

DYNAMICS OF TWO SPRING-CONNECTED MASSES IN ORBIT*

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(Received: 19 July 1995; accepted: 18 March 1996)

Abstract. This paper discusses relative equilibria (or steady motions) and their stability for the dynamics of the system of two spring-connected masses in a central gravitational field. The system can be regarded as a simplified model for the Tethered Satellite System (TSS), where the tether is modeled by a (linear or nonlinear) spring. In the previous studies of the TSS problem, it was typically assumed that the center of mass is located at the massive one of the two end-masses, and moves on a great-circle orbit. However, for the simple system treated in this paper, it is proved that nongreat-circle relative equilibria do exist. Some fundamental concepts of the dynamics of an arbitrary assembly moving in a central gravitational field are discussed. The notion of steady motions used in engineering literature is linked with the notion of relative equilibria in geometric mechanics. Numerical computations show some interesting nongreat-circle relative equilibria for the spring-connected system. Radial relative equilibria, which correspond to the station-keeping mode for TSS, are then introduced. Within the framework of symmetry and reduction, their stability properties are investigated by adopting the reduced energy-momentum method, which takes the advantage of the intrinsic symmetry structure. It is shown that for practical configurations, the system at radial relative equilibria is stable if some conditions are satisfied.

Key words: Spring system, symmetry, relative equilibrium, relative stability, tethered satellite system

1. Introduction

This paper discusses the relative equilibria and their stability for the motions of two spring-connected masses moving in a central gravitational field. The system, composed of two end-point masses connected by an elastic spring, cf. Figure 1, is termed as the *spring system*. It can be regarded as a simplified model for the Tethered Satellite System (TSS), which contains a satellite (or shuttle orbiter) connected to a subsatellite with a long tether. There have been many interesting discussions on this subject, especially after this idea of TSS was put forth by Colombo et al. (1974). According to (Bekey, 1987), the earliest report on such idea was described by Tsiolkovskii in 1895, where an ‘anchored tower’ from the surface of the Earth to the altitude of the geostationary orbit was conceived. The problems regarding dynamics and control of these large systems in orbit have been investigated by many researchers, cf. (Christ and Eisley, 1970; Bainum et al., 1987; Beletskii and Levin, 1981, 1991; Levin, 1983; Liaw and Abed, 1990; de Matteis and de Socio, 1990; Pasca et al., 1991; Breakwell and Janssens, (1992); Penzo

* This work was partially supported by the National Science Council, Republic of China, under grant NSC-83-0208-M-002-082. The authors would like to thank W.-T. Chou for some computational assistance.

et al., 1989), and the references therein. In these literature, either distributed model or lumped system were considered. While the analysis of the distributed system was observed to be quite difficult to deal with, many of previous discussions treated the tether as a massless rigid bar connecting two point masses. Furthermore, it is mostly assumed that the center of mass of the system is located at the massive one of the two end-masses, which moves on a great-circle orbit, i.e. a circular orbit centering at the center of the field. However, even at steady motions, these assumptions may not be valid. In fact, for the simple system treated in this paper, it is proved that the nongreat-circle relative equilibria do exist; namely, the center of field and the circular orbit traced by the center of mass form a cone. Accordingly, the dynamical behavior of the *more* exact model without the classic assumptions becomes interesting.

The assumptions made in this paper are as follows

- (1) the attraction center is at rest in the inertial frame;
- (2) the spring is massless and undergoes extensive or compressive deformation along only one direction;
- (3) the gravitational attraction between the two end bodies is neglected.

After constructing the kinetic energy and the potential energy, it is observed that the system possesses $SO(3)$ -symmetry. In geometric mechanics, cf. (Abraham and Marsden, 1978), such symmetry induces certain reduction of the dynamics and the notion of *relative equilibria* can be defined, which actually corresponds to the notion of steady motions in the literature. The operations of TSS mainly consist of three phases: deployment, station-keeping, and retrieval. In the station-keeping mode, the TSS is controlled such that the tether is straight while pointing toward the Earth. This configuration is in fact at the relative equilibria. With these observations, the techniques in dealing with symmetry, reduction, and stability analysis can be used. Here the Principle of Symmetric Criticality (Palais, 1979) is applied to obtain conditions for relative equilibrium, without the derivation of complete dynamical equations. Similar techniques can be applied to more complete models, such as the one treated in (Wang et al., 1994).

It was proved in the literature, e.g. (Wang et al., 1991), that if the system is a rigid body, the configuration of TSS at the station-keeping mode is stable for an approximate model. This leads toward the so-called gravity-gradient stabilization technique. However, since the tether is not rigid, the stability property may be destroyed. It has been observed in (Arnold, 1987) that numerical simulations showed the instability of systems with very long tethers. Consequently, conditions for the stability of long tethers with certain flexibility are very interesting from either theoretical or practical points of view, cf. (Bainum and Evans, 1974; Beletskii and Levin, 1985; Misra and Modi, 1987; Liu and Bainum, 1988), etc. For the stability of the relative equilibrium, (Wang, et al., 1994) has applied the reduced energy-momentum method on the TSS where the tether is modeled as a nonlinear string. However, due to the complexity of the problem, explicit conditions were not

derived there. In order to gain some insights on the effects of flexibility and make the computations tractable, the tether is modeled as a massless nonlinear spring in this paper.

As discussed in the survey article of Misra and Modi (Misra and Modi, 1987), even the equations of motion governing the dynamics of a long tether system in the station-keeping phase are quite complicated, not mention the stability analysis. As described above, the spring system indeed has $SO(3)$ -symmetry. Accordingly, it is intuitive to take the combination of energy and momentum as a Lyapunov function candidate to determine the nonlinear stability. This leads to the so-called *energy-momentum method*. Moreover, the intrinsic structure of the spring system allows one to apply the *reduced energy-momentum method*, which takes the advantage of the symmetry structure to simplify the computations. The whole geometric framework can be found in (Simo et al., 1991; Simo et al. 1991; Wang and Krishnaprasad, 1992). Here the method is outlined and applied to the spring system.

In the following discussions, the system under consideration is described in Section 2, along with the discussions of some fundamental concepts of the dynamics of an arbitrary assembly moving in orbits. The abstract framework of symmetry is discussed in Section 3, in which the symmetry of the spring system is observed. The aforementioned Principle of Symmetric Criticality is then applied to get the conditions of relative equilibria for the spring system in Section 4. It is proved that with the model treated in this paper, nongreat-circle relative equilibria do exist. It is the *center of gravity* (instead of the center of mass) that must trace a great-circle orbit at the relative equilibria. In particular, if the two end-masses are not equal, there must exist a nongreat-circle relative equilibrium. Numerical computations to find some nongreat-circle relative equilibria were performed. It is observed that while the orbit of the center of mass is in fact very close to a great circle, the system may undergo significant deflections from the vertical (to the orbital plane) at relative equilibria.

The notion of *relative stability* is then introduced in Section 5, along with an outline of the reduced energy-momentum method applicable to the spring system. Section 6 performs the stability analysis for the radial relative equilibria and derives certain conditions for linear or nonlinear springs. In particular, the case of linear spring is further explored, and it is found that for practical configuration (with the ratio of the distances between the outer spacecraft and inner satellite being smaller than 1.74), the system is stable if the spring is stiff enough. Some concluding remarks are given in Section 7.

