First shot at curvilinear section PhL 7.29.12

[This has all my ugly A B stuff that I might need someday. I am not really happy with my frames doc curvilinear section, so let's be sure to keep this doc around.]

Part 1. I took a wrong direction with the A,B stuff in this Section. I wanted to preserve the primes of tensor doc. But then the simple turntable example with r,θ and r',θ' made it obvious that this was the wrong approach, so all that work was for nothing and sits here.

Part 2. Under ***** below, just earlier fiddlings.

13. Rotating Frames in Curvilinear Coordinates

(a) How Curvilinear Coordinates change things

The solution equations to our Original Problem are summarized in Section 12 above, and those to the Inverse Problem are summarized in Section 13 (d). All development work as done assuming that $\mathbf{e}_n$ and $\mathbf{e}'_n$ were Cartesian coordinates. It does not take long to realize that the development is different when curvilinear coordinates are used. For example, compare these equations in frame S where we compare Cartesian cylindrical coordinates r,θ,z :

$(d\mathbf{e}_1/dt)_S = 0$ // Cartesian; $(d\mathbf{e}_n/dt)_S = 0$ is equation (1.27)
$(d\mathbf{e}_2/dt)_S = 0$
$(d\mathbf{e}_3/dt)_S = 0$

$(d\hat{\mathbf{r}}/dt)_S = \dot{\theta}\,\hat{\boldsymbol{\theta}}$ // Cylindrical; seems that (1.27) is no longer true
$(d\hat{\boldsymbol{\theta}}/dt)_S = -\dot{\theta}\,\hat{\mathbf{r}}$
$(d\hat{\mathbf{z}}/dt)_S = 0$

If a point **r** is moving in frame S, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ move with it, while the $\mathbf{e}_n$ do not!

What we really want to do is accept the Cartesian results and *convert* these results to curvilinear coordinates. To be fully general, we would like the ability to use one set of curvilinear coordinates in Frame S (perhaps spherical) and some totally different set of curvilinear coordinates in Frame S' (perhaps toroidal or cylindrical or Cartesian).

We shall first outline a systematic way to do this in the next section few sections.

(b) The Problem with Primes

We have this issue of "primes" to deal with.

On the one hand, we have frame S with $\mathbf{e}_n$ and frame S' with $\mathbf{e}'_n$ and all the associated unprimed and primed variables like **v** and **v'**. The two frames S and S' are related by a rotation.

On the other hand, in our analysis of curvilinear coordinates (see Tensor Analysis...), we want to use unprimed coordinates x_n and Cartesian unit vectors $\hat{\mathbf{u}}_n$ in what we call x-space (the space of the Cartesian coordinates), and we want to use the coordinates x'_n as curvilinear coordinates in what we call x'-space (the space of curvilinear coordinates). Moreover, we want to use non-unit basis vectors in x-space called $\mathbf{e}_n$ which are the "tangent basis vectors" related to the transformation from Cartesian to curvilinear coordinates. These $\mathbf{e}_n$ have nothing to do with the Cartesian $\mathbf{e}_n$ used in the current document! In fact, the $\mathbf{e}_n$ of the current document are really the $\mathbf{u}_n$ of the tensor analysis document. One usually normalizes the tangent base vectors to be unit vectors in which case they are called $\hat{\mathbf{e}}_n$ and the corresponding components are then put in italics. For example, here is a generic vector **a** expanded in three ways within this curvilinear coordinates scenario:

$$\mathbf{a} = a_i \mathbf{u}_i = a'_i \mathbf{e}_i = a'_i \, \hat{\mathbf{e}}_i$$

When one deals with a specific coordinate system, simpler names are usually used for the a'_i components so primes are not needed. For example, if x-space is 2D Cartesian with coordinates x,y and x'-space is 2D polar with coordinates r,θ, the three expansions are

$$\mathbf{a} = a_x \, \mathbf{u}_x + a_y \, \mathbf{u}_y = a_r \, \mathbf{e}_r + a_\theta \, \mathbf{e}_\theta = a_r \, \hat{\mathbf{e}}_r + a_\theta \, \hat{\mathbf{e}}_\theta = a_r \, \hat{\mathbf{r}} + a_\theta \, \hat{\boldsymbol{\theta}} \qquad // = a_x \, \hat{\mathbf{x}} + a_y \, \hat{\mathbf{y}}$$

The notion of x-space and x'-space is very heavily embedded in our Tensor Analysis document and it obviously conflicts with the use or primes in the current document, all based on frames S and S'. One or the other has to "give". We have decided that it is the current document which will yield.

