New Paradox Aug 3 (and its resolution) PhL 8.3.12

At first I thought $v_r = (v')_r$ was impossible, but then I found that it really works. I put this into frames doc only as an exercise for the reader. This is a keeper doc!

Preliminary 1
A. On the One Hand (for our Problem 1) : 1
B. On the Other Hand (for our Problem 1) : 2
C. Statement of Paradox 2
D. Paradox in special case V=0 ? (No paradox.) 3
E. Paradox in the general case ? (No paradox) 4
F. Comments on this seeming Paradox 4

Preliminary

We have the following velocity relation which has been verified by Marion and which seems to give good results in all my ant problems,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \qquad (6.6c)$$

If we dot both sides into $\hat{\mathbf{r}}$, the second term gives 0, and we get

$$v_r = (v')_r \qquad // \ (v_S)_r = (v'_{S'})_r \qquad (1)$$

where

$$v_r \equiv \mathbf{v} \bullet \hat{\mathbf{r}} \qquad (v')_r \equiv \mathbf{v}' \bullet \hat{\mathbf{r}}$$

We are certainly allowed to dot a vector equation into any other vector we want.

A. On the One Hand (for our Problem 1) :

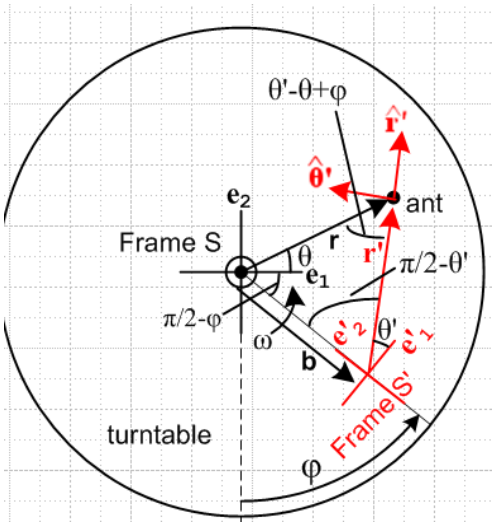
In Problem 1, the ant has this velocity in Frame S' :

$$\mathbf{v}' = -V\hat{\mathbf{r}}' \qquad (15.10)$$

We can compute

$$(v')_r = \hat{\mathbf{r}} \bullet \mathbf{v}' = -V \ \hat{\mathbf{r}} \bullet \hat{\mathbf{r}}' = -V \cos(\theta'-\theta+\varphi) \qquad // \ \theta' \equiv \theta'_0 \text{ at all times in Problem 1}$$

based on this picture



$\sin(\pi/2-\varphi+\theta)/r' = \sin(\theta'-\theta+\varphi)/b$

If we assume V = 0, we get this simple result, valid for any value of b or t :

$(v')_r = 0.$

and the general result is

$(v')_r = -V\cos(\theta'-\theta+\varphi)$

B. On the Other Hand (for our Problem 1) :

We *think* we know from Problem 1 work that

$$(v)_r = \hat{\mathbf{r}} \bullet \mathbf{v} = v_x \hat{\mathbf{r}} \bullet \hat{\mathbf{x}} + v_y \hat{\mathbf{r}} \bullet \hat{\mathbf{y}} = v_x \cos\theta + v_y \sin\theta$$

where

$$v_x = -V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)$$
$$v_y = -V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi) \qquad (15.21)$$

Therefore

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$= [-V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)]\cos\theta$$
$$+ [-V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi)]\sin\theta$$

C. Statement of Paradox

Here is what we showed in Part A and Part B:

$$(v')_r = -V\cos(\theta'-\theta+\varphi) \qquad \text{// Part A}$$

$$v_r = [-V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)]\cos\theta + [-V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi)]\sin\theta \qquad \text{// Part B}$$

The Paradox is that (it seems) these two expressions cannot possibly be equal. First of all, the Part B expression has terms linear in t while the Part A does not. Also the Part B has dependence on ω and on b, whereas Part A does not. So there is your Paradox: two things that should be equal are not equal.