2. System Description

The physical system under investigation, i.e. the *spring system*, is depicted in Figure 1. Let the two end-masses be m_a and m_b , respectively. Denote the vectors from the attraction center O to m_a and m_b by $\mathbf{a}$ and $\mathbf{b}$, respectively. The configu-

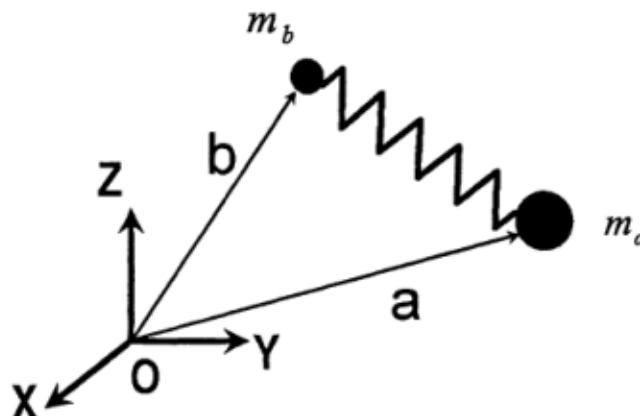


Figure 1. Two spring-connected bodies in a central gravitational field.

ration space of the spring system can be modeled as $Q = \mathbb{R}^3 \times \mathbb{R}^3$. With the only external force – the gravitational force, the kinetic energy and the potential energy of the system can be found to be,

$$T(\dot{\mathbf{a}}, \dot{\mathbf{b}}) = \frac{m_a}{2} \|\dot{\mathbf{a}}\|^2 + \frac{m_b}{2} \|\dot{\mathbf{b}}\|^2, \quad (1)$$

and

$$V(\mathbf{a}, \mathbf{b}) = -\frac{\mu m_a}{\|\mathbf{a}\|} - \frac{\mu m_b}{\|\mathbf{b}\|} + W(\|\mathbf{a} - \mathbf{b}\|), \quad (2)$$

respectively, where the notation $\|\cdot\|$ denotes the Euclidean norm, and μ is the gravitational constant of the field. In the potential energy, the first two terms represent the gravitational potential energy, and the last term W denotes the elastic potential energy stored in the (linear or nonlinear) spring. In particular, for a linearly elastic spring, $W = \frac{1}{2}k(\|\mathbf{a} - \mathbf{b}\| - \ell_0)^2$, with the spring constant k and the reference length ℓ_0 (the initial length of the spring without experiencing any force).

From the kinetic energy and the potential energy, the system Lagrangian can be constructed along with the equations of motion being derived through the Euler–Lagrange’s equation. However, to study the relative equilibria and their properties, it is not necessary to perform the sometimes complicated computations. In fact, it will be shown later that the system has an intrinsic symmetry structure, and thus some modern treatment can be applied.

For an assembly, such as the one described above, moving in a central gravitational field, its *center of gravity* G is defined by the vector $\mathbf{r}_g$ satisfying

$$\frac{\mathbf{r}_g}{\|\mathbf{r}_g\|^3} = \frac{1}{M_{\text{total}}} \int_{\text{assembly}} \frac{\mathbf{r}}{\|\mathbf{r}\|^3} dm, \quad (3)$$

where $\mathbf{r}$ is the vector from the center of the field to a mass element dm , and $M_{\text{total}} = \int_{\text{assembly}} dm$ is the total mass. Let the right-hand side of (3) be denoted by a vector $\boldsymbol{\alpha}$. It can be checked that

$$\mathbf{r}_g = \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|^{3/2}}.$$

Note that while the center of mass of a rigid body is fixed relative to the body, its center of gravity changes with respect to different configurations of the body. For example, consider a system consisting of two equal point masses connected by a 200 km long massless rigid bar. When the system is put in a taut configuration with the lower point mass at the radius 6,800 km Low Earth Orbit (LEO), the center of gravity of the system is at radius 6,897.83 km, which is away from the center of mass about 2.17 km. On the other hand, if the system is positioned about half of the synchronous orbit, e.g. 20,000 km, G is about 20,099.3 km, or about 0.7 km from the center of mass. Clearly, the position of the center of gravity varies with the location of the assembly. Other examples show that it varies also with respect to different orientations, cf. (Rimrott, 1989).

From Newtonian framework, it is not difficult to derive

$$\frac{d^2 \mathbf{r}_c}{dt^2} + \frac{\mu \mathbf{r}_g}{\|\mathbf{r}_g\|^3} = 0, \quad (4)$$

where $\mathbf{r}_c$ denotes the position vector of the center of mass of the assembly. It is also noted that the torque generated by gravitational forces about the center of gravity G is in fact

$$\int_{\text{assembly}} (\mathbf{r} - \mathbf{r}_g) \times \frac{-\mu \mathbf{r}}{\|\mathbf{r}\|^3} dm = 0.$$

In some literature, cf. (Marion and Thornton, 1988), the center of gravity is defined to be the point about which the gravity torque vanishes. On the other hand, the torque about the center of mass is

$$\int_{\text{assembly}} (\mathbf{r} - \mathbf{r}_c) \times \frac{-\mu \mathbf{r}}{\|\mathbf{r}\|^3} dm = \mu M_{\text{total}} \mathbf{r}_c \times \frac{\mathbf{r}_g}{\|\mathbf{r}_g\|^3},$$

which typically does not vanish. In later discussions, the difference between center of mass and center of gravity provides some physical insight on the notion of nongreat-circle relative equilibrium.

3. Symmetry and Relative Equilibrium

A mechanical system may possess some symmetry so that the notion of relative equilibrium can be introduced. For a spacecraft moving in a circular orbit, the steady motions refer to certain configurations with which the assembly is stationary relative to a uniformly rotating frame located at the origin of the field. This corresponds to the symmetry of the system with respect to the rotation group $SO(3)$ in geometric mechanics. The special orthogonal group $SO(3)$ consists of the rotation matrices $\mathbf{B}$, with $\mathbf{B}^T \mathbf{B} = \mathbf{1}$, and $\det(\mathbf{B}) = 1$, where $\mathbf{1}$ denotes the 3×3 identity matrix, and $1 \in \mathbb{R}$. Let the configuration space of the mechanical system be denoted

by Q , with the associated tangent bundle TQ . Actions of $SO(3)$ on Q and TQ may be defined as

$$\Phi: SO(3) \times Q \rightarrow Q, \quad \Phi^T: SO(3) \times TQ \rightarrow TQ,$$

respectively. If the kinetic energy $T: TQ \rightarrow \mathbb{R}$ and the potential energy $V: Q \rightarrow \mathbb{R}$ are invariant under the actions, then it is said that the mechanical system has the $SO(3)$ -symmetry. With this symmetry, the phase space of the system can be reduced. The dynamics in the original phase space can be also reduced to yield the reduced dynamical equations. The equilibrium of the reduced dynamics is termed the *relative equilibrium* of the mechanical system. With respect to the $SO(3)$ -symmetry, the relative equilibrium is nothing more than the steady motion of the assembly. Accordingly, the methodology of finding relative equilibria and investigating their behaviors can be used to study the steady motions. In particular, for the system under consideration, the configuration space is $Q = \mathbb{R}^3 \times \mathbb{R}^3$. The group actions on Q and TQ can be constructed as

$$\begin{aligned}\Phi(B, (a, b)) &= (Ba, Bb), \\ \Phi^T(B, (a, b, \dot{a}, \dot{b})) &= (Ba, Bb, B\dot{a}, B\dot{b}),\end{aligned}$$

respectively, with $B \in SO(3)$. It can be easily checked that the kinetic energy (1) and the potential energy (2) are indeed invariant under the actions, i.e.

$$\begin{aligned}T(\Phi^T(B, (a, b, \dot{a}, \dot{b}))) &= T(a, b, \dot{a}, \dot{b}), \\ V(\Phi(B, (a, b))) &= V(a, b),\end{aligned}$$

for all $B \in SO(3)$. The spring system thus has the $SO(3)$ -symmetry. To find the configurations at the relative equilibria, the Principle of Symmetric Criticality, cf. (Palais, 1979; Wang and Krishnaprasad, 1989), can be applied, which is described below.