(c) Problem Equations in Translated Form for Frame A and Frame B

Therefore, we shall make the following systematic translation of the notation of the current document. We give the main results at the start of the list, then just some random samples for the rest:

S	→	A	Frame A
S'	→	B	Frame B
∂_S	→	∂_A	
$\partial_{S'}$	→	∂_B	
$\mathbf{e}_n$	→	$(\mathbf{u}_{An})$	Cartesian basis vectors in S_A
$\mathbf{e}'_n$	→	$(\mathbf{u}_{Bn})$	Cartesian basis vectors in S_B
$\mathbf{r}$	→	$\mathbf{r}_A$	
$\mathbf{r}'$	→	$\mathbf{r}_B$	
$\mathbf{v}_S$	→	$(\mathbf{r}_A)_A$	
$\mathbf{v}_{S'}$	→	$(\mathbf{r}_A)_B$	
$\mathbf{v}'_S$	→	$(\mathbf{r}_B)_A$	

$\mathbf{v'_{S'}} \quad \rightarrow \quad (\mathbf{r_B})_{\mathbf{B}}$

$\mathbf{a_{SS'}} \quad \rightarrow \quad (a_A)_{AB}$
$\mathbf{a'_{SS'}} \quad \rightarrow \quad (a_B)_{AB}$
$\mathbf{a'_{SS}} = \mathbf{a'_S} \quad \rightarrow \quad (a_B)_{AA} = (a_B)_A$
$\mathbf{a_{SS}} = \mathbf{a_S} = \mathbf{a} \quad \rightarrow \quad (\mathbf{a_A})_{AA} = (\mathbf{a_A})_A = \mathbf{a_A}$
$\mathbf{a'_{S'S'}} = \mathbf{a'_{S'}} = \mathbf{a'} \quad \rightarrow \quad (\mathbf{a_B})_{BB} = (\mathbf{a_B})_B = \mathbf{a_B}$

Things get even more dramatic when one deals with *components* of vectors:

$(\mathbf{r})_i \quad \rightarrow \quad (\mathbf{r_A})_{Ai}$
$(\mathbf{r})'_i \quad \rightarrow \quad (\mathbf{r_A})_{Bi}$
$(\mathbf{r'})_i \quad \rightarrow \quad (\mathbf{r_B})_{Ai}$
$(\mathbf{r'})'_i \quad \rightarrow \quad (\mathbf{r_B})_{Bi}$

The letter A or B before the index i tells whether this is a vector component in Frame A or Frame B. We quietly handled that issue in this document with prime or no prime, but now it comes out into the open! The components of the new basis vectors are especially interesting,

$(\mathbf{e_n})_i \quad \rightarrow \quad (\mathbf{u_{An}})_{Ai}$
$(\mathbf{e_n})'_i \quad \rightarrow \quad (\mathbf{u_{An}})_{Bi}$
$(\mathbf{e_n'})_i \quad \rightarrow \quad (\mathbf{u_{Bn}})_{Bi}$
$(\mathbf{e_n'})'_i \quad \rightarrow \quad (\mathbf{u_{Bn}})_{Bi}$

In a way, this A,B notation is almost more "honest" since it really shows the amount of notational detail one is faced with just in the rotating frames Cartesian discussion, even before curvilinear coordinates are broached.

Here then are the translated solution equations for the Original Problem from Section 12

$\mathbf{r_A}$, $\mathbf{v_A}$, $\mathbf{a_A}$ position, natural velocity and natural acceleration in Frame A
$\mathbf{r_B}$, $\mathbf{v_B}$, $\mathbf{a_B}$ position, natural velocity and natural acceleration in Frame B

$$\mathbf{r_A} = \mathbf{b} + \mathbf{r_B} \qquad (6.1)$$

$$\mathbf{v_A} = \mathbf{v_B} + \boldsymbol{\omega} \times \mathbf{r_B} + \dot{\mathbf{b}}_A \qquad (6.6a)$$

$$\mathbf{v_A} = \mathbf{v_B} + \boldsymbol{\omega} \times \mathbf{r_A} + \dot{\mathbf{b}}_B \qquad (6.6c)$$

$$\mathbf{a_A} = \mathbf{a_B} + \dot{\boldsymbol{\omega}} \times \mathbf{r_B} + 2\,\boldsymbol{\omega} \times \mathbf{v_B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_B}) + \ddot{\mathbf{b}}_A \qquad (7.6a)$$
A B Euler Coriolis centripetal frame

$$\mathbf{a_A} = \mathbf{a_B} + \dot{\boldsymbol{\omega}} \times \mathbf{r_A} + 2\,\boldsymbol{\omega} \times \mathbf{v_B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r_A}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_B + \ddot{\mathbf{b}}_B \qquad (7.6b)$$

