Example 2 PhL 7.9.12

This is similar to the Example 1 doc, but now I let the ant move on the record. Again I use my bad G Rule to compute what things like $d\mathbf{v}_{body}/dt$ "must be" to make things work. I am still using bracket notation, so this Example does no better than the last. I get lots of strange terms in my "must be" expressions and it seems hopeless that one could figure out why they are what they are.

A phonograph record rotates at $\boldsymbol{\omega}(t) = \omega(t)\hat{\mathbf{z}}$ where a prescribed $\omega(t) = d\varphi/dt$ varies in time. Frame S' rotates with the record, frame S is at rest. An ant is on the record's surface.

Ant in frame S' has Cartesian coordinates x',y' and polar coordinates r',θ'.
Frame S' has Cartesian unit vectors $\mathbf{e'}_1$, $\mathbf{e'}_2$ and polar unit vectors $\hat{\mathbf{r}}'$,$\hat{\boldsymbol{\theta}}'$.

Ant in frame S has Cartesian coordinates x,y and polar coordinates r,θ
Frame S has Cartesian unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and polar unit vectors $\hat{\mathbf{r}}$,$\hat{\boldsymbol{\theta}}$.

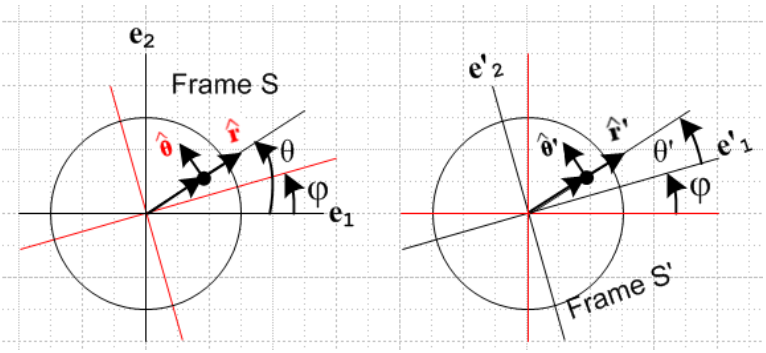
At time t = 0, the two frames are lined up.

- An ant wanders around on the record with some prescribed path

$\mathbf{r'}_{body}(t) = (r'(t),\theta'(t)) = r'_{body}(t)\,\hat{\mathbf{r}}'$ in frame S'.

Describe everything that happens.

(a) Here is a picture:



Both pictures show the ant as a black dot at some time t. Frame S' at this time t is rotated relative to frame S by amount φ as shown. The paper is aligned with frame S and graphically all vectors are therefore displayed in terms of their frame S coordinates. The $\hat{\mathbf{z}}$ axis emerges from the plane of paper. In each picture the heavy arrow from the spindle shows the position of the ant at time t.

From the pictures it seems clear that

$\mathbf{e'}_n = R(\varphi)\,\mathbf{e}_n \qquad n = 1, 2$

$\theta = \theta' + \varphi \quad$ so $\quad \dot{\theta} = \dot{\theta}' + \omega \qquad \omega = d\varphi/dt$
$r = r'$ // the radius of the ant's position is the same in the two frames.

$\hat{\mathbf{r}}' = R(\theta')\mathbf{e'}_1 = R_z(\varphi + \theta')\,\mathbf{e}_1 = \hat{\mathbf{r}}$
$\hat{\boldsymbol{\theta}}' = R(\theta')\,\mathbf{e'}_2 = R_z(\theta + \theta')\mathbf{e}_2 = \hat{\boldsymbol{\theta}}$

We will abbreviate $R_z(\psi)$ as $R(\psi)$. The picture shows the above equations with frame S components, but of course these vector equations are valid in any frame including frame S', with different numerical values for the components. We can rewrite some of the above equations in this manner

$\mathbf{e'}_n = R\,\mathbf{e}_n \qquad n = 1, 2$

$\hat{\boldsymbol{\theta}}' = \hat{\boldsymbol{\theta}}$
$\hat{\mathbf{r}}' = \hat{\mathbf{r}}$

where all polar unit vectors are based on the ant's position. Since vectors projected onto basis vectors transform inversely to the way the basis vectors transform, if **V** is some arbitrary vector, then we have

$\mathbf{V} = R\,\mathbf{V}'$
$\mathbf{e'}_n = R\,\mathbf{e}_n$

$\mathbf{V} = V_n\mathbf{e}_n = V'_n\mathbf{e'}_n$

Question 1. What is the meaning of the claim made above that $\hat{\mathbf{r}}' = \hat{\mathbf{r}}$? Well, for the pictures as I have drawn them, it means that $[\,\hat{\mathbf{r}}'\,]_S = [\,\hat{\mathbf{r}}\,]_S$. This does NOT say that $[\,\hat{\mathbf{r}}'\,]_{S'} = [\,\hat{\mathbf{r}}\,]_S$, [still using that annoying bracket notation] and in fact the latter equation is false. But still this seems confusing because one thinks of **V** as representing the frame S coordinates of a vector **V**, and **V'** as representing the S' coordinates of vector **V**. So **V'** = **V** would be wrong, and we would want to say **V** = R**V'**.

