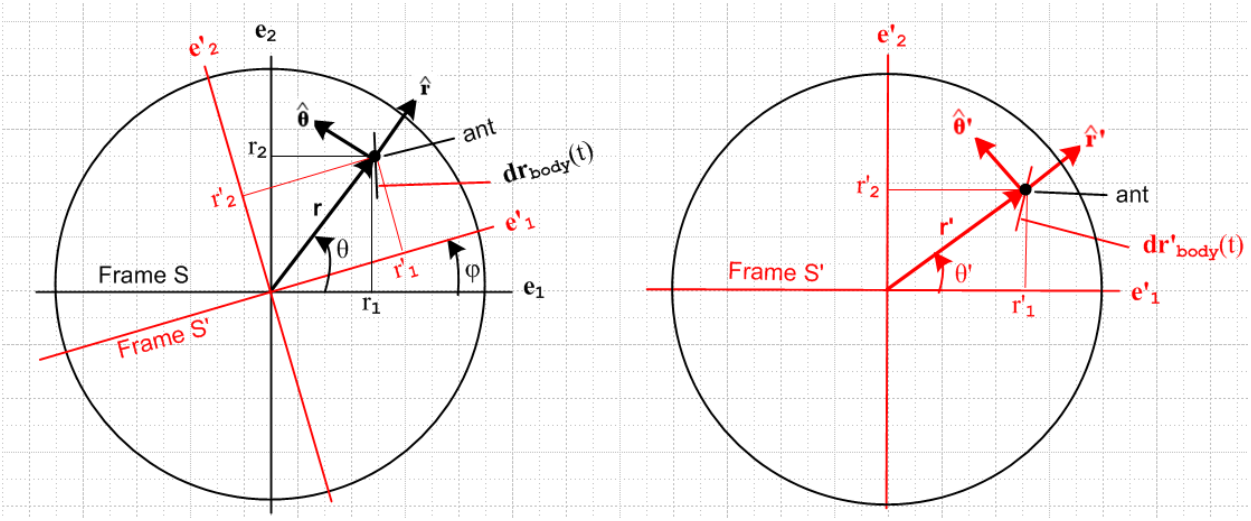
Ant on Phono Record PhL 7.12.12

Here I work *at once* in polar coordinates r,θ in Frame S and r',θ' in Frame S. I don't have a general framework at this time. I am still in the older context where the G Rule involves dr_{body}. I seem to get some separated equations $\ddot{r} = \ddot{r}'$ and $\ddot{\theta} - \omega = \ddot{\theta}'$ which are each ODE's. I do simple cases and make a few spiral plots. I then try to find r,v,a equations in a more general sense, but it all ends badly. This is now all systematized in frames doc so I will just regard this as an early attempt at doing ant on record problems.

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1. Kinematics and variables.

Here is the picture showing the situation in frames S and S' : ($\hat{z} = \hat{e}_3$ out of plane of paper) (the picture on the left is valid for a specific time t)



The convention here is for symbol R is as follows:

$\mathbf{e'}_n = R\mathbf{e}_n$ where $R = R_z(\varphi)$ expansions: $\mathbf{V} = V_n\mathbf{e}_n = V'_n\mathbf{e'}_n$

$\mathbf{V} = R\mathbf{V'}$.

With components define relative to the axes of paper, we can see that the second line above can be applied to four vectors in the pictures:

$\mathbf{r} = R\mathbf{r}'$
$\hat{\mathbf{r}} = R\hat{\mathbf{r}}'$
$\hat{\boldsymbol{\theta}} = R\hat{\boldsymbol{\theta}}'$
$d\mathbf{r}_{body} = R\, d\mathbf{r}'_{body}(t)$

We agree from discussion elsewhere that, although we shall think of $\mathbf{r}'_{body}(t)$ as a full path of some sort of the ant in frame S', we shall never give consideration to any object named $\mathbf{r}_{body}(t)$. However, we shall give consideration to objects $d\mathbf{r}'_{body}(t)$ and $d\mathbf{r}_{body}(t)$.

2. Equation of Motion using the Goldstein r equation

The main equations of interest are these.