$\mathbf{L_A} = \mathbf{r_A} \times [\mathbf{v_A}]$ where $\mathbf{v_A}$ = (6.6a) or (6.6c) above *(11.4)*

$\dot{\mathbf{L}}_\mathbf{A} = \mathbf{r_A} \times [\mathbf{a_A}]$ where $\mathbf{a}$ = (7.6a) or (7.6b) above *(11.7)*

The equations for the inverse problem are then obtained from these swap rules (see ***)

A ↔ B and all that entails $\mathbf{b} \leftrightarrow -\mathbf{b}$ and all that entails $\boldsymbol{\omega} \leftrightarrow -\boldsymbol{\omega}$

For any vector we can do four expansions as shown in these samples:

$$\mathbf{r_A} = (r_A)_{Ai}(\mathbf{u_A})_i = (r_A)_{Bi}(\mathbf{u_B})_i = (r_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (r_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\mathbf{r_B} = (r_B)_{Ai}(\mathbf{u_A})_i = (r_B)_{Bi}(\mathbf{u_B})_i = (r_B)_{Ai}(\hat{\mathbf{e}}_A)_i = (r_B)_{Bi}(\hat{\mathbf{e}}_B)_i$$

$$\mathbf{v_A} = (v_A)_{Ai}(\mathbf{u_A})_i = (v_A)_{Bi}(\mathbf{u_B})_i = (v_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (v_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\mathbf{v_B} = (v_B)_{Ai}(\mathbf{u_A})_i = (v_B)_{Bi}(\mathbf{u_B})_i = (v_B)_{Ai}(\hat{\mathbf{e}}_A)_i = (v_B)_{Bi}(\hat{\mathbf{e}}_B)_i$$

In polar coordinates, for example we would have

$(v_A)_{B1} = (v_A)_{Br}$ = the component of vector $\mathbf{v_A}$ when expanded on the Frame B unit vector $\hat{\mathbf{r}}$ $(\hat{\mathbf{r}}_B)$

$(v_A)_{B2} = (v_A)_{B\theta}$ = the component of vector $\mathbf{v_A}$ when expanded on the Frame B unit vector $\hat{\boldsymbol{\theta}}$ $(\hat{\boldsymbol{\theta}}_B)$

Here are some other sample expansions

$$\mathbf{a_A} = (a_A)_{Ai}(\mathbf{u_A})_i = (a_A)_{Bi}(\mathbf{u_B})_i = (a_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (a_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\mathbf{L_A} = (L_A)_{Ai}(\mathbf{u_A})_i = (L_A)_{Bi}(\mathbf{u_B})_i = (L_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (L_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\dot{\mathbf{L}}_\mathbf{A} = (\dot{L}_A)_{Ai}(\mathbf{u_A})_i = (\dot{L}_A)_{Bi}(\mathbf{u_B})_i = (\dot{L}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\dot{L}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$

$$\mathbf{b} = (b)_{Ai}(\mathbf{u_A})_i = (\mathbf{b})_{Bi}(\mathbf{u_B})_i = (b)_{Ai}(\hat{\mathbf{e}}_A)_i = (b)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\dot{\mathbf{b}}_\mathbf{A} = (\dot{b}_A)_{Ai}(\mathbf{u_A})_i = (\dot{b}_A)_{Bi}(\mathbf{u_B})_i = (\dot{b}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\dot{b}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\ddot{\mathbf{b}}_\mathbf{A} = (\ddot{b}_A)_{Ai}(\mathbf{u_A})_i = (\ddot{b}_A)_{Bi}(\mathbf{u_B})_i = (\ddot{b}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\ddot{b}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\boldsymbol{\omega} = (\boldsymbol{\omega})_{Ai}(\mathbf{u_A})_i = (\boldsymbol{\omega})_{Bi}(\mathbf{u_B})_i = (\omega)_{Ai}(\hat{\mathbf{e}}_A)_i = (\omega)_{Bi}(\hat{\mathbf{e}}_B)_i$$

(d) Solution equations in curvilinear coordinates

Consider one of our equations from above,

$\mathbf{v_A} = \mathbf{v_B} + \boldsymbol{\omega} \times \mathbf{r_B} + \dot{\mathbf{b}}_\mathbf{A}$ *(6.6a)*