THEOREM 1. *Consider a simple mechanical system with symmetry (Q, T, V, G) , cf. (Abraham and Marsden, 1978), where Q is a Riemannian manifold with metric K , T is the kinetic energy, V denotes the potential, and G is the symmetry (Lie) group. We have the relation*

$$K(x)(v_x, v_x) = 2T(v_x),$$

with $v_x \in TQ$. Given $\xi \in \mathcal{G}$, the Lie algebra of the Lie group G , the position element of a relative equilibrium corresponding to ξ is a critical point of the augmented potential $V_\xi: Q \rightarrow \mathbb{R}$,

$$V_\xi(x) = V(x) - \frac{1}{2}K(x)(\xi_Q(x), \xi_Q(x)),$$

where ξ_Q is the infinitesimal generator associated with ξ .

For the spring system, letting the vector ξ denote the angular velocity of the uniformly rotating frame, the augmented potential is

$$\begin{aligned} V_{\xi}(\mathbf{a}, \mathbf{b}) &= V(\mathbf{a}, \mathbf{b}) - T((\xi \times \mathbf{a}, \xi \times \mathbf{b})) \\ &= -\frac{\mu m_a}{\|\mathbf{a}\|} - \frac{\mu m_b}{\|\mathbf{b}\|} + W(\|\mathbf{a} - \mathbf{b}\|) - \\ &\quad - \frac{m_a}{2} \langle \xi \times \mathbf{a}, \xi \times \mathbf{a} \rangle - \frac{m_b}{2} \langle \xi \times \mathbf{b}, \xi \times \mathbf{b} \rangle. \end{aligned} \quad (5)$$

The critical points of V_{ξ} give rise to relative equilibria or steady motions.

The same $\text{SO}(3)$ -symmetry can be found for an arbitrary assembly moving in the orbit, with the corresponding relative equilibria being classified as follows.

DEFINITION 1. The relative equilibrium is called *great-circle* if the center of mass moves on a great-circle orbit, i.e., the center of the field resides on the plane of motion, or $\mathbf{r}_c \cdot \xi = 0$. Otherwise, it is called *nongreat-circle*.

Let the operator ' $\hat{\cdot}$ ' denote the isomorphism from $\mathbb{R}^3$ to $\text{so}(3)$ (the space of 3×3 skew-symmetric matrices), defined by

$$\hat{\mathbf{w}} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3. \quad (6)$$

A uniformly rotating frame with constant angular velocity ξ can be then represented by matrix exponential $e^{\hat{\xi}t}$, which maps skew-symmetric matrices to rotation matrices. With these notations, the relative equilibria can be characterized by finding all $\mathbf{r}_o$'s (constant vectors in the inertial frame) such that $\mathbf{r}(t) = e^{\hat{\xi}t} \mathbf{r}_o$ for all $\mathbf{r}(t)$'s in the assembly. It is not difficult to show that there exist $\mathbf{r}_{co}$ and $\mathbf{r}_{go}$ (independent of time) such that $\mathbf{r}_c(t) = e^{\hat{\xi}t} \mathbf{r}_{co}$, and $\mathbf{r}_g(t) = e^{\hat{\xi}t} \mathbf{r}_{go}$. Substituting the above expressions in (4), we obtain

$$\hat{\xi} \hat{\xi} \mathbf{r}_{co} + \frac{\mu \mathbf{r}_{go}}{\|\mathbf{r}_{go}\|^3} = 0. \quad (7)$$

Taking inner product on (7) with ξ , we find

$$\mathbf{r}_{go} \cdot \xi = 0, \quad (8)$$

from which the following theorem is proved.

THEOREM 2. *At relative equilibria, the center of gravity $\mathbf{r}_g$ is perpendicular to the constant angular velocity of the assembly, i.e., $\mathbf{r}_g$ traces a great circle.*

On the other hand, taking inner product on (7) with $\mathbf{r}_{go}$, we have

$$\mu = \|\xi\|^2 \|\mathbf{r}_{go}\| (\mathbf{r}_{go} \cdot \mathbf{r}_{co}) = \|\xi\|^2 \|\dot{\mathbf{r}}_g\| (\mathbf{r}_g \cdot \mathbf{r}_c), \quad (9)$$

which is the modified Kepler's third law at the relative equilibria. When the assembly contains only one point mass, the points G and C coincide with each other; hence the above formula (9) reduces to the classical Kepler's third law for circular orbits.

While the center of gravity always traces a great-circle orbit at relative equilibrium, the center of mass may not. This fact is shown in the following section. We remark that the notion of nongreat-circle relative equilibrium was termed 'oblique regular motions' in (Aboelnaga and Barkin, 1979; Barkin, 1985), where a rigid body system was considered. It was also described in (Wang et al., 1991) and numerically justified in (Wang et al., 1992).

4. Relative Equilibria of the Spring System

This section describes interesting relative equilibria for the spring system. In particular, the existence of the nongreat-circle relative equilibria is proved analytically, along with some numerical examples.

As discussed in the previous section, we seek the critical points of the augmented potential V_ξ . The first derivative of the augmented potential is obtained as follows,

$$\begin{aligned} DV_\xi(\mathbf{a}, \mathbf{b}) \cdot (\delta\mathbf{a}, \delta\mathbf{b}) \\ = \mu m_a \frac{\mathbf{a}}{\|\mathbf{a}\|^3} \cdot \delta\mathbf{a} + \mu m_b \frac{\mathbf{b}}{\|\mathbf{b}\|^3} \cdot \delta\mathbf{b} + W'(\|\mathbf{a} - \mathbf{b}\|) \frac{\mathbf{a} - \mathbf{b}}{\|\mathbf{a} - \mathbf{b}\|} \cdot (\delta\mathbf{a} - \delta\mathbf{b}) + \\ + m_a (\widehat{\xi\xi}\mathbf{a}) \cdot \delta\mathbf{a} + m_b (\widehat{\xi\xi}\mathbf{b}) \cdot \delta\mathbf{b}, \end{aligned}$$

where the prime $'$ denotes the derivative of W with respect to its argument. By the Principle of Symmetric Criticality, the conditions of relative equilibria are as follows,

$$\frac{\mu m_a \mathbf{a}_e}{a_e^3} + m_a \widehat{\xi\xi} \mathbf{a}_e + W' \frac{\mathbf{a}_e - \mathbf{b}_e}{\|\mathbf{a}_e - \mathbf{b}_e\|} = 0, \quad (10)$$

$$\frac{\mu m_b \mathbf{b}_e}{b_e^3} + m_b \widehat{\xi\xi} \mathbf{b}_e - W' \frac{\mathbf{a}_e - \mathbf{b}_e}{\|\mathbf{a}_e - \mathbf{b}_e\|} = 0, \quad (11)$$

where $a_e = \|\mathbf{a}_e\|$ and $b_e = \|\mathbf{b}_e\|$.

The terms containing W' represent the elastic forces on the spring. When the spring becomes more and more rigid and the length $\|\mathbf{a}_e - \mathbf{b}_e\|$ at relative equilibrium approaches the reference length ℓ_0 , the spring system becomes a rigid bar system.

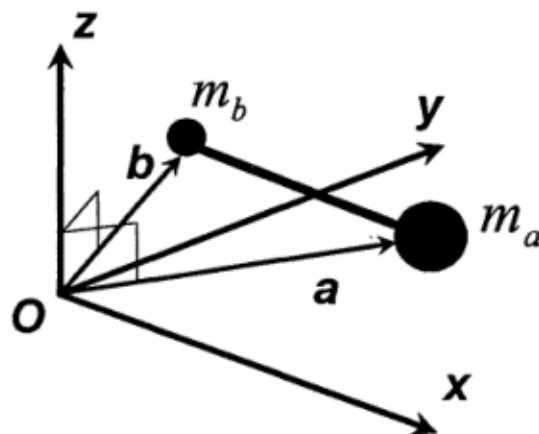


Figure 2. The frame system.