(b) Comments on the ant's body path. The ant wanders around on the record with some prescribed path

$\mathbf{r'}_{body}(t) = r'(t)\,\hat{\mathbf{r}}'(t)$ in frame S'.

In polar coordinates, we can denote the body path this way

$\mathbf{r'}_{body}(t) = (r'(t),\theta'(t))$.

Since r' and θ' are unambiguous, there is no need to put subscripts "body" on them. Similarly, we could just denote $\mathbf{r'}_{body}(t)$ by **r'**(t), since **r'**(t) is unambiguous. But we keep this label anyway.

Expressed in frame S coordinates, this exact same path is given by

$\mathbf{r}_{\mathbf{body}}(t) = R(\varphi)\,\mathbf{r}'_{\mathbf{body}}(t) = (r_{\mathbf{body}}(t),\theta_{\mathbf{body}}(t))$.

Note that this *differs* from the actual path of the ant in frame S, which we call $\mathbf{r}(t)$ or $(r(t),\theta(t))$, so we need the body subscript on everything here.

Similarly to the above we have this rate of change of the body path in frame S' :

$d\mathbf{r}'_{\mathbf{body}}(t)/dt = d_t(r'(t)\,\hat{\mathbf{r}}'(t))$ in frame S'.

Expressed in frame S coordinates, this exact same body velocity object is given by

$(d\mathbf{r}_{\mathbf{body}}(t)/dt) = R(\varphi)(d\mathbf{r}'_{\mathbf{body}}(t)/dt)$

(c) In frame S the polar unit vectors are moving and it is easy to show that

$d\hat{\boldsymbol{\theta}}/dt = -\dot{\theta}\,\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt = \dot{\theta}\,\hat{\boldsymbol{\theta}}$.

(d) The angular location $\theta(t)$ of the ant in frame S can be found from $\omega = d\varphi/dt$ as follows:

$\varphi(t) = \int_0^t \omega(t)dt$ $\qquad\qquad$ $\omega = d\varphi/dt$

$\theta(t) = \theta'(t) + \varphi(t)$ $\qquad$ => $\qquad$ $\dot{\theta} = \dot{\theta}' + \omega = \dot{\theta}' + \dot{\varphi}$.

All angles are taken to be in $(-\infty,\infty)$. Since $\omega(t)$ is prescribed, $\varphi(t)$ is known. Since $\theta'(t)$ is also prescribed, $\theta(t)$ is also known.

(e) Now with r(t) as the true trajectory of the ant in frame S, we compute everything as usual:

$\mathbf{r}(t) = r\hat{\mathbf{r}}$ $\qquad\qquad$ // true position of the ant in frame S

$\mathbf{v}(t) = d\mathbf{r}/dt = r\,d\hat{\mathbf{r}}/dt + \dot{r}\,\hat{\mathbf{r}} = r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$\mathbf{a}(t) = d\mathbf{v}/dt = (\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}[-\dot{\theta}\hat{\mathbf{r}}]) + (\ddot{r}\hat{\mathbf{r}} + \dot{r}[\dot{\theta}\hat{\boldsymbol{\theta}}]) = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}$

$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \mathbf{v} = m\,r\hat{\mathbf{r}} \times [r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}] = mr^2\dot{\theta}\,\hat{\mathbf{z}}$

$d\mathbf{L}/dt = mr^2\ddot{\theta}\hat{\mathbf{z}} + 2m\,r\,\dot{r}\,\dot{\theta}\,\hat{\mathbf{z}} = mr\,(r\ddot{\theta} + 2\dot{r}\,\dot{\theta})\hat{\mathbf{z}}$.

(f) Our general equation (**) applied to the position vector $\mathbf{r}$ is this

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + d\mathbf{r}_{\mathbf{body}}/dt$

We can compute $d\mathbf{r}_{body}/dt$ from the above rule:

$$d\mathbf{r}_{body}/dt = d\mathbf{r}/dt - \boldsymbol{\omega} \times \mathbf{r}$$

$$= (r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}) - \boldsymbol{\omega} \times (r\hat{\mathbf{r}}) = (r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}) - \omega r\hat{\boldsymbol{\theta}} = r(\dot{\theta} - \dot{\varphi})\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$$

$$= r\dot{\theta}'\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}} \quad = r'\dot{\theta}'\,\hat{\boldsymbol{\theta}}' + \dot{r}'\,\hat{\mathbf{r}}'$$

In frame S', the ant has tangential speed $r'\dot{\theta}'$ and radial speed $\dot{r}'$, so the above expression for quantity $d\mathbf{r}_{body}/dt$ seems quite reasonable.

Note added: Let's try to compute $d\mathbf{r}_{body}/dt$ *a priori*. From section (b) above we can say

$$d\mathbf{r}'_{body}(t)/dt = d_t(r'(t)\,\hat{\mathbf{r}}'(t)) = \dot{r}'\,\hat{\mathbf{r}}' + r'\,d_t\hat{\mathbf{r}}'(t) = \dot{r}'\,\hat{\mathbf{r}}' + r'\dot{\theta}'\,\hat{\boldsymbol{\theta}}'$$

Now $d_t\hat{\mathbf{r}}'(t) = \dot{\theta}'\,\hat{\boldsymbol{\theta}}'$ giving the rightmost result above. Now how do we apply R(φ) to the above equation?