$d\mathbf{r} = \mathbf{d\varphi} \times \mathbf{r} + d\mathbf{r}_{body}$
$d\mathbf{r}_{body} = R\, d\mathbf{r}'_{body}$

The first is pretty obvious in that you just add the two displacements to get the total displacement. The first displacement $\mathbf{d\varphi} \times \mathbf{r}$ is in the $\hat{\boldsymbol{\theta}}$ direction as right hand rule shows and is of course caused by the rotational motion of S' relative to S. The second displacement is $d\mathbf{r}'_{body}$ which has been mapped into Frame S as shown above. A key idea to keep in mind is that $d\mathbf{r}_{body}$ appears in the equation, and not $d\mathbf{r}'_{body}(t)$.

The second equation was already stated above.

Upon division by dt, may be written

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + d\mathbf{r}_{body}/dt$ where $\boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$
$d\mathbf{r}_{body}/dt = R\, d\mathbf{r}'_{body}/dt$

which we can also write as

$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}$ where $\boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$
$\mathbf{v}_{body} = R\, \mathbf{v}'_{body}$

The first equation here shows clearly why we had to use the "body" subscript on $\mathbf{v}_{body}$ etc.

We may regard our first equation as a first order PDE in variable $\mathbf{r}(t)$

$d\mathbf{r}/dt - \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}_{body}$

where $\boldsymbol{\omega}$, R, and $\mathbf{v}'_{body}$ are all functions of time. In any problem we shall consider, we shall be given the following facts: $\boldsymbol{\omega}(t)$, R(t), and $\mathbf{r}'_{body}(t)$, and some reasonable initial conditions.

Notice that, in official parlance, and in Cartesian coordinates, the above equation is in fact a system of three coupled, first-order, partial differential equations for functions x,y,z with variable time t:

$dx_i/dt - \varepsilon_{ijk}\omega_j(t)x_k = [\mathbf{v}_{body}(t)]_i$

The equations are inhomogeneous in that there is a driving term on the right side. As shown below, for our special case this equation, in polar coordinates, separates into a pair of decoupled first-order ODEs.

3. Separating and then Solving the Equation of Motion.

We work first on the RHS driving term in the above equation:

$$\mathbf{v'}_{\mathbf{body}} = d\mathbf{r'}_{\mathbf{body}}/dt$$

Let's assume that someone has specified $\mathbf{r'}_{\mathbf{body}}(t)$, and we express it in this manner in terms of our polar (or cylindrical) unit vectors,

$$\mathbf{r'}_{\mathbf{body}}(t) = \mathbf{r'}(t) = r'(t)\,\hat{\mathbf{r}}'(t) \ .$$

In Frame S' we know that

$$d\hat{\mathbf{r}}'/dt = \dot{\theta}'\,\hat{\boldsymbol{\theta}}' \qquad\qquad d\hat{\boldsymbol{\theta}}'/dt = -\dot{\theta}'\,\hat{\mathbf{r}}'$$

Therefore, we compute

$$\mathbf{v'}_{\mathbf{body}} = d\mathbf{r'}_{\mathbf{body}}/dt = d_t[r'(t)\,\hat{\mathbf{r}}'(t)] = \dot{r}'\ \hat{\mathbf{r}}' + r'\ \dot{\theta}'\,\hat{\boldsymbol{\theta}}'$$

We then find that

$$\mathbf{v}_{\mathbf{body}} = R\,\mathbf{v'}_{\mathbf{body}} = R\,[\dot{r}'\ \hat{\mathbf{r}}' + r'\,\dot{\theta}'\,\hat{\boldsymbol{\theta}}'] = \dot{r}'\,R\,\hat{\mathbf{r}}' + r'\,\dot{\theta}'\,R\hat{\boldsymbol{\theta}}' = \dot{r}'\,\hat{\mathbf{r}} + r'\,\dot{\theta}'\,\hat{\boldsymbol{\theta}}$$

Looking at the left picture above, and noting the r' = r, this result seems very reasonable.