It is easily checked that (10) and (11) are invariant under the transformation $R_a = B a_e$, $R_b = B b_e$, and $\Omega = B \xi$, where $B \in \text{SO}(3)$ is the transformation matrix from the inertial frame to a new frame and R_a, R_b, Ω are vectors in the new frame. With this observation, a suitable frame is sought to make the problem tractable. The frame adopted here is such that the x -axis being parallel to the spring, with the z -axis being perpendicular to both a_e and b_e , and the y -axis completing the triad, cf. Figure 2. As for the case that a_e and b_e are parallel, we may arbitrarily choose a z -axis without loss of generality.

With respect to the chosen frame, the vectors can be expressed as $R_a = (x_a, y_c, 0)^T$, $R_b = (x_b, y_c, 0)^T$, and $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$. It is further assumed that x_a is greater than x_b . Under this setting, (10) and (11) are rewritten as six equations,

$$-(\Omega_2^2 + \Omega_3^2)x_a + \Omega_1\Omega_2y_c + \mu \frac{x_a}{R_a^3} = -\frac{W'}{m_a}, \quad (12)$$

$$\Omega_1\Omega_2x_a - (\Omega_1^2 + \Omega_3^2)y_c + \mu \frac{y_c}{R_a^3} = 0, \quad (13)$$

$$(\Omega_1x_a + \Omega_2y_c)\Omega_3 = 0, \quad (14)$$

$$-(\Omega_2^2 + \Omega_3^2)x_b + \Omega_1\Omega_2y_c + \mu \frac{x_b}{R_b^3} = \frac{W'}{m_b}, \quad (15)$$

$$\Omega_1\Omega_2x_b - (\Omega_1^2 + \Omega_3^2)y_c + \mu \frac{y_c}{R_b^3} = 0, \quad (16)$$

$$(\Omega_1x_b + \Omega_2y_c)\Omega_3 = 0, \quad (17)$$

where $R_a = \|R_a\|$, and $R_b = \|R_b\|$. The solutions of the above equations correspond to the relative equilibria of the spring system. Let $\ell = x_a - x_b > 0$. Define $R_c = (x_c, y_c, 0)^T$ with $(m_a + m_b)x_c = m_ax_a + m_bx_b$. It is easily derived that

$$x_a = x_c + m_b \ell / (m_a + m_b), \quad x_b = x_c - m_a \ell / (m_a + m_b). \quad (18)$$

Given μ , m_a , m_b , ℓ , and $R_c = \|\mathbf{R}_c\|$, the process of solving the equations (12)–(17) can be divided into the following two cases.

I. $\Omega_3 \neq 0$

The equations of relative equilibria can be simplified as,

$$\Omega_1 = 0, \quad \Omega_2 y_c = 0, \quad \Omega_3 \neq 0, \quad (19)$$

$$\left[-(\Omega_2^2 + \Omega_3^2) + \frac{\mu}{R_a^3} \right] x_a = -\frac{W'}{m_a}, \quad (20)$$

$$\left(-\Omega_3^2 + \frac{\mu}{R_a^3} \right) y_c = 0, \quad (21)$$

$$\left[-(\Omega_2^2 + \Omega_3^2) + \frac{\mu}{R_b^3} \right] x_b = \frac{W'}{m_b}, \quad (22)$$

$$\left(-\Omega_3^2 + \frac{\mu}{R_b^3} \right) y_c = 0. \quad (23)$$

Subtracting (21) from (23), we get

$$y_c \left(\frac{1}{R_a^3} - \frac{1}{R_b^3} \right) = 0.$$

The discussions may be further separated depending on whether $y_c = 0$ or not.

First, for $y_c \neq 0$, Ω_2 must be 0, and $R_a = R_b \equiv R$. The solution can be easily obtained from (19–23) as

$$\mathbf{R}_a = (\tfrac{1}{2}\ell, y_c, 0)^T, \quad \mathbf{R}_b = (-\tfrac{1}{2}\ell, y_c, 0)^T, \quad \text{and} \quad \boldsymbol{\Omega} = (0, 0, \Omega_3)^T,$$

with

$$y_c = \left[R_c^2 - \frac{\ell^2}{4} \left(\frac{m_a - m_b}{m_a + m_b} \right)^2 \right]^{1/2}, \quad \Omega_3^2 = \frac{\mu}{R^3}.$$

This is a great-circle relative equilibrium with $W' = 0$, which indicates that there is no force in the spring, cf. Figure 3. At such relative equilibria, the two point masses move on the same circular orbit.

Next, we turn to the situation $y_c = 0$. Since $x_b \leq 0 \leq x_a$ is not physically interesting, it is further assumed that $x_a > x_b > 0$. Thus,

$$\mathbf{R}_a = (x_a, 0, 0)^T, \quad \mathbf{R}_b = (x_b, 0, 0)^T, \quad \mathbf{R}_c = (x_c, 0, 0)^T.$$

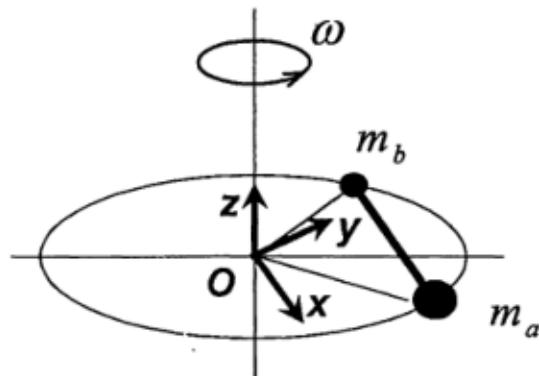


Figure 3. Relative equilibrium with zero elastic force.

This is in fact the case of *radial relative equilibrium*, i.e. the spring deforming along a radial axis. The frame can be then selected such that $\Omega_2 = 0$. With these observations, the conditions for relative equilibria are further simplified as

$$\Omega_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 \neq 0, \quad y_c = 0, \quad (24)$$

$$\left(-\Omega_3^2 + \frac{\mu}{x_a^3}\right) x_a = -\frac{W'}{m_a}, \quad (25)$$

$$\left(-\Omega_3^2 + \frac{\mu}{x_b^3}\right) x_b = \frac{W'}{m_b}. \quad (26)$$

As a consequence, the configuration of a radial relative equilibrium is derived, with

$$\begin{aligned} \mathbf{R}_a &= (x_a, 0, 0)^T, \quad \mathbf{R}_b = (x_b, 0, 0)^T, \quad \boldsymbol{\Omega} = (0, 0, \Omega_3)^T, \\ \Omega_3^2 &= \frac{\mu}{(m_a + m_b)R_c} \left(\frac{m_a}{x_a^2} + \frac{m_b}{x_b^2} \right), \\ W' &= \frac{\mu m_a m_b (x_a^3 - x_b^3)}{(m_a + m_b)x_a^2 x_b^2 R_c} > 0. \end{aligned} \quad (27)$$

This is a great-circle relative equilibrium with tensile elastic forces, cf. Figure 4.

II. $\Omega_3 = 0$

For such case, the equations of relative equilibria become

$$-\Omega_2^2 x_a + \Omega_1 \Omega_2 y_c + \mu \frac{x_a}{R_a^3} = -\frac{W'}{m_a}, \quad (28)$$

$$\Omega_1 \Omega_2 x_a - \Omega_1^2 y_c + \mu \frac{y_c}{R_a^3} = 0, \quad (29)$$

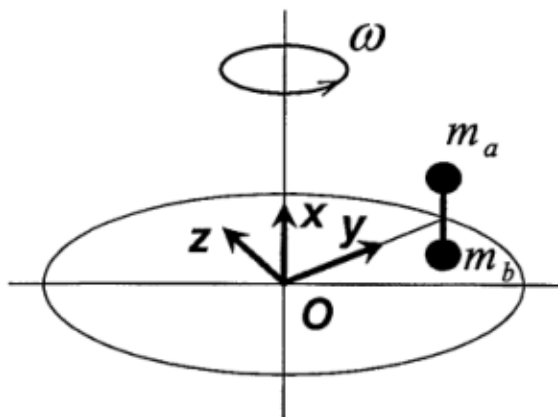


Figure 5. Relative equilibrium with compressive elastic force and $m_a = m_b$.

