Example 1 PhL 7.9.12

I set up the ant problem and then compute everything by direct differentiation (things like **v**, **a**, **L** etc) . Probably this is done correctly, and I am using polar coordinates. My goal seems to be to see if my results are consistent with the G Rule or not. I work backwards from the G Rules to see what things like $d\mathbf{v}_{body}/dt$ "must be" to make things work. But I am never able to give good reasons why such things "must be". It all seems fuzzy and footloose. My idea at least was to study a very simple example before pondering "the general case", and that was a good idea. I think nobody on the web was offering me a clean understanding of the G Rule or I would not have flailed for so wrong. It is not an easy topic to search on.

A phonograph record rotates at $\boldsymbol{\omega}(t) = \omega(t)\hat{\mathbf{z}}$ where a prescribed ω(t) varies in time.
Frame S' rotates with the record, frame S is at rest.

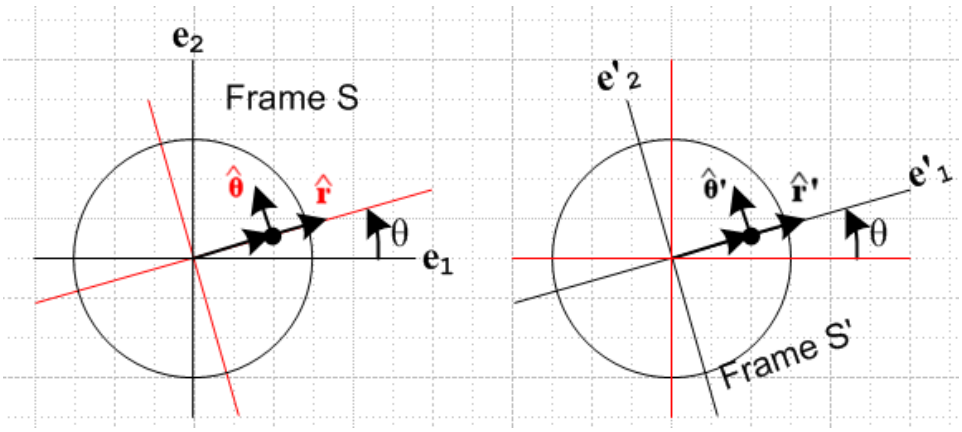
Ant in frame S' has Cartesian coordinates x',y' and polar coordinates r',θ'.
Frame S' has Cartesian unit vectors $\mathbf{e'}_1$, $\mathbf{e'}_2$ and polar unit vectors $\hat{\mathbf{r}}'$,$\hat{\boldsymbol{\theta}}'$.

Ant in frame S has Cartesian coordinates x,y and polar coordinates r,θ .
Frame S has Cartesian unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and polar unit vectors $\hat{\mathbf{r}}$,$\hat{\boldsymbol{\theta}}$.

At time t = 0, the two frames are lined up.

An ant is glued to the record at location (r',θ') = (a,0).
Describe everything that happens.

(a) Here is a picture:



Both pictures show the ant as a black dot at some time t. Frame S' at this time t is rotated relative to frame S by amount θ as shown. The paper is aligned with frame S and graphically all vectors are therefore displayed in terms of their frame S coordinates. The $\hat{\mathbf{z}}$ axis emerges from the plane of paper. The location of the ant in frame S is given by $\mathbf{r}(t) = a\hat{\mathbf{r}}(t)$ and this is the black arrow shown emerging from the origin.

From the pictures it seems clear that

$$\hat{\boldsymbol{\theta}}' = \mathbf{e}'_2 = R_z(\theta)\mathbf{e}_2 = \hat{\boldsymbol{\theta}}$$
$$\hat{\mathbf{r}}' = \mathbf{e}'_1 = R_z(\theta)\mathbf{e}_1 = \hat{\mathbf{r}}$$

