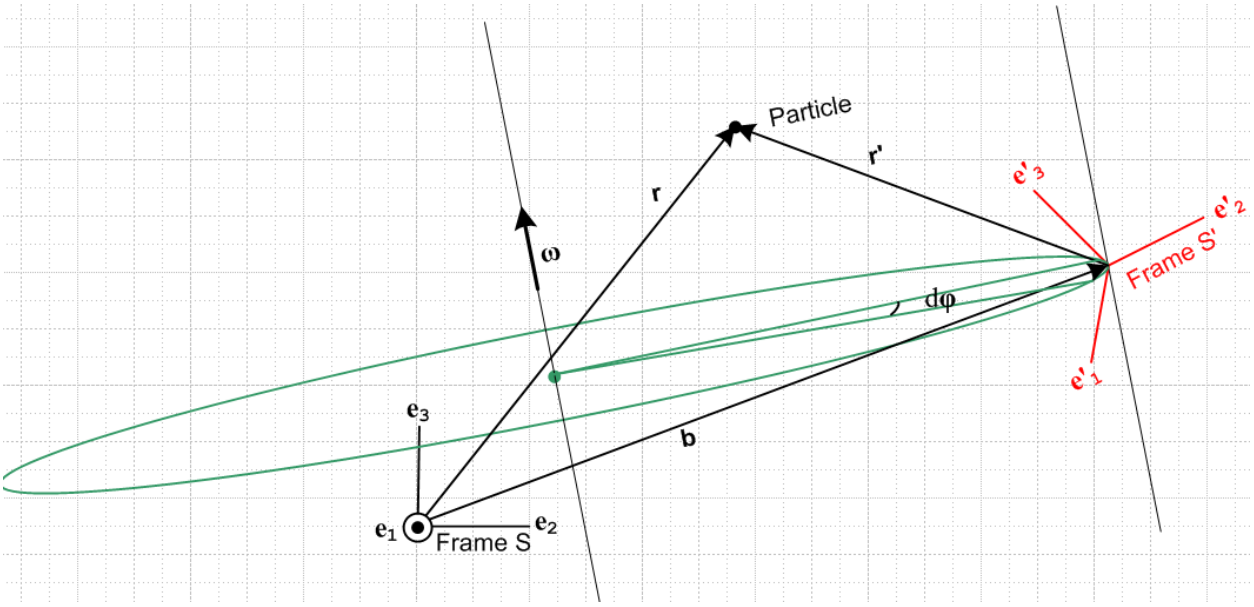
Making the relative motion of the frames more general? PhL 7.20.12

[Here I generalize the location of the **ω** axis and then define my two special cases, all as now appear in frames doc.

[Also, I state the system of ODE's problem of how to calculate R(θ) given **ω**(t) which I don't think has an analytic solution even though only first order. You need to solve a system of 3 first order ODE's with variable coefficients.]

Consider this drawing showing Frame S and Frame S' :



Frame S is aligned with the paper, with unit vector $\mathbf{e}_3$ coming directly out of the plane of paper.

The origin of frame S' lies on the end of a vector **b** whose tail lies on the origin of frame S. This vector **b** may be varying in time.

Frame S' is instantaneously rotating about some axis defined by vector $\boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$.

We make the following computation to learn how the basis vectors $\mathbf{e}'_n$ are rotating relative to the axes of frame S (see Appendix A for the notation used here),

$\mathbf{e}'_n(t+dt) = R(d\boldsymbol{\varphi})\ \mathbf{e}'_n(t)$ where $R(d\boldsymbol{\varphi})$ = some small rotation about the **ω** vector axis

$d\mathbf{e}'_n = \mathbf{e}'_n(t+dt) - \mathbf{e}'_n(t) = [R(d\boldsymbol{\varphi}) - 1]\ \mathbf{e}'_n(t) \approx -id\boldsymbol{\varphi} \bullet \mathbf{J}\ \mathbf{e}'_n(t) = -id\varphi_k J_k\ \mathbf{e}'_n(t)$

$\Rightarrow (d\mathbf{e}'_n)_i = -id\varphi_k (J_k)_{ij}\ (\mathbf{e}'_n(t))_j = -d\varphi_k \varepsilon_{kij}\ (\mathbf{e}'_n(t))_j = \varepsilon_{ikj}\ d\varphi_k\ (\mathbf{e}'_n(t))_j$

$\Rightarrow \; d\mathbf{e'}_n = d\boldsymbol{\varphi} \times \mathbf{e'}_n$

$\Rightarrow \; (d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega}(t) \times \mathbf{e'}_n \qquad n = 1,2,3 \qquad \text{where} \qquad \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt \qquad$ [usual result]

Statement of the problem of finding θ(t) given ω(t)