This relative equilibrium is also great-circle, at which the spring with compressive elastic force is perpendicular to the orbital plane and m_a and m_b are equidistant to the attraction center, cf. Figure 5.

On the other hand, for $R_a \neq R_b$, from (29) and (31), we obtain

$$(x_a - x_b)\Omega_1\Omega_2 = -\mu y_c \left(\frac{1}{R_a^3} - \frac{1}{R_b^3} \right) \neq 0, \quad (36)$$

which implies $\Omega_1 \neq 0$ and $\Omega_2 \neq 0$. By adding $m_a \times (29)$ and $m_b \times (31)$, it is found that

$$-(m_a + m_b)(\Omega_2 x_c - \Omega_1 y_c)\Omega_1 = \mu \left(\frac{m_a}{R_a^3} + \frac{m_b}{R_b^3} \right) y_c \neq 0, \quad (37)$$

which implies $\Omega_2 x_c - \Omega_1 y_c \neq 0$. From (28) and (30), we have

$$(m_a + m_b)(\Omega_2 x_c - \Omega_1 y_c)\Omega_2 = \mu \left(\frac{m_a x_a}{R_a^3} + \frac{m_b x_b}{R_b^3} \right) \neq 0. \quad (38)$$

Define

$$f_x = \frac{m_a x_a}{R_a^3} + \frac{m_b x_b}{R_b^3} \neq 0, \quad \text{and} \quad f_y = \left(\frac{m_a}{R_a^3} + \frac{m_b}{R_b^3} \right) y_c \neq 0.$$

From (37) and (38), it is observed that

$$\frac{\Omega_2}{\Omega_1} = -\frac{f_x}{f_y}.$$

Since

$$f_x y_c - f_y x_c = \frac{m_a m_b}{m_a + m_b} \left(\frac{1}{R_a^3} - \frac{1}{R_b^3} \right) (x_a - x_b) y_c \neq 0,$$

we know that

$$x_c \Omega_1 + y_c \Omega_2 \neq 0, \quad (39)$$

which indicates that there is no great-circle relative equilibrium for this case. Moreover, it can be easily verified that the vectors $\mathbf{r}_g$ and $\mathbf{r}_c$ defined in Section 2 are not parallel to each other.

Eliminating Ω_1 and Ω_2 from (36), (37), and (38), one obtains

$$f = f_x f_1 + f_y f_2 = 0, \quad (40)$$

where

$$f_1 = \frac{x_b}{R_a^3} - \frac{x_a}{R_b^3}, \quad \text{and} \quad f_2 = \left(\frac{1}{R_a^3} - \frac{1}{R_b^3} \right) y_c.$$

Now if $m_a = m_b = m$, then

$$f = m \left(\frac{1}{R_a^2} - \frac{1}{R_b^2} \right) \times \\ \times \left[\frac{1}{2} (R_a^2 + R_b^2 - \ell^2) \left(\frac{1}{R_a^4} + \frac{1}{R_a^2 R_b^2} + \frac{1}{R_b^4} \right) + \frac{1}{R_a R_b} \right].$$

For natural configurations, we have $R_a^2 + R_b^2 > R_c^2 > \ell^2$, and thus (40) implies $R_a = R_b$. Consequently, the following theorem is concluded.

THEOREM 3. *At the relative equilibrium of a natural configuration, where the constraint force is compressive, $R_a = R_b$ if and only if $m_a = m_b$.*

In particular, the case of $m_a \neq m_b$ leads toward the nongreat-circle relative equilibria, in which

$$\mathbf{R}_a = (x_a, y_c, 0)^T, \quad \mathbf{R}_b = (x_b, y_c, 0)^T, \quad \boldsymbol{\Omega} = (\Omega_1, \Omega_2, 0)^T,$$

which are all on the xy -plane. For such case, equation (40) needs to be satisfied. Let $x_c = R_c \cos \theta$ and $y_c = R_c \sin \theta$. Given μ , R_c , ℓ , m_a , and m_b , the formula (40) can be written as an equation of θ . Consequently, nongreat-circle relative equilibria can be obtained by solving $f(\theta) = 0$. Note that θ is independent of μ or $\boldsymbol{\Omega}$.

Since $f(0) < 0$, $f(\pi) > 0$, and that $f(\theta)$ is a continuous function for $R_c > \ell$ (which is true for natural configuration), there exists a solution for $f(\theta) = 0$. On the other hand, the first derivative of f can be found to be

$$\frac{df}{d\theta}(\theta) = \frac{\ell y_c}{m_a + m_b} \left\{ m_a^2 \left(\frac{3}{R_a R_b^5} + \frac{1}{R_a^6} \right) + m_b^2 \left(\frac{3}{R_a^5 R_b} + \frac{1}{R_b^6} \right) + \right. \\ \left. + m_a m_b \left[3(R_a^2 + R_b^2 - \ell^2) \left(\frac{1}{R_a^8} + \frac{1}{R_b^8} \right) - \left(\frac{1}{R_a^3} + \frac{1}{R_b^3} \right)^2 \right] \right\},$$

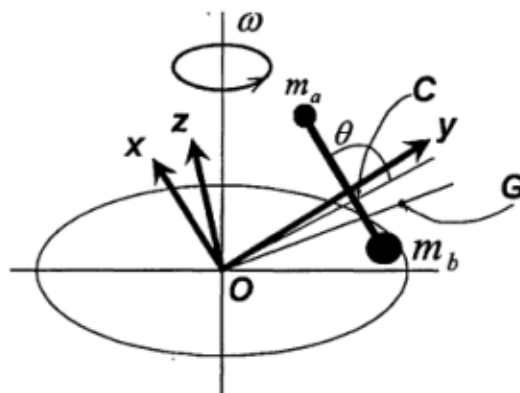


Figure 6. Nongreat-circle relative equilibrium.

which can be proved to be positive. As a consequence, the equation $f(\theta) = 0$ has one and only one solution in the domain $\theta \in [0, \pi]$ for $m_a \neq m_b$ and $R_c > \ell$.

With the value of θ , the variables Ω_1 and Ω_2 can be computed from the formula

$$\Omega_1 = f_y \beta, \quad \Omega_2 = -f_x \beta,$$

where

$$\beta = \sqrt{\frac{\mu y_c}{\ell f_x f_y} \left(\frac{1}{R_a^3} - \frac{1}{R_b^3} \right)}.$$

The elastic force on the spring can be then obtained from either (28) or (30). This leads immediately to the following theorem.

THEOREM 4. *For natural configurations ($R_c > \ell$), the dynamics of the spring system has exactly one nongreat-circle relative equilibrium if and only if $m_a \neq m_b$.*

The nongreat-circle relative equilibria are depicted in Figure 6, where the spring also undergoes compressive forces.

Note that when the spring becomes more and more rigid, the spring system approaches a rigid bar system, which consists of two point masses connected by a massless rigid link. The analyses also show that there are nongreat-circle relative equilibria for the rigid bar system.

While the nongreat-circle relative equilibrium for a rigid body was numerically obtained in (Wang et al., 1992), it is analytically proved in this paper. To justify the existence of nongreat-circle relative equilibria, some numerical computations were performed to solve $f(\theta) = 0$ for different R_c/ℓ and $m_a/(m_a + m_b)$ (it may be assumed that $m_a < m_b$ without loss of generality). The results are presented in Table I.