In Cartesian Frame A components this says (recall that $\boldsymbol{\omega}_\mathbf{A} = \boldsymbol{\omega}_\mathbf{B} = \boldsymbol{\omega}$ from *****)

$$(v_A)_{Ai} = (v_B)_{Ai} + \varepsilon_{ijk}\,\omega_{Aj}\,(v_B)_{Ak} + (\dot{b}_A)_{Ai}$$

Since our vector equation above is "covariant with respect to rotations", we may write

$$(v_A')_{Ai} = (v_B')_{Ai} + \varepsilon_{ijk}\,\omega'_{Aj}\,(v_B')_{Ak} + (\dot{b}_A')_{Ai}$$

where the italicized components are those in the curvilinear coordinate system selected for Frame A. Another way to understand this is the following. Since the equation *(6.6a)* is entirely stated in vector notation, it will be valid in any orthogonal basis one chooses, and we are assuming that the $\hat{e}_{Ai}$ basis vectors are orthogonal because we assume an orthogonal curvilinear coordinate system. An example of such an orthogonal curvilinear basis would be $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{z}}$ for cylindrical coordinates.

Therefore, if we take any of our rotating-frames equations (all of them are in vector notation), we can interpret them in curvilinear coordinates if we do the following. For every vector **s** :

(1) italicize the component and add a prime to it (so then have s')
(2) If we want the i^{th} component in terms of the curvilinear coordinates associated with Frame A, then write that i^{th} component as $(s')_{Ai}$.
(3) If we want the i^{th} component in terms of the curvilinear coordinates associated with Frame B, then write that i^{th} component as $(s')_{Bi}$.

That is really all there it to it. The formalism and notation is heavy duty, but the application is light.

(e) Turntable example

********** scraps **********

Try to go with

$\mathbf{a}$, $\mathbf{a}'$ → $(\mathbf{a}_A)$, $(\mathbf{a}_B)$

$\mathbf{e}_n$, $\mathbf{e}'_n$ → $(\mathbf{u}_A)_n$, $(\mathbf{u}_B)_n$

$\mathbf{a} = (a)_i\mathbf{e}_i$ → $\mathbf{a}_A = (\mathbf{a}_A)_{Ai}(\mathbf{u}_A)_i = (\boldsymbol{a}_A)_{Ai}(\hat{\mathbf{e}}_A)_i$

$\mathbf{a} = (a)'_i\mathbf{e}'_i$ → $\mathbf{a}_A = (\mathbf{a}_A)_{Bi}(\mathbf{u}_B)_i = (\boldsymbol{a}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$

$\mathbf{a}' = (a')_i\mathbf{e}_i$ → $\mathbf{a}_B = (\mathbf{a}_B)_{Ai}(\mathbf{u}_A)_i = (\boldsymbol{a}_B)_{Ai}(\hat{\mathbf{e}}_A)_i$

$\mathbf{a'} = (a')'_i \mathbf{e'}_i \quad \rightarrow \quad \mathbf{a}_B = (\mathbf{a}_B)_{Bi}(\mathbf{u}_B)_i = (\alpha_B)_{Bi}(\hat{\mathbf{e}}_B)_i$

Here are the previous equations

$\mathbf{r}, \mathbf{v}, \mathbf{a}$	position, natural velocity and natural acceleration in Frame S	
$\mathbf{r'}, \mathbf{v'}, \mathbf{a'}$	position, natural velocity and natural acceleration in Frame S'	
$\mathbf{r} = \mathbf{b} + \mathbf{r'}$		*(6.1)*

$\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S$ *(6.6a)*

$\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}$ *(6.6c)*

$\mathbf{a} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + \ddot{\mathbf{b}}_S$ *(7.6a)*

S S' Euler Coriolis centripetal frame

$\mathbf{a} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'}$ *(7.6b)*

$\mathbf{L} = \mathbf{r} \times [\mathbf{v}]$	where $\mathbf{v}$ = (6.6a) or (6.6c) above	*(11.4)*
$\dot{\mathbf{L}} = \mathbf{r} \times [\mathbf{a}]$	where $\mathbf{a}$ = (7.6a) or (7.6b) above	*(11.7)*

Here are the translated equations

$\mathbf{r}_A, \mathbf{v}_A, \mathbf{a}_A$	position, natural velocity and natural acceleration in Frame A
$\mathbf{r}_B, \mathbf{v}_B, \mathbf{a}_B$	position, natural velocity and natural acceleration in Frame B

$\mathbf{r}_A = \mathbf{b} + \mathbf{r}_B$ *(6.1)*

$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_B + \dot{\mathbf{b}}_A$ *(6.6a)*

