Consistency of the Cartesian G Rules PhL 7.14.12

Here I want to show that, if you stay in Cartesian coordinates, the four "G rules" for **r**, **v**, **a** and **L** are all consistent. I will do this in the "ant on phono record" context.

[This is the first document (by creation date) that shows the $(d\mathbf{r}/dt)_S$ notation, July 14.] [a keeper]

[Despite using the ∂_S stuff correctly, at this time I was unaware of the fact that there are really 8 different accelerations and here is what happens below: (1) I compute $\mathbf{a} = \mathbf{a}_S$ by differentiation of **v** and I notice that there are $\dot{\omega}$ terms present. (2) I compute the cross acceleration $a_{S'}$ and notice that there are no $\dot{\omega}$ terms present. (3) I wrongly think of these two results as being the same acceleration, and I then have a paradox since one has $\dot{\omega}$ stuff and the other does not !]

1. Start with the G rule for r:

[I think this entire section is OK, though the result is not expressed in a useful manner.]

$\mathbf{v} = (d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'}$. [correct !!]

Our ant makes his voyage **r**(t) in frame S' which we write as

$\mathbf{r}(t) = r'_1(t)\, \mathbf{e}'_1 + r'_2(t)\, \mathbf{e}'_2$, [ok if components mean $(r)'_i$]

where everything is S'-frame stuff. Then it is quite easy to compute

$(d\mathbf{r}/dt)_{S'} = d_t\, \mathbf{r}(t) = \dot{r}'_1(t)\, \mathbf{e}'_1 + \dot{r}'_2(t)\, \mathbf{e}'_2$. [since $\mathbf{e}'_i$ constants; again $(\dot{r})'_i(t)$ components]

Meanwhile,

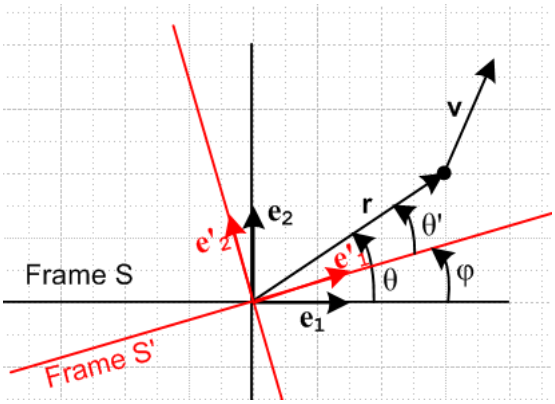
$\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} = [\omega \mathbf{z}] \times [r'_1(t)\, \mathbf{e}'_1 + r'_2(t)\, \mathbf{e}'_2] = \omega\, r'_1\, \mathbf{e}'_2 - \omega\, r'_2\, \mathbf{e}'_1$. [ok]

We then end up with

$\mathbf{v} = (d\mathbf{r}/dt)_S = \omega\, r'_1\, \mathbf{e}'_2 - \omega\, r'_2\, \mathbf{e}'_1 + \dot{r}'_1(t)\, \mathbf{e}'_1 + \dot{r}'_2(t)\, \mathbf{e}'_2$ [ok]

$= (\dot{r}'_1 - \omega\, r'_2)\, \mathbf{e}'_1 + (\dot{r}'_2 + \omega\, r'_1)\, \mathbf{e}'_2$ [ok, components must be $(v)'_i$]

But now we want this in terms of frame S basis vectors,



$$\mathbf{e}_1' = R(\varphi)\,\mathbf{e}_1 = (\mathbf{e}_1' \bullet \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{e}_1' \bullet \mathbf{e}_2)\,\mathbf{e}_2$$
$$\mathbf{e}_2' = R(\varphi)\,\mathbf{e}_2 = (\mathbf{e}_2' \bullet \mathbf{e}_1)\,\mathbf{e}_1 + (\mathbf{e}_2' \bullet \mathbf{e}_2)\,\mathbf{e}_2$$

$$\mathbf{e}_1' = R(\varphi)\,\mathbf{e}_1 = \cos\varphi\,\mathbf{e}_1 + \cos(\pi/2-\varphi)\,\mathbf{e}_2$$
$$\mathbf{e}_2' = R(\varphi)\,\mathbf{e}_2 = \cos(\pi/2+\varphi)\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2$$

$$\mathbf{e}_1' = R(\varphi)\,\mathbf{e}_1 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2$$
$$\mathbf{e}_2' = R(\varphi)\,\mathbf{e}_2 = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2$$

or

$$\begin{pmatrix}\mathbf{e}_1'\\ \mathbf{e}_2'\end{pmatrix} = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix} = R_z(\varphi)^T\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{pmatrix}$$