Let ϕ denote the angle between the rotating axis and the x -axis of the frame system, c.f. Figure 6. The angle between Ω and R_c is then $\theta - \phi$. The configurations for some nongreat-circle relative equilibria are next computed, and shown in

Table I. Solutions of $f(\theta) = 0$

θ (deg)	$R_c/\ell = 2$	$R_c/\ell = 20$	$R_c/\ell = 200$	$R_c/\ell = 2000$
$\frac{m_a}{m_a + m_b} = 0.01$	100.36489817	91.05265856	90.10528084	90.01052809
$\frac{m_a}{m_a + m_b} = 0.1$	98.41353280	90.85926471	90.08594349	90.00859436
$\frac{m_a}{m_a + m_b} = 0.2$	96.27869712	90.64441681	90.06445759	90.00644577
$\frac{m_a}{m_a + m_b} = 0.3$	94.17070995	90.42959610	90.04297171	90.00429718
$\frac{m_a}{m_a + m_b} = 0.4$	92.08080378	90.21479351	90.02148585	90.00214859

Table II. Nongreat-circle relative equilibria

m_a (kg)	m_b (kg)	θ (deg)	ϕ (deg)	δ (deg)
1	9999	91.073934	1.073934	-0.00000026839
10	9990	91.071999	1.072002	-0.0000026767
100	9900	91.052659	1.052684	-0.000026048
1000	9000	90.858264	0.859458	-0.00019335
2000	8000	90.644416	0.644674	-0.00025784

Table II. Here $\delta = \theta - \phi - 90^\circ$, and it is assumed that $R_c = 7,000$ km, $\ell = 350$ km, and $\mu = 4 \times 10^{14} (\text{m}^3/\text{s}^2)$, the gravitational constant of the Earth.

Note that $\delta \neq 0$ implies that the corresponding relative equilibrium is nongreat-circle. Although this angle is rather small, the deflection of the spring system from the vertical (of the orbital plane), represented by ϕ , is significant. For longer tether at LEO, this attitude drift may reach tens of degrees. This effects of orbit-attitude coupling can be significant for systems with long tethers such as the orbit tower or the skyhook.

5. Relative Stability and the Reduced Energy-momentum Method

Relative equilibria are the *equilibria* of the reduced dynamical equations after performing the reduction process with respect to the symmetry. They are called *relatively stable* if the corresponding equilibria are stable for the reduced dynamics in the sense of Lyapunov. Recall that in classical mechanics, cf. (Goldstein, 1980), the Routhian reduction process can be performed for systems having cyclic coordinates. In geometric language, this reduction corresponds to a symmetry with

Euclidean group $\mathbb{R}^n$. The stability of the steady motion ignoring the cyclic coordinates is in fact equivalent to the relative stability of relative equilibrium.

For general reductions, such as the one treated here, the symmetry group may not be $\mathbb{R}^n$. The classical Routhian approach is thus not directly applicable. On the other hand, recent development in geometric mechanics enables one to assess the stability of steady motions via the so-called *reduced energy-momentum method*, cf. (Simo et al., 1991; Wang et al., 1992). There the sum of energy map and momentum map is used to obtain conditions for relative stability. With the terminologies in geometric mechanics, cf. (Abraham and Marsden, 1978), the method applying to the system under consideration can be briefly outlined as follows.

- (i) Pick $\xi \in \mathbb{R}^3$.
- (ii) Find $\mathbf{a}_e, \mathbf{b}_e$ such that $DV_\xi(\mathbf{a}_e, \mathbf{b}_e) = 0$, cf. (5).
- (iii) Compute the *premomentum map* $\tilde{\mathbf{J}}: Q \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$\tilde{\mathbf{J}}(\mathbf{a}, \mathbf{b}, \xi) = \mathbf{I}_{\text{lock}}^0(\mathbf{a}, \mathbf{b})\xi,$$

where $\mathbf{I}_{\text{lock}}^0$ is the *locked inertia tensor*, defined through the formula,

$$\langle \zeta, \mathbf{I}_{\text{lock}}^0(\mathbf{a}, \mathbf{b})\eta \rangle = K(\mathbf{a}, \mathbf{b})((\zeta \times \mathbf{a}, \zeta \times \mathbf{b}), (\eta \times \mathbf{a}, \eta \times \mathbf{b})),$$

for $\zeta, \eta \in \mathbb{R}^3$. Then let $\mu_e = \tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \xi)$.

- (iv) Find the kernel of $D\tilde{\mathbf{J}}$ at $(\mathbf{a}_e, \mathbf{b}_e, \xi)$.
- (v) Compute the subgroup

$$G_{\mu_e} = \{B \in \text{SO}(3): B\mu_e = \mu_e\}$$

and the associated tangent space of the group orbit, $T_{(\mathbf{a}_e, \mathbf{b}_e, \xi)}(G_{\mu_e} \cdot (\mathbf{a}_e, \mathbf{b}_e, \xi))$, where the action of G_{μ_e} on $(\mathbf{a}, \mathbf{b}, \xi)$ is defined to be

$$B, (\mathbf{a}, \mathbf{b}, \xi) \mapsto (B\mathbf{a}, B\mathbf{b}, B\xi).$$

- (vi) Find the subspace S such that

$$\ker D\tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \xi) = S \oplus T_{(\mathbf{a}_e, \mathbf{b}_e, \xi)}(G_{\mu_e} \cdot (\mathbf{a}_e, \mathbf{b}_e, \xi)).$$

- (vii) Define the bilinear form

$$\begin{aligned} & \Gamma(\mathbf{a}_e, \mathbf{b}_e, \xi) \cdot (\delta\mathbf{a}_1, \delta\mathbf{b}_1, \eta_1) \cdot (\delta\mathbf{a}_2, \delta\mathbf{b}_2, \eta_2) \\ &= \langle \eta_1, \mathbf{I}_{\text{lock}}^0(\mathbf{a}_e, \mathbf{b}_e)\eta_2 \rangle + D^2V_\xi(\mathbf{a}_e, \mathbf{b}_e) \cdot (\delta\mathbf{a}_1, \delta\mathbf{b}_1) \cdot (\delta\mathbf{a}_2, \delta\mathbf{b}_2). \end{aligned}$$

Then the relative equilibrium is relatively stable if the associated quadratic form on S is positive definite.

Here the bilinear form is in fact the second variation of the energy-momentum map induced on an appropriate space. For more complicated systems, the general form of the reduced energy-momentum method can be found in the above-

mentioned references. The method listed above shall be followed step by step to determine the stability of radial relative equilibrium in the next section.

6. Relative Stability of Radial Relative Equilibria

The radial relative equilibria discussed in Section 4 correspond to the station-keeping mode for TSS. The stability of such mode is important during the operation of TSS. However, the classical energy method is not adequate for the system under consideration. Accordingly, the reduced energy-momentum method is employed in this section to prove the stability of radial relative equilibria.

For simplicity, we rewrite the conditions for radial relative equilibrium (27) as

$$\mathbf{R}_a = (a, 0, 0)^T, \quad \mathbf{R}_b = (b, 0, 0)^T, \quad a > b > 0,$$

and $\boldsymbol{\Omega} = (0, 0, \omega)^T$, with

$$\mu = \frac{m_a a + m_b b}{m_a/a^2 + m_b/b^2} \omega^2, \quad (41)$$

and

$$W' = \frac{(a^2 + ab + b^2)m_a m_b \omega^2}{m_a b^2 + m_b a^2} \cdot (a - b) > 0. \quad (42)$$

Without loss of generality, we choose $\boldsymbol{\xi} = (0, 0, \omega)$. Then Step (ii) described in Section 5 is fulfilled by the radial relative equilibria. In Step (iii), the metric K can be found to be

$$K(\mathbf{a}, \mathbf{b})((\mathbf{v}_1, \mathbf{v}_2), (\mathbf{w}_1, \mathbf{w}_2)) = m_a \langle \mathbf{v}_1, \mathbf{w}_1 \rangle + m_b \langle \mathbf{v}_2, \mathbf{w}_2 \rangle.$$

Thus the *locked inertia tensor* $\mathbf{I}_{\text{lock}}^0(\mathbf{a}, \mathbf{b})$ can be derived as

$$\mathbf{I}_{\text{lock}}^0(\mathbf{a}, \mathbf{b}) = -m_a \widehat{\mathbf{a}}\widehat{\mathbf{a}} - m_b \widehat{\mathbf{b}}\widehat{\mathbf{b}}.$$