Meanwhile, we have of course that

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \boldsymbol{\omega} = \omega\hat{\mathbf{z}} \qquad \Rightarrow \qquad \boldsymbol{\omega} \times \mathbf{r} = [\omega\hat{\mathbf{z}}] \times [r\,\hat{\mathbf{r}}] = \omega\, r\,\hat{\boldsymbol{\theta}}$$

Now, analogous to the above, we have in Frame S,

$$d\hat{\mathbf{r}}/dt = \dot{\theta}\,\hat{\boldsymbol{\theta}} \qquad\qquad d\hat{\boldsymbol{\theta}}/dt = -\dot{\theta}\,\hat{\mathbf{r}}$$

so

$$d\mathbf{r}/dt = d_t(r\,\hat{\mathbf{r}}) = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

which has the exact same form as (**) above. We now insert these pieces into our equation of motion,

$$d\mathbf{r}/dt - \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v}_{\mathbf{body}}$$

$$\dot{r}\,\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} - \omega\, r\,\hat{\boldsymbol{\theta}} = \dot{r}'\,\hat{\mathbf{r}} + r'\,\dot{\theta}'\,\hat{\boldsymbol{\theta}}$$

We can isolate from this two separate equations:

$$\frac{\dot{r}}{c} = \frac{\dot{r}'}{c}$$

But for our ant, r = r', so these equations simplify to

$$\frac{\dot{r}}{c} = \frac{\dot{r}'}{c}$$

$$\dot{\theta} - \omega = \dot{\theta}'$$

These equations are fully "separated" and may be solved by single quadrature:

$$r(t) = r(0) + \int_0^t \dot{r}'(\tau)\, d\tau$$

$$\theta(t) = \theta(0) + \int_0^t [\dot{\theta}'(\tau) + \omega(\tau)]\, d\tau$$

But of course we can do the integrals of the dotted objects to get

$$r(t) = r(0) + [\, r'(t) - r'(0)\,]$$

$$\theta(t) = \theta(0) + [\, \theta'(t) - \theta'(0)] + \int_0^t \omega(\tau)\, d\tau$$

But since r = r' in general, the first equation is a trivial identity. The second says

$$\theta(t) = \varphi_o + \theta'(t) + \int_0^t \omega(\tau)\, d\tau$$

So here are our general solutions

$$r(t) = r'(t)$$

$$\theta(t) = \varphi_o + \theta'(t) + \int_0^t \omega(\tau)\, d\tau$$

4. Special Cases

Here we do a few "sanity checks" on the above result.

Case 1: Ant is glued to the phono record so that $\dot{r}'(\tau) = 0$ and $\dot{\theta}'(t) = 0$. The solution is then

$r(t) = r'(t)$

$\theta(t) = \varphi_o + \theta' + \int_0^t \omega(\tau)\ d\tau = \theta(0) + \int_0^t \omega(\tau)\ d\tau$

In the even more specialized case the ω = constant, this says

$r(t) = r'(t)$

$\theta(t) = \varphi_o + \theta' + \omega t = \theta(0) + \omega t$

This solution is in exact agreement (including signs) with the above kinematics picture.

Case 2: Ant is only allowed to move radially in frame S' . Then of course $\theta'(t) = \theta'_0$ and

$r(t) = r'(t)$

$\theta(t) = \varphi_o + \theta'_0 + \int_0^t \omega(\tau)\ d\tau = \theta(0) + \int_0^t \omega(\tau)\ d\tau$

The angular equation is the same as Case 1. while the radial equation does the obvious.

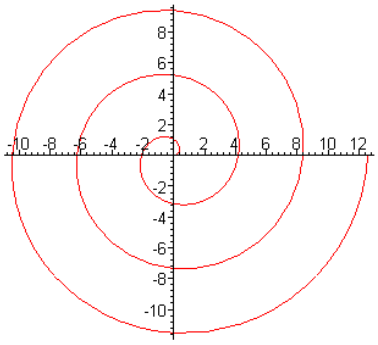
So the ant moves out radially say if $\dot{r}' > 0$, and picks up the CCW motion of the record. If would be fun to plot this case in Maple.

$x(t) = r(t)\cos(\theta(t)) = r'(t)\cos(\theta(0) + \omega t)$

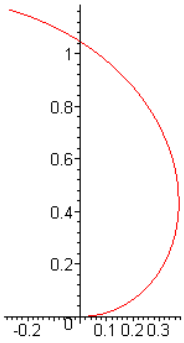
$y(t) = r(t)\sin(\theta(t)) = r'(t)\sin(\theta(0) + \omega t)$

Here it is, where ant starts at $r_0 = 0$ and $\theta_0 = 0$ and has $\dot{r}' = 2$ and $\omega = 3$

```
restart;
r := 2*t:
theta := 3*t:
x := r*cos(theta):
y := r*sin(theta):
plot([x,y,t=0..2*Pi], scaling = CONSTRAINED );
```




```
plot([x,y,t=0..0.6], scaling = CONSTRAINED );
```



The plot on the right is a blowup of that of the left near the origin, and you can see $r_0 = 0$ and $\theta_0 = 0$ a bit more clearly.