Then we find

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\dot{r}'_1 - \omega\, r'_2)\,\mathbf{e}'_1 + (\dot{r}'_2(t) + \omega\, r'_1)\,\mathbf{e}'_2$$

$$= (\dot{r}'_1 - \omega\, r'_2)\,[\cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2]$$
$$+ \quad (\dot{r}'_2 + \omega\, r'_1)\,[-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2]$$

$$= [\,(\dot{r}'_1 - \omega\, r'_2)\cos\varphi - (\dot{r}'_2 + \omega\, r'_1)\sin\varphi\,]\,\mathbf{e}_1$$
$$+ \quad [\,(\dot{r}'_1 - \omega\, r'_2)\sin\varphi + (\dot{r}'_2 + \omega\, r'_1)\cos\varphi]\,\mathbf{e}_2$$

[ok unless algebra errors]

Now let's try for more compact vector notation. How about

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}'_1, \mathbf{e}'_2)\begin{pmatrix}\dot{r}'_1 - \omega\, r'_2\\ \dot{r}'_2 + \omega\, r'_1\end{pmatrix}$$

$$(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\,R(\varphi) = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}$$

Then we get

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix}$$

which agrees with the written-out result above. Notice there are no $\dot{\omega}$ terms here.

Test Case: Suppose the ant is at rest in frame S'. This then says $[(d\mathbf{r}/dt)_{S'} = 0\ ]$

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} -\omega\, r'_2 \\ \omega\, r'_1 \end{pmatrix} = (\mathbf{e}'_1, \mathbf{e}'_2)\begin{pmatrix} -\omega\, r'_2 \\ \omega\, r'_1 \end{pmatrix}$$

$= -\omega r'_2\, \mathbf{e}'_1 + \omega r'_1\, \mathbf{e}'_2$ [would rather express this in terms of $\mathbf{e}_n$]

Now meanwhile

$\boldsymbol{\omega} \times \mathbf{r} = [\omega \mathbf{z}] \times [r'_1(t)\, \mathbf{e}'_1 + r'_2(t)\, \mathbf{e}'_2\ ] = \omega\, r'_1\, \mathbf{e}'_2 - \omega\, r'_2\, \mathbf{e}'_1$ = the above $\mathbf{v}$.

In polar coordinates, $\mathbf{r} = r\, \hat{\mathbf{r}}$ and we then get

$$\boldsymbol{\omega} \times \mathbf{r} = \omega r\, \hat{\boldsymbol{\theta}} = \mathbf{v}$$

which seems right.

2. Differentiate v to get a

[my general plan in this doc is to compare the "differentiation method" to the "G Rule method" to see if the two methods agree or not. So here we are doing the first method on **v** .]

[apart from possible algebra errors, I think this section is also OK.]

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix}$$

$$d_t\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \omega\begin{pmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}$$

$$d_t\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix} = \begin{pmatrix} \ddot{r}'_1 - \omega\, \dot{r}'_2 - \dot{\omega}\, r'_2 \\ \ddot{r}'_2 + \omega\, \dot{r}'_1 + \dot{\omega}\, r'_1 \end{pmatrix}$$

// for access later: $\ddot{r}'_1 - \omega\, \dot{r}'_2 - \dot{\omega}\, r'_2$ $\quad$ $\ddot{r}'_2 + \omega\, \dot{r}'_1 + \dot{\omega}\, r'_1$

$\dot{r}'_1 - \omega\, r'_2$ $\quad$ $\dot{r}'_2 + \omega\, r'_1$

Then our result seems to be

$$a = (\mathbf{e}_1, \mathbf{e}_2)\left[\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2 \\ \ddot{r}'_2 + \omega \dot{r}'_1 + \dot{\omega} r'_1 \end{pmatrix} + \omega \begin{pmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix} \begin{pmatrix} \dot{r}'_1 - \omega r'_2 \\ \dot{r}'_2 + \omega r'_1 \end{pmatrix} \right]$$

An impressive mess!