The pre-momentum map $\tilde{\mathbf{J}}: Q \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is then

$$\tilde{\mathbf{J}}(\mathbf{a}, \mathbf{b}, \boldsymbol{\xi}) = \mathbf{I}_{\text{lock}}^0(\mathbf{a}, \mathbf{b})\boldsymbol{\xi} = -m_a \widehat{\mathbf{a}}\widehat{\mathbf{a}}\boldsymbol{\xi} - m_b \widehat{\mathbf{b}}\widehat{\mathbf{b}}\boldsymbol{\xi}.$$

Accordingly, the derivative of $\tilde{\mathbf{J}}$ is

$$\begin{aligned} D\tilde{\mathbf{J}}(\mathbf{a}, \mathbf{b}, \boldsymbol{\xi}) \cdot (\delta\mathbf{a}, \delta\mathbf{b}, \boldsymbol{\eta}) \\ = m_a (2\widehat{\mathbf{a}}\widehat{\boldsymbol{\xi}} - \widehat{\boldsymbol{\xi}}\widehat{\mathbf{a}})\delta\mathbf{a} + m_b (2\widehat{\mathbf{b}}\widehat{\boldsymbol{\xi}} - \widehat{\boldsymbol{\xi}}\widehat{\mathbf{b}})\delta\mathbf{b} - (m_a \widehat{\mathbf{a}}\widehat{\mathbf{a}} + m_b \widehat{\mathbf{b}}\widehat{\mathbf{b}})\boldsymbol{\eta}. \end{aligned}$$

At radial relative equilibrium, $\mathbf{a}_e = a\mathbf{e}_1$, $\mathbf{b}_e = b\mathbf{e}_1$ and $\boldsymbol{\xi} = \omega\mathbf{e}_3$, where $\mathbf{e}_i$'s denote the unit vectors of the coordinate axes of the uniformly rotating frame. Thus,

$$\mathbf{I}_{\text{lock}}^0(\mathbf{a}_e, \mathbf{b}_e) = I_e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with } I_e = m_a a^2 + m_b b^2.$$

Since the locked inertia tensor is singular at such radial relative equilibrium, the block diagonalization result in (Simo et al., 1991; Wang and Krishnaprasad, 1992) cannot be directly used; however, the reduced energy-momentum method still applies. At radial relative equilibrium, we have

$$\tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi}) = I_e \omega \mathbf{e}_3 = \boldsymbol{\mu}_e.$$

Next it is required to find the kernel space of $D\tilde{\mathbf{J}}$ at the radial relative equilibria. Let $\delta\mathbf{a} = (\delta a_1, \delta a_2, \delta a_3)^T$, $\delta\mathbf{b} = (\delta b_1, \delta b_2, \delta b_3)^T$ and $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)^T$. It can be checked that the kernel of $D\tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})$ is

$$\ker(D\tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})) = \left\{ (\delta\mathbf{a}, \delta\mathbf{b}, \boldsymbol{\eta}) : \begin{aligned} &m_a a \delta a_3 + m_b b \delta b_3 = 0, \\ &\eta_2 = 0, \quad \eta_3 = -\frac{2\omega}{I_e} (m_a a \delta a_1 + m_b b \delta b_1) \end{aligned} \right\}. \quad (43)$$

In Step (v), we need to find $T_{(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})}(G_{\boldsymbol{\mu}_e} \cdot (\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi}))$, the tangent space on the group orbit, where

$$G_{\boldsymbol{\mu}_e} = \{e^{\hat{\mathbf{q}}}: \mathbf{q} // \mathbf{e}_3\}.$$

From the above observation, the tangent space is immediately obtained,

$$T_{(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})}(G_{\boldsymbol{\mu}_e} \cdot (\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})) = \{(\alpha a \mathbf{e}_2, \alpha b \mathbf{e}_2, \mathbf{0}) : \alpha \in \mathbb{R}\}. \quad (44)$$

Obviously, $T_{(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi})}(G_{\boldsymbol{\mu}_e} \cdot (\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi}))$ is a subspace of $\ker(D\tilde{\mathbf{J}}(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi}))$. From (43) and (44), the space S in Step (vi) can be expressed as

$$S = \left\{ (\delta\mathbf{a}, \delta\mathbf{b}, \boldsymbol{\eta}) : \begin{aligned} &\delta a_2 = \delta b_2 = \eta_2 = 0, \quad m_a a \delta a_3 + m_b b \delta b_3 = 0, \\ &\eta_3 = -\frac{2\omega}{I_e} (m_a a \delta a_1 + m_b b \delta b_1) \end{aligned} \right\}. \quad (45)$$

Step (vii) is next performed. The quadratic form corresponding to Γ on S can be obtained as

$$\begin{aligned}
& \Gamma(\mathbf{a}_e, \mathbf{b}_e, \boldsymbol{\xi}) \cdot (\delta \mathbf{a}, \delta \mathbf{b}, \boldsymbol{\eta}) \cdot (\delta \mathbf{a}, \delta \mathbf{b}, \boldsymbol{\eta}) \\
&= \frac{4\omega^2}{I_e} (m_a a \delta a_1 + m_b b \delta b_1)^2 + m_a \left[-\left(\frac{2\mu}{a^3} + \omega^2\right) \delta a_1^2 + \frac{\mu}{a^3} \delta a_3^2 \right] + \\
&+ m_b \left[-\left(\frac{2\mu}{b^3} + \omega^2\right) \delta b_1^2 + \frac{\mu}{b^3} \delta b_3^2 \right] + \\
&+ W'' \cdot (\delta a_1 - \delta b_1)^2 + W' \cdot (\delta a_3 - \delta b_3)^2 / (a - b). \tag{46}
\end{aligned}$$

Since $W' > 0$ and $a > b > 0$, it is easily checked that the terms containing δa_3 and δb_3 in the above expression are always positive. Hence to determine the positive-definiteness, it is only necessary to consider the following terms

$$\begin{aligned}
F &= \left[\frac{4\omega^2}{I_e} m_a^2 a^2 - \left(\frac{2\mu}{a^3} + \omega^2\right) m_a + W'' \right] \delta a_1^2 + \\
&+ \left[\frac{4\omega^2}{I_e} m_b^2 b^2 - \left(\frac{2\mu}{b^3} + \omega^2\right) m_b + W'' \right] \delta b_1^2 + \\
&+ 2 \left(\frac{4\omega^2}{I_e} m_a m_b a b - W'' \right) \delta a_1 \delta b_1. \tag{47}
\end{aligned}$$

Defining

$$u = \frac{4m_a a^2}{I_e} - \frac{2\mu}{a^3 \omega^2} - 1, \quad v = \frac{4m_b b^2}{I_e} - \frac{2\mu}{b^3 \omega^2} - 1,$$

F in (47) is simplified as

$$\begin{aligned}
F &= (m_a \omega^2 u + W'') \delta a_1^2 + (m_b \omega^2 v + W'') \delta b_1^2 + \\
&+ 2(4\omega^2 m_a m_b a b / I_e - W'') \delta a_1 \delta b_1.
\end{aligned}$$

The necessary and sufficient condition for F being positive-definite is that

$$m_a \omega^2 u + W'' > 0, \tag{48}$$

and

$$\begin{aligned}
& \left(m_a u + m_b v + \frac{8m_a m_b a b}{I_e} \right) \omega^2 W'' + \\
&+ \left(uv - \frac{16m_a m_b a^2 b^2}{I_e^2} \right) m_a m_b \omega^4 > 0. \tag{49}
\end{aligned}$$

The results are summarized in the following theorem.