We will abbreviate $R_z(\theta)$ as $R(\theta)$ and as R. The picture shows the above equations with frame S components, but of course these vector equations are valid in any frame including frame S', with different numerical values for the components. We can rewrite some of the above equations in this manner

$$\mathbf{e}'_n = R\,\mathbf{e}_n \qquad n = 1, 2$$

$$\hat{\boldsymbol{\theta}}' = \hat{\boldsymbol{\theta}}$$
$$\hat{\mathbf{r}}' = \hat{\mathbf{r}}$$

where all polar unit vectors are based on the ant's position. Since vectors projected onto basis vectors transform inversely to the way the basis vectors transform, if **V** is some arbitrary vector, then we have

$$\mathbf{V} = R\,\mathbf{V}'$$
$$\mathbf{e}'_n = R\,\mathbf{e}_n$$

$$\mathbf{V} = V_n\mathbf{e}_n = V'_n\mathbf{e}'_n$$

(b) In frame S the polar unit vectors are moving and it is easy to show that

$$d\hat{\boldsymbol{\theta}}/dt = -\omega\hat{\mathbf{r}} \qquad \text{and} \qquad d\hat{\mathbf{r}}/dt = \omega\hat{\boldsymbol{\theta}}$$

(c) The angular location of the ant in frame S can be found from $\omega = d\theta/dt$ as follows:

$$\theta(t) = \int_0^t \omega(t)dt$$

The range of angle $\theta(t)$ is taken to be $(-\infty,\infty)$. Since $\omega(t)$ is a prescribed function, $\theta(t)$ is known function.

(d) We then find that (all quantities are in frame S)

$$\mathbf{r}(t) = a\hat{\mathbf{r}} \qquad \text{// position of the ant}$$

$$\mathbf{v}(t) = d\mathbf{r}/dt = a\,d\hat{\mathbf{r}}/dt = a\omega\hat{\boldsymbol{\theta}}$$

$$\mathbf{a}(t) = d\mathbf{v}/dt = a\dot{\omega}\hat{\boldsymbol{\theta}} + a\omega(-\omega\hat{\mathbf{r}}) = a\dot{\omega}\hat{\boldsymbol{\theta}} - a\omega^2\hat{\mathbf{r}}$$

$$d\mathbf{a}/dt = r\ddot{\omega}\hat{\boldsymbol{\theta}} + r\dot{\omega}(-\omega\hat{\mathbf{r}}) - 2r\omega\dot{\omega}\hat{\mathbf{r}} - r\omega^2(\omega\hat{\boldsymbol{\theta}}) = -3r\omega\dot{\omega}\,\hat{\mathbf{r}} + (r\ddot{\omega} - r\omega^3)\,\hat{\boldsymbol{\theta}}$$

$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \mathbf{v} = m\,a\hat{\mathbf{r}} \times a\omega\hat{\boldsymbol{\theta}} = ma^2\omega\,\hat{\mathbf{z}}$

$d\mathbf{L}/dt = ma^2\dot{\omega}\hat{\mathbf{z}}$ [probably all the above is OK to this point]

(b) Our general equation (**) applied to the position vector **r** is this

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + d\mathbf{r}_{body}/dt$

But $\mathbf{r}_{body}$ = constant since the ant is glued to the platter in S' space. This then says

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} = [\omega(t)\hat{\mathbf{z}}] \times [a\hat{\mathbf{r}}] = \omega a\hat{\boldsymbol{\theta}}$

which agrees with the above calculation.

(b) Our general equation (**) applied to the velocity vector **v** is this [non sequitor section number]

$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + d\mathbf{v}_{body}/dt$ [this is my G Rule interpretation for **v**]

Installing $\mathbf{v} = a\omega\hat{\boldsymbol{\theta}}$ from above, we find that

$\boldsymbol{\omega} \times \mathbf{v} = [\omega(t)\hat{\mathbf{z}}] \times [a\omega\hat{\boldsymbol{\theta}}] = a\omega^2\,[-\hat{\mathbf{r}}] = -a\omega^2\hat{\mathbf{r}}$

and so this term accounts for the centripetal (center seeking) acceleration due to the rotation at rate ω. We then have

$d\mathbf{v}/dt = -a\omega^2\hat{\mathbf{r}} + d\mathbf{v}_{body}/dt$

Comparison with the computed result above that $d\mathbf{v}/dt = a\dot{\omega}\hat{\boldsymbol{\theta}} - a\omega^2\hat{\mathbf{r}}$ shows that

$d\mathbf{v}_{body}/dt = a\dot{\omega}\hat{\boldsymbol{\theta}} = a\dot{\omega}\hat{\boldsymbol{\theta}}'$

The ant in the rotating frame S' feels the centripetal acceleration just noted, but in addition feels a tangential acceleration because the rotation rate ω is changing in time. The ant's tangential speed is aω and this acceleration is then $a\dot{\omega}$. So we obtain a reasonable interpretation of $d\mathbf{v}_{body}/dt$.