Case 3: Ant is only allowed to move tangentially in frame S' :

$$r(t) = r'(t) = r_0$$
$$\theta(t) = \varphi_o + \theta'(t) + \int_0^t \omega(\tau)\, d\tau$$

If both ω and $\dot{\theta}'$ are constants, we get

$$r(t) = r_0$$
$$\theta(t) = \varphi_o + [\theta'(0) + \dot{\theta}'\, t] + \omega t = \theta_0 + (\dot{\theta}' + \omega)t$$

where the angular velocities just add together.

5. Solving the same problem in a trivial fashion.

The position of the ant in frame S is given by

$$\mathbf{r(t)} = R(t)\mathbf{r'} = R(t)\, \mathbf{r'}_{\mathbf{body}}(t)$$

which we can write as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$x(t) = \cos\varphi(t)\, x'(t) - \sin\varphi(t)\, y'(t)$$
$$y(t) = \sin\varphi(t)\, x'(t) + \cos\varphi(t)\, y'(t)$$

and this certainly is a full solution to the problem where

$$\varphi(t) = \varphi(0) + \int_0^t \omega(\tau) d\tau \ .$$

How would we convert the above solution to polar coordinates? We know from our picture (or from the above equations) that

$$r(t) = r'(t)$$
$$\theta(t) = \theta'(t) + \varphi(t)$$

Thus, our equations are

$$r(t) = r'(t)$$
$$\theta(t) = \varphi(0) + \theta'(t) + \int_0^t \omega(\tau) d\tau$$

and these are the same equations obtained in the previous section using the Goldstein **r** equation.

6. Finding velocity v, acceleration a, and angular momentum L.

Above (using two methods) we found the solution $\mathbf{r}(t)$ in terms of r(t) and θ(t).

$r(t) = r'(t)$

$\theta(t) = \varphi_o + \theta'(t) + \int_0^t \omega(\tau)\, d\tau$.

We can certainly write

$\mathbf{r}(t) = r(t)\,\hat{\mathbf{r}}(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)) = r\hat{\mathbf{r}} = (r\cos\theta, r\sin\theta)$.

Then the following lines follow from what we have done above, including the facts that

$d\hat{\boldsymbol{\theta}}/dt = -\dot{\theta}\,\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt = \dot{\theta}\,\hat{\boldsymbol{\theta}}$.

We get this sequence:

$\mathbf{r}(t) = r\hat{\mathbf{r}}$ // true position of the ant in frame S

$\mathbf{v}(t) = d\mathbf{r}/dt = r\, d\hat{\mathbf{r}}/dt + \dot{r}\,\hat{\mathbf{r}} = r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$\mathbf{a}(t) = d\mathbf{v}/dt = (\dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}[-\dot{\theta}\hat{\mathbf{r}}]) + (\ddot{r}\hat{\mathbf{r}} + \dot{r}[\dot{\theta}\hat{\boldsymbol{\theta}}])$

$= (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}$

$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\,\mathbf{r} \times \mathbf{v} = m\, r\hat{\mathbf{r}} \times [r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}] = mr^2\dot{\theta}\,\hat{\mathbf{z}}$

$d\mathbf{L}/dt = mr^2\ddot{\theta}\hat{\mathbf{z}} + 2m\, r\dot{r}\,\dot{\theta}\,\hat{\mathbf{z}} = mr\,(r\ddot{\theta} + 2\dot{r}\,\dot{\theta})\hat{\mathbf{z}}$.