Write out as

$$a = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2$$

$$a_1 = \cos\varphi\,(\ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2) - \sin\varphi(\ddot{r}'_2 + \omega \dot{r}'_1 + \dot{\omega} r'_1) - \omega\sin\varphi\,(\dot{r}'_1 - \omega r'_2) - \omega\cos\varphi\,(\dot{r}'_2 + \omega r'_1)$$
$$a_2 = \sin\varphi\,(\ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2) + \cos\varphi(\ddot{r}'_2 + \omega \dot{r}'_1 + \dot{\omega} r'_1) + \omega\cos\varphi\,(\dot{r}'_1 - \omega r'_2) - \omega\sin\varphi\,(\dot{r}'_2 + \omega r'_1)$$

$$a_1 = \cos\varphi\,(\ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2) - \sin\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) - \omega\sin\varphi\,(-\omega r'_2) - \omega\cos\varphi\,(\dot{r}'_2 + \omega r'_1)$$
$$a_2 = \sin\varphi\,(\ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2) + \cos\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) + \omega\cos\varphi\,(-\omega r'_2) - \omega\sin\varphi\,(\dot{r}'_2 + \omega r'_1)$$

$$a_1 = \cos\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2) - \sin\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) - \omega\sin\varphi\,(-\omega r'_2) - \omega\cos\varphi\,(\omega r'_1)$$
$$a_2 = \sin\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2) + \cos\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) + \omega\cos\varphi\,(-\omega r'_2) - \omega\sin\varphi\,(\omega r'_1)$$

$$a_1 = \cos\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2) - \sin\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) + \omega^2\sin\varphi\, r'_2 - \omega^2\cos\varphi\, r'_1$$
$$a_2 = \sin\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2) + \cos\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1) - \omega^2\cos\varphi\, r'_2 - \omega^2\sin\varphi\, r'_1$$

$$a_1 = \cos\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2 - \omega^2 r'_1) - \sin\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1 - \omega^2 r'_2)$$
$$a_2 = \sin\varphi\,(\ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2 - \omega^2 r'_1) + \cos\varphi(\ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1 - \omega^2 r'_2)$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \ddot{r}'_1 - 2\omega \dot{r}'_2 - \dot{\omega} r'_2 - \omega^2 r'_1 \\ \ddot{r}'_2 + 2\omega \dot{r}'_1 + \dot{\omega} r'_1 - \omega^2 r'_2 \end{pmatrix}$$

Here is an alternate way to get the above result. Start again with

$$\mathbf{a} = (\mathbf{e}_1, \mathbf{e}_2)\left[\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \ddot{r}'_1 - \omega \dot{r}'_2 - \dot{\omega} r'_2 \\ \ddot{r}'_2 + \omega \dot{r}'_1 + \dot{\omega} r'_1 \end{pmatrix} + \omega \begin{pmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix} \begin{pmatrix} \dot{r}'_1 - \omega r'_2 \\ \dot{r}'_2 + \omega r'_1 \end{pmatrix} \right]$$

Rewrite the second term as (you have to stare carefully to verify this, check all 4 items!)

$$-\omega \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \dot{r}'_2 + \omega r'_1 \\ -\dot{r}'_1 + \omega r'_2 \end{pmatrix}$$

Then we have

$$\mathbf{a} = (\mathbf{e}_1, \mathbf{e}_2)\left[\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \ddot{r}'_1 - \omega\,\dot{r}'_2 - \dot{\omega}\,r'_2 \\ \ddot{r}'_2 + \omega\,\dot{r}'_1 + \dot{\omega}\,r'_1 \end{pmatrix} - \omega \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \dot{r}'_2 + \omega\,r'_1 \\ -\dot{r}'_1 + \omega\,r'_2 \end{pmatrix} \right]$$

$$= (\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \ddot{r}'_1 - 2\omega\,\dot{r}'_2 - \dot{\omega}\,r'_2 - \omega^2\,r'_1 \\ \ddot{r}'_2 + 2\omega\,\dot{r}'_1 + \dot{\omega}\,r'_1 - \omega^2 r'_2 \end{pmatrix}$$

which replicates the result obtained above. Note the presence of $\dot{\omega}$ terms!