THEOREM 5. *For a spring system whose spring characteristics are governed by the elastic potential energy function W (possibly nonlinear), the radial relative equilibrium is stable if (48) and (49) are satisfied.*

Checking the conditions on stability thus requires the knowledge of the elastic potential energy W . It is noted that in the classical approach, such as the one in (Beletskii and Levin, 1991), the reduced system is obtained by deriving the equations of motion relative to a constantly rotating frame. The stability analysis is then performed with an appropriate Hamiltonian. It has been checked that such method leads to a similar quadratic form as in (46), however, without the *first* positive term. As a consequence, the conditions obtained by the reduced energy-momentum method are *weaker* than those by classical energy method.

Next we consider a linearly elastic spring whose elastic potential energy is

$$W(a-b) = \frac{1}{2}k(a-b-\ell_0)^2,$$

where k is the spring constant and ℓ_0 is the reference length. From (42), we have

$$W'' = \frac{\mu m_a m_b (a^2 + ab + b^2)}{a^2 b^2 (m_a a + m_b b)} \cdot \chi,$$

where $\chi = (a-b)/(a-b-\ell_0) > 1$. A sufficient condition for stability can be then obtained as

$$A^* = u + \frac{r^2 + r + 1}{m + r^2} \chi > 0, \quad (50)$$

and

$$B^* = C^* \frac{r^2 + r + 1}{m + r^2} \chi + uv - \frac{16mr^2}{(mr^2 + 1)^2} > 0, \quad (51)$$

where

$$C^* = mu + v + \frac{8mr}{mr^2 + 1}, \quad m = \frac{m_a}{m_b}, \quad r = \frac{a}{b}.$$

It can be checked that

$$A^* = \{(r^3)m^2 + [(r^5 + r^4 + r^3)\chi + (3r^5 - 2r^2 - 3r)]m + \\ + [(r^3 + r^2 + r)\chi - (r^3 + 2)]\}/r^3h,$$

where

$$h = (m + 1/r^2)(m + r^2) > 0.$$

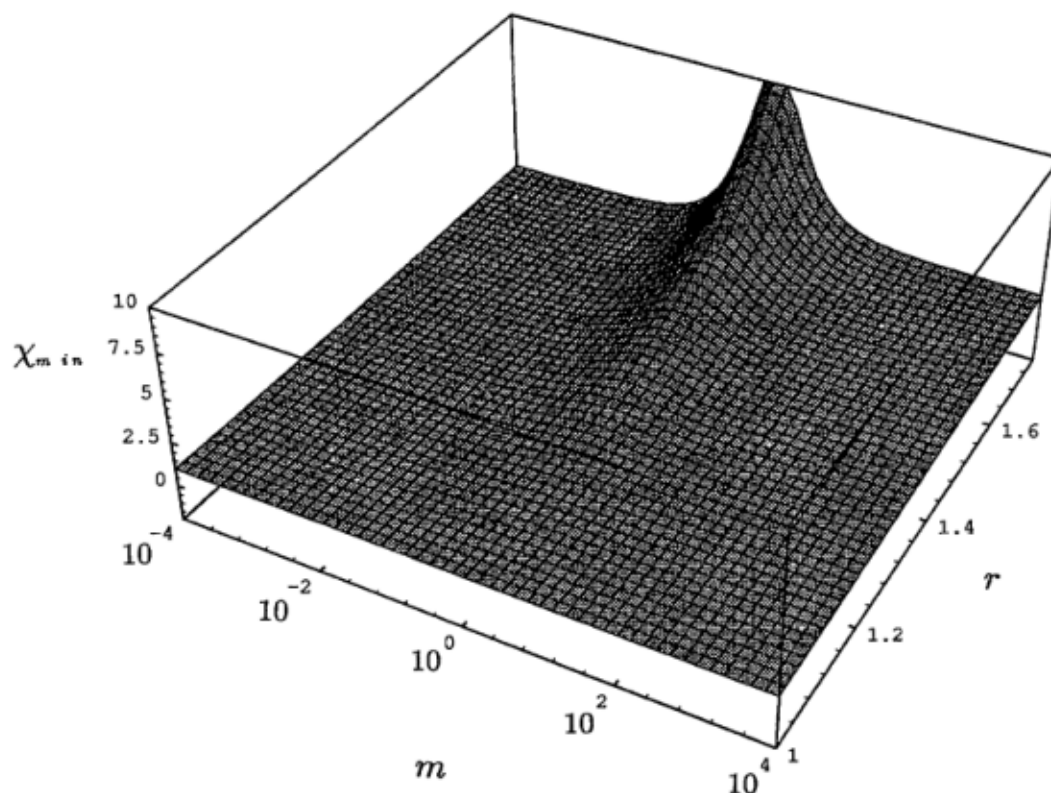


Figure 7. $\chi_{\min}$ with respect to m and r .

Since $r > 1$ and $\chi > 1$, the coefficients of m^2 , m^1 and m^0 in the above expression of A^* are all positive. Thus A^* is always positive. On the other hand, we have

$$C^* = [r^3 m^3 + (-2r^6 + 3r^5 - r^3 + 6r^2 - 3r)m^2 + (-3r^5 + 6r^4 - r^3 + 3r - 2)m + r^3]/r^3 h,$$

which can be checked numerically to be positive if $1 < r (= a/b) < 1.74$, which is a physically accepted range. Within such range, B^* is positive if and only if

$$\chi > \frac{(m + r^2) [16mr^2/(mr^2 + 1)^2 - uv]}{C^*(r^2 + r + 1)} = \chi_{\min}(m, r).$$

Some values of $\chi_{\min}(m, r)$ with respect to m and r are shown in Figure 7, where it is found that practically χ is greater than $\chi_{\min}$. Consider, for example, a radial relative equilibrium of the TSS with $m_a = 10,000$ kg, $m_b = 100$ kg, and $a = 7,350,000$ m, $b = 7,000,000$ m. Then $r = 1.05$ and $m = 100$. Since $1 < r < 1.74$, the radial relative equilibrium is stable if $\chi > \chi_{\min}(m, r)$, where $\chi_{\min}(m, r) = 1.05138$. One equivalent condition is that the reference length

$$\ell_0 > \left(1 - \frac{1}{\chi_{\min}}\right) (a - b) = 17103.7 \text{ m}.$$

If $\ell_0 = 340,000$ m, we have $\chi = 35$, and thus the relative equilibrium is stable. With respect to the Earth's gravitational field, the corresponding spring constant must be $k = 0.011$ (Nt/m), which implies that the product of Young's modulus E of the tether material and the cross section area A of the tether is equal to $EA = k\ell_0 = 3,730.45$ (Nt). Accordingly, for the material Alumina with $E = 69$ GPa, the cross-section area should be $A = 5.4 \times 10^{-8}$ m².

For the rigid bar system, which is the limiting case of the spring system with $\chi \rightarrow \infty$, the condition is always satisfied and thus the radial relative equilibrium is stable.

7. Conclusions

In this paper, the dynamical behaviors of the system of two spring-connected point masses moving in a central gravitational field were discussed. It was observed that the spring system in orbit possesses the SO(3)-symmetry. At the relative equilibrium corresponding to this symmetry, it is the center of gravity (instead of the center of mass) that must trace a great-circle orbit. This leads to the notion of nongreat-circle relative equilibria, in which the center of mass traces a nongreat circle. For the spring systems under consideration, such nongreat-circle relative equilibria exist if and only if the two end-masses are unequal. To study the stability of the radial relative equilibria, the reduced energy-momentum method was applied to successfully obtain some stability conditions. For general (linear or nonlinear) springs, the conditions are in terms of the second derivatives of the elastic potential energy functions. In particular, for a linear spring, explicit conditions were obtained such that, for certain configuration, the radial relative equilibrium of TSS become stable if the tether is chosen to be stiff enough with suitable material constants. Although the radial relative equilibria are stable for most physically acceptable configurations, the analysis in this paper may become essential for more challenging designs such as the orbital tower (Pearson, 1975) or the skyhook (Moravec, 1977).

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