e}'_i + a'_i \boldsymbol{\omega} \times \mathbf{e}'_i && \text{(because the } \mathbf{e}'_i \text{ are fixed in } S') \\
&= \dot{a}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times (a'_i \mathbf{e}'_i) \\
&= \dot{a}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}.
\end{aligned}$$

Introduce the notations

$$\left(\frac{d\mathbf{a}}{dt}\right)_S \equiv \dot{a}_i \mathbf{e}_i, \quad \left(\frac{d\mathbf{a}}{dt}\right)_{S'} \equiv \dot{a}'_i \mathbf{e}'_i.$$

Note that an observer in S' who does not know that it is rotating would measure the rate of change of $\mathbf{a}$ as $(d\mathbf{a}/dt)_{S'}$, because he would not include the contribution of $\dot{\mathbf{e}}'_i$; he only notices the rate of change of the components a'_i in his own frame. We have

$$\left(\frac{d\mathbf{a}}{dt}\right)_S = \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$$

for *any* vector $\mathbf{a}$.

We can apply the same method again:

$$\begin{aligned}
\ddot{\mathbf{a}} &= \frac{d}{dt}(\dot{a}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}) \\
&= \ddot{a}'_i \mathbf{e}'_i + \dot{a}'_i \dot{\mathbf{e}}'_i + \dot{\boldsymbol{\omega}} \times \mathbf{a} + \boldsymbol{\omega} \times \dot{\mathbf{a}} \\
&= \ddot{a}'_i \mathbf{e}'_i + 2\boldsymbol{\omega} \times (\dot{a}'_i \mathbf{e}'_i) + \dot{\boldsymbol{\omega}} \times \mathbf{a} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{a}).
\end{aligned}$$

So

$$\left(\frac{d^2\mathbf{a}}{dt^2}\right)_S = \left(\frac{d^2\mathbf{a}}{dt^2}\right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{a}}{dt}\right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{a} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{a}).$$

7.2 The Equation of Motion in a Rotating Frame

Since S is an inertial frame, $\mathcal{NII}$ holds in that frame:

$$\mathbf{F} = m \left(\frac{d^2\mathbf{x}}{dt^2}\right)_S.$$

Hence

$$\mathbf{F} = m \left\{ \left(\frac{d^2 \mathbf{x}}{dt^2} \right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt} \right)_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) \right\}. \quad (7.1)$$

Example: a particle is suspended by a string in a laboratory on the Earth's surface, with position vector $\mathbf{R}$ relative to the centre of the Earth which rotates with (constant) angular velocity $\boldsymbol{\omega}$. The particle is hanging at equilibrium in the lab frame. What is the tension in the string?

Since $(d\mathbf{x}/dt)_{S'} = \mathbf{0}$, because the particle is at rest in the lab,

$$\mathbf{T} + m\mathbf{g} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}),$$

i.e.,

$$\mathbf{T} = -m\{\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})\}.$$

Without rotation, the answer would have been just $-m\mathbf{g}$; so we call $\mathbf{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ the *apparent gravity*.

Example: in a fairground ride on Midsummer Common, a large circular drum of radius a rotates about its own axis, which is vertical. People pay good money to get pinned to the wall as the floor drops away. What is the minimum safe angular velocity of the drum?

In the rotating frame of the drum, the position vector $\mathbf{x}$ of a person is fixed, so $(d\mathbf{x}/dt)_{S'}$ and $(d^2\mathbf{x}/dt^2)_{S'}$ both vanish. The forces on the person are $m\mathbf{g}$, a reaction $\mathbf{N}$ from the wall and friction $\mathbf{R}$, so

$$\begin{aligned} m\mathbf{g} + \mathbf{N} + \mathbf{R} &= m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) \\ &= m(\boldsymbol{\omega} \cdot \mathbf{x})\boldsymbol{\omega} - m(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{x} \\ &= -m\omega^2\mathbf{x}. \quad (\text{Because } \boldsymbol{\omega} \text{ is perpendicular to } \mathbf{x}.) \end{aligned}$$

Resolving vertically, $\mathbf{R} = -m\mathbf{g}$, so friction is the only thing holding the person up, while horizontally, $\mathbf{N} = -m\omega^2\mathbf{x}$. Recall that $|\mathbf{R}| \leq \mu|\mathbf{N}|$ from §1.6.2, where μ is the coefficient of static friction; so

$$mg \leq \mu m\omega^2 a, \quad \text{i.e.,} \quad \omega \geq \sqrt{g/(\mu a)}.$$

If ω drops below this, the person slides off the ride.

The equation of motion is sometimes rearranged (by engineers and physicists) as

$$m \left(\frac{d^2 \mathbf{x}}{dt^2} \right)_{S'} = \mathbf{F} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt} \right)_{S'} - m\dot{\boldsymbol{\omega}} \times \mathbf{x} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}).$$

The various terms on the RHS are then called *fictitious forces*; they do not really exist but an observer in S' (who does not know that S' is rotating) feels an acceleration caused by them just as if they were real.

For example, in the fairground example above, the fictitious force is $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$ which equals $m\omega^2 \mathbf{x}$ (radially *outwards*) — this is called the “centrifugal force”. As far as the observer in S' is concerned, the “centrifugal force” is cancelled by $\mathbf{N}$.

The most physically important rotating system is the Earth itself. Our weather patterns are strongly influenced by the Earth’s rotation, and in particular by the “Coriolis force”, i.e., the fictitious force $-2m\boldsymbol{\omega} \times (d\mathbf{x}/dt)_{S'}$.