(c) Our general equation (**) applied to the acceleration vector **a** is this

$d\mathbf{a}/dt = \boldsymbol{\omega} \times \mathbf{a} + d\mathbf{a}_{body}/dt$. [this is my G Rule interpretation for **a**]

As in the previous case, we will first calculate $d\mathbf{a}_{body}/dt$ from the above equation, then we will attempt to interpret it. So :

$$d\mathbf{a}_{body}/dt = d\mathbf{a}/dt - \boldsymbol{\omega} \times \mathbf{a}$$

Inserting **a** from calculation above, we find that

$$-\boldsymbol{\omega} \times \mathbf{a} = -\boldsymbol{\omega} \times (a\dot{\omega}\hat{\boldsymbol{\theta}} - a\omega^2\hat{\mathbf{r}}) = a\omega\dot{\omega}\hat{\mathbf{r}} + a\omega^3\hat{\boldsymbol{\theta}} \ .$$

And above we also computed d**a**/dt, so we then have

$$d\mathbf{a}_{body}/dt = d\mathbf{a}/dt - \boldsymbol{\omega} \times \mathbf{a}$$

$$= [-3a\omega\dot{\omega}\ \hat{\mathbf{r}} + (a\ddot{\omega} - a\omega^3)\ \hat{\boldsymbol{\theta}}\ ] + a\omega\dot{\omega}\hat{\mathbf{r}} + a\omega^3\hat{\boldsymbol{\theta}}$$

$$= a\ddot{\omega}\hat{\boldsymbol{\theta}} - 2a\omega\dot{\omega}\hat{\mathbf{r}}$$

$$= a\ddot{\omega}\hat{\boldsymbol{\theta}}' - 2a\omega\dot{\omega}\hat{\mathbf{r}}'$$

In this case, the $d\mathbf{a}_{body}/dt$ term includes a piece that we would lose if we were to set ω = 0, as possibility mentioned in the general discussion above.

We can now interpret the two terms of $d\mathbf{a}_{body}/dt$. The first term arises because the tangential acceleration magnitude $a\dot{\omega}$ has the derivative $a\ddot{\omega}$. The second term arises because the radial centripetal acceleration has magnitude $-a\omega^2$ and this has derivative $-2a\omega\dot{\omega}$.

(d) Our general equation (**) applied to the angular momentum vector **L** is this

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + d\mathbf{L}_{body}/dt \ .$$ [this is my G Rule interpretation for **a**]

Once again we indirectly compute $d\mathbf{L}_{body}/dt$, then attempt to interpret that result

$$d\mathbf{L}_{body}/dt = d\mathbf{L}/dt - \boldsymbol{\omega} \times \mathbf{L}$$

$$= ma^2\dot{\omega}\hat{\mathbf{z}} - m\ \boldsymbol{\omega} \times [ma^2\omega\ \hat{\mathbf{z}}] = ma^2\dot{\omega}\hat{\mathbf{z}} - \mathbf{0} = ma^2\dot{\omega}\hat{\mathbf{z}}$$

In this example we have **ω** x **L** = 0 and we find that

$$d\mathbf{L}_{body}/dt = d\mathbf{L}/dt = ma^2\dot{\omega}\hat{\mathbf{z}} = ma^2\dot{\omega}\hat{\mathbf{z}}'$$

The ant's angular momentum magnitude is L = ap = amv = am(aω) = $ma^2\omega$, so the derivative then gives dL/dt = $ma^2\dot{\omega}$. In frame S', the ant has $\mathbf{L}_{body} = 0$, but the ant still feels $dL_{body}/dt = ma^2\dot{\omega}$.