D. Paradox in special case V=0 ? (No paradox.)

The Preliminary equation says that

$$v_r = (v')_r$$

If V = 0, we certainly have

$$(v')_r = -V\cos(\theta'-\theta+\varphi) = 0 \qquad \text{// Part A result}$$

For Part B in this V = 0 case we have

$$v_r = [+\omega b\cos\varphi - \omega r'_0\sin(\theta'+\varphi)]\cos\theta + [\omega b\sin\varphi + \omega r'_0\cos(\theta'+\varphi)]\sin\theta$$

$$= \omega b\{\cos\varphi\cos\theta + \sin\varphi\sin\theta\} - \omega r'_0\{\sin(\theta'+\varphi)\cos\theta - \cos(\theta'+\varphi)\sin\theta\}$$

$$= \omega b\cos(\varphi-\theta) - \omega r'_0\sin(\theta'+\varphi-\theta)$$

$$= \omega b\cos(\varphi-\theta) - \omega r'\sin(\theta'+\varphi-\theta) \qquad \text{// since } r' = r'_0 \text{ when } V = 0$$

Then writing out our preliminary $v_r = (v')_r$ we get

$$\omega b\cos(\varphi-\theta) - \omega r'\sin(\theta'+\varphi-\theta) = 0 \qquad \text{// Part B = Part A} \qquad (*)$$

or

$$b\cos(\varphi-\theta)\} - r'\sin(\theta'+\varphi-\theta) = 0$$

Offhand this seems not to be true, but there *is* a tricky relationship between the angles. Rewrite

$$\cos(\varphi-\theta) = \sin(\pi/2 - \varphi + \theta)$$

Then our Paradox (in this V = 0 limit) goes away if we have

$$b\sin(\pi/2 - \varphi + \theta) = r'\sin(\theta'+\varphi-\theta)$$

or

$$\sin(\pi/2 - \varphi + \theta)/ r' = \sin(\theta'+\varphi-\theta)/b$$

Looking at the picture above, we see that this is a law of sines for our triangle. I was hoping to show the paradox in this limit just to make it easier to study, but it seems there is no paradox in this limit! So we have to go do the general case.

E. Paradox in the general case ? (No paradox)

Now look in the general case. We have from Part A

$$(v')_r = -V \cos(\theta'-\theta+\varphi)$$

and we have from Part B,

$$\begin{aligned} v_r &= [-V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)]\cos\theta \\ &\quad + [-V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi)]\sin\theta \\ &= \omega b \{\cos\varphi\cos\theta + \sin\varphi\sin\theta\} - \omega(r'_0 - Vt)\{\sin(\theta'+\varphi)\cos\theta - \cos(\theta'+\varphi)\sin\theta\} \\ &\quad - V\{\cos(\theta'+\varphi)\cos\theta + \sin(\theta'+\varphi)\sin\theta\} \\ &= \omega b\cos(\varphi-\theta) - \omega(r'_0 - Vt)\sin(\theta'+\varphi-\theta) - V\cos(\theta'+\varphi-\theta) \\ &= \omega b\cos(\varphi-\theta) - \omega r'_0\sin(\theta'+\varphi-\theta) + \omega Vt\sin(\theta'+\varphi-\theta) - V\cos(\theta'+\varphi-\theta) \end{aligned}$$

Now write another fact we know for our Problem 1,

$$r' = r'_0 - Vt \qquad \Rightarrow \qquad r'_0 = r' + Vt$$

Use this in the second term to get

$$\begin{aligned} &= \omega b\cos(\varphi-\theta) - \omega r'\sin(\theta'+\varphi-\theta) - \omega Vt\sin(\theta'+\varphi-\theta) + \omega Vt\sin(\theta'+\varphi-\theta) - V\cos(\theta'+\varphi-\theta) \\ &= \omega b\cos(\varphi-\theta) - \omega r'\sin(\theta'+\varphi-\theta) - V\cos(\theta'+\varphi-\theta) \end{aligned}$$

Now the two ωVt terms cancel, AND the first two terms cancel by that law of sines quoted in the last section as (*) and we are left with only one term

$$v_r = -V\cos(\theta'+\varphi-\theta)$$

And so we have $(v')_r = v_r$ and there IS no Paradox even in the general case.

F. Comments on this seeming Paradox

(1) It seemed that the terms linear in t could not cancel but they did. And the dependence of the Part B side of the equation on both b and ω went completely away! All very non-obvious.

(2) This "exercise" consumed a lot of time, but it did testing of many things and built my confidence in those things. I list those things here:

- The velocity equation

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \qquad (6.6c)$$

is indeed valid, even though this particular form of the equation in special case #1 is not verified in Marion. I verified it today for this case.

- You are allowed to dot this into $\hat{\mathbf{r}}$ and reach this valid conclusion

$$v_{\mathbf{r}} = (v')_{\mathbf{r}} \qquad // \; (v_{\mathbf{S}})_{\mathbf{r}} = (v'_{\mathbf{S}'})_{\mathbf{r}} \qquad (1)$$

- Using the velocity and position results obtained in Problem 1 verified this equation after a lot of pain. This serves to add credibility to those Problem 1 equations and to the Problem 1 solution in general.

- There is no subtle difference set of curvilinear frames you have to worry about.

- I think the velocity equation is valid even if you use curvilinear unit vectors to view it.

- I think the G Rule is valid if you express things using curvilinear unit vectors.