Now we can also write

$r(t) = r_r\hat{\mathbf{r}}$ where $r_r = r$

$\mathbf{v}(t) = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$ where $v_r = \dot{r}$ $v_\theta = r\dot{\theta}$

$\mathbf{a}(t) = a_r\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}}$ where $a_r = \ddot{r} - r\dot{\theta}^2$ $a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$\mathbf{L}(t) = L_z\hat{\mathbf{z}}$ where $L_z = mr^2\dot{\theta}$

7. Examine the Goldstein equation in v

That equation says this (I think the second line is correct, just another vector)

$$d\mathbf{v} = d\boldsymbol{\varphi} \times \mathbf{v} + d\mathbf{v}_{body}$$
$$d\mathbf{v}_{body} = R\, d\mathbf{v}'_{body}$$

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + d\mathbf{v}_{body}/dt \qquad \text{where} \qquad \boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$$
$$d\mathbf{v}_{body}/dt = R\, d\mathbf{v}'_{body}/dt$$

which we can also write as

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} + \mathbf{a}_{body} \quad \text{where} \qquad \boldsymbol{\omega} = d\boldsymbol{\varphi}/dts$$
$$\mathbf{a}_{body} = R\, \mathbf{a}'_{body}$$

Again, we are happy to have our body subscript on $\mathbf{a}_{body}$.

We may regard our first equation as a first order PDE in variable $\mathbf{v}(t)$

$$d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v} = \mathbf{a}_{body}$$

where $\boldsymbol{\omega}$, R, and $\mathbf{a}'_{body}$ are all functions of time. In any problem we shall consider, we shall be given the following facts: $\boldsymbol{\omega}(t)$, R(t), and $\mathbf{r}'_{body}(t)$, and some reasonable initial conditions.

We work first on the RHS driving term in the above equation:

$$\mathbf{a}'_{body} = d\mathbf{v}'_{body}/dt = d_t[\dot{r}'\ \hat{\mathbf{r}}' + r'\ \dot{\theta}'\ \hat{\boldsymbol{\theta}}'] = \text{five terms}$$

$$= \ddot{r}'\ \hat{\mathbf{r}}' + \dot{r}'\ \dot{\theta}'\ \hat{\boldsymbol{\theta}}' + \dot{r}'\ \dot{\theta}'\ \hat{\boldsymbol{\theta}}' + r'\ddot{\theta}'\ \hat{\boldsymbol{\theta}}' + r'\ \dot{\theta}'\ [-\dot{\theta}'\ \hat{\mathbf{r}}']$$

$$= (\ddot{r}' - r'\dot{\theta}'^2)\ \hat{\mathbf{r}}' + (r'\ddot{\theta}' + 2\,\dot{r}'\ \dot{\theta}')\ \hat{\boldsymbol{\theta}}' \ .$$

Then the next step is as above to get (primes go away on the unit vectors)

$$\mathbf{a}_{body} = (\ddot{r}' - r'\dot{\theta}'^2)\ \hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\,\dot{r}'\ \dot{\theta}')\ \hat{\boldsymbol{\theta}}$$

Meanwhile, we have of course that

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} \qquad \boldsymbol{\omega} = \omega\hat{\mathbf{z}} \qquad \Rightarrow \qquad \boldsymbol{\omega} \times \mathbf{v} = [\omega\hat{\mathbf{z}}] \times [v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}] = \omega\, v_r\, \hat{\boldsymbol{\theta}} - \omega\, v_\theta\hat{\mathbf{r}}$$

Also, we have

$$d\mathbf{v}/dt = \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\theta}\ \hat{\boldsymbol{\theta}} + \dot{v}_\theta\hat{\boldsymbol{\theta}} + v_\theta[-\dot{\theta}\ \hat{\mathbf{r}}] = (\dot{v}_r - v_\theta\dot{\theta})\hat{\mathbf{r}} + (v_r\dot{\theta} + \dot{v}_\theta)\ \hat{\boldsymbol{\theta}}$$

We now insert these pieces into our equation of motion,

$$d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v} = \mathbf{a}_{body}$$

$$(\dot{v}_r - v_\theta\dot{\theta})\hat{\mathbf{r}} + (v_r\dot{\theta} + \dot{v}_\theta)\,\hat{\boldsymbol{\theta}} - \omega v_r\,\hat{\boldsymbol{\theta}} + \omega v_\theta\hat{\mathbf{r}} = (\ddot{r}' - r'\dot{\theta}'^2)\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}')\,\hat{\boldsymbol{\theta}}$$

$$(\dot{v}_r - v_\theta\dot{\theta} + \omega v_\theta)\hat{\mathbf{r}} + (v_r\dot{\theta} + \dot{v}_\theta - \omega v_r)\,\hat{\boldsymbol{\theta}} = (\ddot{r}' - r'\dot{\theta}'^2)\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}')\,\hat{\boldsymbol{\theta}}$$