Consider the motion of air masses in the atmosphere. Air close to the surface is warmer than air higher up, which produces an upwards force that more or less balances the downwards pull of gravity: hence we can ignore vertical motion as it is essentially static. However, there are significant effects in horizontal directions.

As an example, consider a region of low atmospheric pressure p , known as a *depression*. Weather maps usually include *isobars* which are contours of constant p , and the (real) forces acting on air currents push them in the direction of $-\nabla p$ (i.e., from high to low pressure). Therefore, we might naïvely expect air flow to be orthogonal to the isobars (because ∇f is always perpendicular to a curve of constant f for any function $f(\mathbf{x})$).

However, in the Northern hemisphere, $\boldsymbol{\omega}$ has a component vertically upwards and $-\boldsymbol{\omega} \times (d\mathbf{x}/dt)_{S'}$ therefore pushes to the right when viewed from above. So the fictitious Coriolis force causes winds to be deflected rightwards. (In the Southern hemisphere, winds are deflected to the left.) This deflection continues until the pressure force $-\nabla p$ is in balance with the Coriolis force; this happens when the wind direction $(d\mathbf{x}/dt)_{S'}$ is along the contour lines. The result is that winds actually circle anticlockwise *around* a depression, parallel to the isobars, rather than orthogonal to them. (In reality, other effects such as friction cause the air flow to move slightly inwards as it circulates.)

7.3 The Foucault Pendulum

This is just a simple pendulum, with a bob of mass m and a string of length l attached to the point $(0, 0, l)$. We assume that the displacements are small, i.e., $|\mathbf{x}| \ll l$. We choose our axes such that the x -axis points East and the y -axis North.

We note that $x^2 + y^2 + (l - z)^2 = l^2$, so

$$z = \frac{1}{2l}(x^2 + y^2 + z^2).$$

Since x, y, z are all small, z is actually very small (second order), and we therefore take it to be zero. So we can use plane polar coordinates (r, ϕ) in the (x, y) plane. Note that θ (the angle of the pendulum from the vertical) is small and that $\sin \theta = r/l$; so the components of $\mathbf{T} = (T_x, T_y, T_z)$ are

$$T_z = T \cos \theta \approx T,$$

$$T_x = -T \sin \theta \cos \phi$$

$$= -T(r/l)(x/r)$$

$$= -Tx/l,$$

$$T_y = -Ty/l.$$

Now our co-ordinate frame is rotating with the Earth at (constant) angular velocity $\boldsymbol{\omega}$. Since $\omega = |\boldsymbol{\omega}|$ is small, we may ignore terms of order ω^2 ; hence, from (7.1),

$$\mathbf{T} + m\mathbf{g} = m \left\{ \left(\frac{d^2 \mathbf{x}}{dt^2} \right)_{S'} + 2\boldsymbol{\omega} \times \left(\frac{d\mathbf{x}}{dt} \right)_{S'} \right\}.$$

Since $(d/dt)_{S'}$ refers to the rate of change measured *in this frame*, $(d\mathbf{x}/dt)_{S'} = (\dot{x}, \dot{y}, \dot{z})$ and $(d^2 \mathbf{x}/dt^2)_{S'} = (\ddot{x}, \ddot{y}, \ddot{z})$. We also have $\boldsymbol{\omega} = (0, \omega \cos \lambda, \omega \sin \lambda)$ where λ is the latitude. Hence

$$T_x/m = \ddot{x} + 2\omega(\dot{z} \cos \lambda - \dot{y} \sin \lambda),$$

$$T_y/m = \ddot{y} + 2\omega\dot{x} \sin \lambda,$$

$$T_z/m - g = \ddot{z} - 2\omega\dot{x} \cos \lambda.$$

Since $z \approx 0$, the last of these three equations gives $T_z \approx mg + O(\omega)$; since $T \approx T_z$, we obtain $T_x = -(mg/l)x + O(\omega x/l)$. As both ω and x/l are small, we can ignore the second-order correction term to T_x and obtain

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y} \sin \lambda, \tag{7.2}$$

$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x} \sin \lambda. \tag{7.3}$$

Now let $\zeta = x + iy$. Taking (7.2) + i(7.3),

$$\ddot{\zeta} = -\frac{g}{l}\zeta - 2i\omega\dot{\zeta} \sin \lambda.$$

The auxiliary equation is

$$\alpha^2 + 2i\omega\alpha \sin \lambda + \frac{g}{l} = 0,$$

i.e.,

$$\alpha = -i\omega \sin \lambda \pm \sqrt{-\omega^2 \sin^2 \lambda - \frac{g}{l}} \approx -i\omega \sin \lambda \pm i\sqrt{g/l}.$$

The general solution is therefore

$$\zeta = e^{-i\omega t \sin \lambda} (A \cos(\sqrt{g/l} t) + B \sin(\sqrt{g/l} t)).$$

In particular,

$$\arg \zeta = -\omega t \sin \lambda + \arg(A \cos(\sqrt{g/l} t) + B \sin(\sqrt{g/l} t)).$$

We note that $\phi = \arg \zeta$, because the (x, y) -plane is simply the Argand plane for ζ . The only effect of ω is to cause ϕ to decrease at a constant rate $\omega \sin \lambda$; that is, the direction of swing of the pendulum moves clockwise at constant angular speed $\omega \sin \lambda$ while the pendulum continues to swing to and fro with frequency $\sqrt{g/l}$.

(This rotation is slow: its period is $2\pi/(\omega \sin \lambda)$, and $\omega = 2\pi/(1 \text{ day})$, so at $\lambda = 52^\circ \text{ N}$ the period is around $30\frac{1}{2}$ hours.)