3. Do the G rule for v:

We want to obtain the same result as above for a by using the G rule for **v**. That rule is

$$\mathbf{a} = (d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'} \quad . \qquad \text{[ok]}$$

So, ant makes this voyage [normally would have used **r**', but OK to give voyage in terms of **r**]

$$\mathbf{r}_{ant}(t) = r'_1(t)\,\mathbf{e}'_1 + r'_2(t)\,\mathbf{e}'_2 \qquad [\ \mathbf{r} = (r)'_i\,\mathbf{e}'_i, \text{ seems valid }]$$

$$\mathbf{v}_{ant}(t) = \dot{r}'_1(t)\,\mathbf{e}'_1 + \dot{r}'_2(t)\,\mathbf{e}'_2 \qquad [\ (d\mathbf{r}/dt)_{S'} = \mathbf{v}_{S'} = (\dot{r})'_i\,\mathbf{e}'_i\ ]$$

$$(d\mathbf{v}_{ant}/dt)_{S'} = \ddot{r}'_1(t)\,\mathbf{e}'_1 + \ddot{r}'_2(t)\,\mathbf{e}'_2 = (\mathbf{e}'_1, \mathbf{e}'_2)\begin{pmatrix} \ddot{r}'_1 \\ \ddot{r}'_2 \end{pmatrix} = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \ddot{r}'_1 \\ \ddot{r}'_2 \end{pmatrix}$$

[$\partial_{S'}{}^2\mathbf{r} = \partial_{S'}\,\mathbf{v}_{S'} = \mathbf{a}_{S',S'} = \mathbf{a}_{S'}$ = as shown on right above = one of the 6 cross accelerations!]

and boom, we have the second [first?] term right away. There are no $\dot{\omega}$ (or ω) factors anywhere. After all, this calculation is not supposed to know about the rotation.

STOP. Right away we have a problem Houston. Recall from above that we had

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\,r'_2 \\ \dot{r}'_2 + \omega\,r'_1 \end{pmatrix}$$

which contains no $\dot{\omega}$ terms. Thus $\boldsymbol{\omega} \times \mathbf{v}$ contains no $\dot{\omega}$ terms . And we just showed that $(d\mathbf{v}/dt)_{S'}$ has no ω or $\dot{\omega}$ terms. Therefore, our G Rule will give a result for a which has no $\dot{\omega}$ terms. This then disagrees with the previous section's calculation of a. So we have now arrived at this paradox staying entirely with Cartesian coordinates, avoiding possible confusing extra issues with the polar coordinates.

[A good paradox! Here is the resolution: the derivative method correctly computed $\mathbf{a} = \mathbf{a}_S$, but in the G Rule method we computed the cross acceleration $\mathbf{a}_{S'}$ which is a different object!!!]

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In the first term, we shall use the exact **v** from the previous problem, which was this

$$\mathbf{v} = (d\mathbf{r}/dt)_S = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix}$$

Then,

$$\boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix}$$

$$= \omega\, (\mathbf{e}_2, -\mathbf{e}_1)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix}$$

and then our full result becomes

$$\mathbf{a} = (d\mathbf{v}/dt)_S = \omega\, (\mathbf{e}_2, -\mathbf{e}_1)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \dot{r}'_1 - \omega\, r'_2 \\ \dot{r}'_2 + \omega\, r'_1 \end{pmatrix} + (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \ddot{r}'_1 \\ \ddot{r}'_2 \end{pmatrix}$$

But now compare this with our result from Section 2 above

$$\mathbf{a} = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \ddot{r}'_1 - 2\omega\, \dot{r}'_2 - \dot{\omega}\, r'_2 - \omega^2\, r'_1 \\ \ddot{r}'_2 + 2\omega\, \dot{r}'_1 + \dot{\omega}\, r'_1 - \omega^2 r'_2 \end{pmatrix}$$

This Section 2 result includes factors $\dot{\omega}$, but our G Rule for **v** produces no such factors. So the results are in fact NOT consistent. Why is this? (My 3 week long problem continues, even in Cartesians)

[see resolution above]