This last equation is in fact a system of nine first-order, homogeneous, ordinary, linear differential equations in variable t in which there are nine unknown functions $(e'_n)_i$, where n = 1,2,3 and i = 1,2,3,

$$[(d\mathbf{e'}_n/dt)_S]_i - \varepsilon_{ijk}\omega_j(t)\,(e'_n)_k = 0 \qquad n = 1,2,3 \qquad i = 1,2,3$$

In principle, given **ω**(t) and given some initial conditions, one could solve this to determine $\mathbf{e'}_n(t)$ at any time t. This result could then be written in the form

$$\mathbf{e'}_n(t) = R[\boldsymbol{\theta}(t)]\,\mathbf{e}_n$$

since at any time t the frame S' axes are just some rotation applied to those of frame S. That is to say, by solving the problem as just described, we would then know the parameter vector **θ**(t) and it would then be a function of **ω**(t) and the initial (boundary) conditions. Here is the problem in more conventional form

$$f'^{(n)}{}_i(t) + \Sigma_k A_{ik}(t)\, f^{(n)}{}_k(t) = 0 \qquad A_{ik}(t) = \Sigma_j \varepsilon_{ijk}\omega_j(t) = \text{antisym}$$

$$\frac{df_{ni}(t)}{dt} + \Sigma_k A_{ik}(t)\, f_{nk}(t) = 0 \qquad n,i = 1..3$$

For some general **ω**(t), this is a *non-trivial problem*! Well, for each fixed n, it is a system of three equations. If we suppress the n, it becomes

$$\frac{df_i(t)}{dt} + \Sigma_k A_{ik}(t)\, f_k(t) = 0 \qquad n,i = 1..3$$

I think if you write this all out and play with it, it is solvable.

$$\dot{u}_1 + A_{12}u_2 + A_{13}u_3 = 0 \qquad A_{12} = \varepsilon_{1j2}\omega_j = \varepsilon_{132}\omega_3 = -\omega_3$$
$$\dot{u}_2 + A_{21}u_1 + A_{23}u_3 = 0 \qquad A_{13} = \varepsilon_{1j3}\omega_j = \varepsilon_{123}\omega_2 = \omega_2$$
$$\dot{u}_3 + A_{31}u_1 + A_{32}u_2 = 0 \qquad A_{23} = \varepsilon_{2j3}\omega_j = \varepsilon_{213}\omega_1 = -\omega_1$$

$$\dot{u}_1 - \omega_3 u_2 + \omega_2 u_3 = 0$$
$$\dot{u}_2 + \omega_3 u_1 - \omega_1 u_3 = 0$$
$$\dot{u}_3 - \omega_2 u_1 + \omega_1 u_2 = 0 \qquad \dot{\mathbf{u}} - \boldsymbol{\omega} \times \mathbf{u} = 0$$

$$\omega_1\dot{u}_1 - \omega_1\omega_3 u_2 + \omega_1\omega_2 u_3 = 0$$
$$\omega_2\dot{u}_2 + \omega_2\omega_3 u_1 - \omega_2\omega_1 u_3 = 0$$

Add to get u_3 eliminated.

$$\omega_1\dot{u}_1 + \omega_2\dot{u}_2 + \omega_2\omega_3 u_1 - \omega_1\omega_3 u_2 = 0$$
$$\omega_1\dot{u}_1 + \omega_2\dot{u}_2 + \omega_3[\omega_2 u_1 - \omega_1 u_2] = 0$$

Maybe define

$$v = \omega_2 u_1 - \omega_1 u_2$$
$$\dot{v} = \dot{\omega}_2 u_1 - \dot{\omega}_1 u_2 + \omega_2\dot{u}_1 - \omega_1\dot{u}_2$$

Well, I don't see how to get things separated here. Since A_{ij} is not symmetric, the equation is probably not diagonalizable. We are off-topic, so let this be "till a rainy day". Web scan suggests that there is no analytic solution.

Special cases of ω positioning

The picture supports many different scenarios, here are two: [my special cases]

In one scenario, the **ω** axis always passes through the origin of frame S' so frame S' just tumbles about its origin in some manner determined by **ω**(t) while the frame S' origin simultaneously translates relative to frame S according to some **b**(t) . In this scenario, one might imagine that Frame S' is a self-propelled camera platform that is observing some activity in frame S.

In another scenario, suppose the **ω** axis passes through the origin of frame S, is aligned with $\mathbf{e}_3$, and **ω** is a constant. If the origin of frame S were put at the center of the earth and if **b** were a constant-length vector to the surface of the earth, and if perhaps e'_1 pointed "east" when e'_2 pointed "north" and e'_3 pointed "up", the scenario would be suitable for studying Coriolis forces for physics on the surface of the earth.