We can isolate from this two separate equations:

$$\dot{v}_r - v_\theta\dot{\theta} + \omega v_\theta = \ddot{r}' - r'\dot{\theta}'^2$$

$$v_r\dot{\theta} + \dot{v}_\theta - \omega v_r = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}'$$

or

$$\dot{v}_r + (\omega - \dot{\theta})v_\theta = \ddot{r}' - r'\dot{\theta}'^2$$

$$\dot{v}_\theta - (\omega - \dot{\theta})v_r = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}'$$

And now the two equations are not decoupled! Things are much more complicated than before. No doubt we could solve these equations somehow, but I am not going to try. My main interest is to verify that these equations are true!!

From the previous section we found that

$$\mathbf{v}(t) = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} \quad \text{where} \quad v_r = \dot{r} \quad v_\theta = r\dot{\theta}$$

So we can say

$$v_r = \dot{r}$$

$$\dot{v}_r = \ddot{r}$$

$$v_\theta = r\dot{\theta}$$

$$\dot{v}_\theta = r\ddot{\theta} + \dot{r}\dot{\theta}$$

Then our two coupled equations above become

$$\dot{v}_r + (\omega - \dot{\theta})v_\theta = \ddot{r}' - r'\dot{\theta}'^2$$

$$\dot{v}_\theta - (\omega - \dot{\theta})v_r = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}'$$

or

$$\ddot{r} + (\omega - \dot{\theta})\,r\dot{\theta} = \ddot{r}' - r'\dot{\theta}'^2$$

$$r\ddot{\theta} + \dot{r}\dot{\theta} - (\omega - \dot{\theta})\dot{r} = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}'$$

Let's whittle away on the *first* equation

$\ddot{r} + (\omega - \dot{\theta})\, r\, \dot{\theta} = \ddot{r}' - r'\dot{\theta}'^2$

$\ddot{r} - \dot{\theta}'\, r\, \dot{\theta} = \ddot{r}' - r\dot{\theta}'^2$ // since r = r' and since $\varphi - \theta = -\theta'$ so $\omega - \dot{\theta} = -\dot{\theta}'$

$\dot{\theta}'\, r\, \dot{\theta} = r\dot{\theta}'^2$

$\dot{\theta} = \dot{\theta}'$

$\dot{\theta} - \dot{\theta}' = 0$

$\omega = 0$ // something bad happened somewhere

Let's whittle away on the *second* equation

$r\,\ddot{\theta} + \dot{r}\,\dot{\theta} - (\omega - \dot{\theta})\dot{r} = r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}'$

$r\,\ddot{\theta} + \dot{r}\,\dot{\theta} - (\omega - \dot{\theta})\dot{r} = r\ddot{\theta}' + 2\,\dot{r}\,\dot{\theta}'$

$r\,(\ddot{\theta} - \ddot{\theta}') + \dot{r}\,\dot{\theta} - (\omega - \dot{\theta})\dot{r} = 2\,\dot{r}\,\dot{\theta}'$

$r\,(\ddot{\theta} - \ddot{\theta}') + \dot{r}\,\dot{\theta} - (-\dot{\theta}')\,\dot{r} = 2\,\dot{r}\,\dot{\theta}'$

$r\,(\ddot{\theta} - \ddot{\theta}') + \dot{r}\,\dot{\theta} + \dot{\theta}'\,\dot{r} = 2\,\dot{r}\,\dot{\theta}'$

$r\,(\ddot{\theta} - \ddot{\theta}') + \dot{r}\,\dot{\theta} = \dot{r}\,\dot{\theta}'$

$r\,(\ddot{\theta} - \ddot{\theta}') + \dot{r}\,(\dot{\theta} - \dot{\theta}') = 0$

$r\,\dot{\omega} + \dot{r}\,\omega = 0$ // something is wrong here as well

So both our equations come out wrong! This is certainly telling me something. I think this is exactly the problem I have been puzzling over for the last 2 weeks. Something is wrong in my understanding here because I think the algebra has been done right.