(g) Our general equation (**) applied to the velocity vector $\mathbf{v}$ is this

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + d\mathbf{v}_{body}/dt$$

As usual, we first compute $d\mathbf{v}_{body}/dt$ indirectly from the above, then interpret it.

$$d\mathbf{v}_{body}/dt = d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v}$$

The second term is

$$-\boldsymbol{\omega} \times \mathbf{v} = -\boldsymbol{\omega} \times (r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}) = -\omega r\dot{\theta}\,[-\hat{\mathbf{r}}] - \omega\dot{r}\,\hat{\boldsymbol{\theta}} = \omega r\dot{\theta}\,\hat{\mathbf{r}} - \omega\dot{r}\,\hat{\boldsymbol{\theta}}$$

Then inserting $d\mathbf{v}/dt$ as well, we find

$$d\mathbf{v}_{body}/dt = d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v}$$

$$= [(\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}] + (\omega r\dot{\theta}\,\hat{\mathbf{r}} - \omega\dot{r}\,\hat{\boldsymbol{\theta}})$$

$$= (\ddot{r} - r\dot{\theta}^2 + \omega r\dot{\theta})\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - \omega\dot{r})\hat{\boldsymbol{\theta}}$$

$$= (\ddot{r} - r[\dot{\theta}' + \omega]^2 + \omega r[\dot{\theta}' + \omega])\hat{\mathbf{r}} + (2\dot{r}[\dot{\theta}' + \omega] + r[\ddot{\theta}' + \dot{\omega}] - \omega\dot{r})\hat{\boldsymbol{\theta}}$$

$$= (\ddot{r} - r\dot{\theta}'^2 - r\omega^2 - 2r\omega\dot{\theta}' + \omega r\dot{\theta}' + r\omega^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta}' + 2\dot{r}\omega + r\ddot{\theta}' + r\dot{\omega} - \omega\dot{r})\hat{\boldsymbol{\theta}}$$

$= (\ddot{r} - r\dot{\theta}'^2 - r\omega\dot{\theta}')\hat{\mathbf{r}} + (r\ddot{\theta}' + 2\dot{r}\dot{\theta}' + \dot{r}\omega + r\dot{\omega})\hat{\boldsymbol{\theta}}$

$= (\ddot{r} - r\dot{\theta}'^2 - r\dot{\phi}\dot{\theta}')\hat{\mathbf{r}} + (r\ddot{\theta}' + 2\dot{r}\dot{\theta}' + \dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\theta}}$

$= (\ddot{r} - r\dot{\theta}'[\dot{\theta}' + \dot{\phi}])\hat{\mathbf{r}} + (r[\ddot{\theta}' + \ddot{\phi}] + 2\dot{r}\dot{\theta}' + \dot{r}\dot{\phi})\hat{\boldsymbol{\theta}}$

$= (\ddot{r} - r\dot{\theta}'\dot{\theta})\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}' + \dot{r}\dot{\phi})\hat{\boldsymbol{\theta}}$

I just don't know how to interpret all these terms. Some I can, but there are fancy cross terms and defy interpretation, you would have to do a lot of work!

Suppose $\theta' = 0$ so ant just moves radially. Then $\dot{\theta} = \omega$ the above becomes

$= (\ddot{r})\hat{\mathbf{r}} + (r\dot{\omega} + \dot{r}\omega)\hat{\boldsymbol{\theta}} \quad = (\ddot{r})\hat{\mathbf{r}} + (d(r\omega)/dt)\hat{\boldsymbol{\theta}} \quad = (\ddot{r})\hat{\mathbf{r}} + (\dot{v}_{\phi})\hat{\boldsymbol{\theta}}$

and this $d\mathbf{v}_{\mathbf{body}}/dt$ seems pretty reasonable. And finally, if the ant stays put, this becomes

$= r\dot{\omega}\hat{\boldsymbol{\theta}}$

which at least replicates our Example 1 result.

(h) Our general equation (**) applied to the angular momentum vector **L** is this

$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + d\mathbf{L}_{\mathbf{body}}/dt$.

Once again we indirectly compute $d\mathbf{L}_{\mathbf{body}}/dt$, then attempt to interpret that result

$d\mathbf{L}_{\mathbf{body}}/dt = d\mathbf{L}/dt - \boldsymbol{\omega} \times \mathbf{L}$

$= mr(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{z}} - m\,\boldsymbol{\omega} \times [mr^2\dot{\theta}\,\hat{\mathbf{z}}] = mr(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{z}}$

so we end up with a result that is not very mysterious

$d\mathbf{L}_{\mathbf{body}}/dt = d\mathbf{L}/dt = d(mr^2\dot{\theta}\,\hat{\mathbf{z}})/dt$

The ant's angular momentum magnitude is $L = rp = rmv = rm(r\dot{\theta}) = mr^2\dot{\theta}$, so the derivative then gives $dL/dt = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$.