$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_A + \dot{\mathbf{b}}_B$ *(6.6c)*

$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_B + 2\,\boldsymbol{\omega} \times \mathbf{v}_B + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_B) + \ddot{\mathbf{b}}_A$ *(7.6a)*

S S' Euler Coriolis centripetal frame

$\mathbf{a}_A = \mathbf{a}_B + \dot{\boldsymbol{\omega}} \times \mathbf{r}_A + 2\,\boldsymbol{\omega} \times \mathbf{v}_B + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_A) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_B + \ddot{\mathbf{b}}_B$ *(7.6b)*

$\mathbf{L}_A = \mathbf{r}_A \times [\mathbf{v}_A]$	where $\mathbf{v}_A$ = (6.6a) or (6.6c) above	*(11.4)*
$\dot{\mathbf{L}}_A = \mathbf{r}_A \times [\mathbf{a}_A]$	where $\mathbf{a}$ = (7.6a) or (7.6b) above	*(11.7)*

Now we can rewrite expansions for some of our vectors of interest:

$\mathbf{r}_A = (r_A)_{Ai}(\mathbf{u}_A)_i = (r_A)_{Bi}(\mathbf{u}_B)_i = (r_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (r_A)_{Bi}(\hat{\mathbf{e}}_B)_i$

$$\mathbf{r}_B = (r_B)_{Ai}(\mathbf{u}_A)_i = (r_B)_{Bi}(\mathbf{u}_B)_i = (r_B)_{Ai}(\hat{\mathbf{e}}_A)_i = (r_B)_{Bi}(\hat{\mathbf{e}}_B)_i$$

$$\mathbf{v}_A = (v_A)_{Ai}(\mathbf{u}_A)_i = (v_A)_{Bi}(\mathbf{u}_B)_i = (v_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (v_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\mathbf{v}_B = (v_B)_{Ai}(\mathbf{u}_A)_i = (v_B)_{Bi}(\mathbf{u}_B)_i = (v_B)_{Ai}(\hat{\mathbf{e}}_A)_i = (v_B)_{Bi}(\hat{\mathbf{e}}_B)_i$$

In polar coordinates, for example we would have

$(v_A)_{B1} = (v_A)_{Br}$ = the component of vector $\mathbf{v}_A$ when expanded on the Frame B unit vector $\hat{\mathbf{r}}$ ($\hat{\mathbf{r}}_B$)

$(v_A)_{B2} = (v_A)_{B\theta}$ = the component of vector $\mathbf{v}_A$ when expanded on the Frame B unit vector $\hat{\boldsymbol{\theta}}$ ($\hat{\boldsymbol{\theta}}_B$)

Here are some other sample expansions

$$\mathbf{a}_A = (a_A)_{Ai}(\mathbf{u}_A)_i = (a_A)_{Bi}(\mathbf{u}_B)_i = (a_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (a_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\mathbf{L}_A = (L_A)_{Ai}(\mathbf{u}_A)_i = (L_A)_{Bi}(\mathbf{u}_B)_i = (L_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (L_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\dot{\mathbf{L}}_A = (\dot{L}_A)_{Ai}(\mathbf{u}_A)_i = (\dot{L}_A)_{Bi}(\mathbf{u}_B)_i = (\dot{L}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\dot{L}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$

$$\mathbf{b} = (b)_{Ai}(\mathbf{u}_A)_i = (\mathbf{b})_{Bi}(\mathbf{u}_B)_i = (b)_{Ai}(\hat{\mathbf{e}}_A)_i = (b)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\dot{\mathbf{b}}_A = (\dot{b}_A)_{Ai}(\mathbf{u}_A)_i = (\dot{b}_A)_{Bi}(\mathbf{u}_B)_i = (\dot{b}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\dot{b}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\ddot{\mathbf{b}}_A = (\ddot{b}_A)_{Ai}(\mathbf{u}_A)_i = (\ddot{b}_A)_{Bi}(\mathbf{u}_B)_i = (\ddot{b}_A)_{Ai}(\hat{\mathbf{e}}_A)_i = (\ddot{b}_A)_{Bi}(\hat{\mathbf{e}}_B)_i$$
$$\boldsymbol{\omega} = (\boldsymbol{\omega})_{Ai}(\mathbf{u}_A)_i = (\boldsymbol{\omega})_{Bi}(\mathbf{u}_B)_i = (\omega)_{Ai}(\hat{\mathbf{e}}_A)_i = (\omega)_{Bi}(\hat{\mathbf{e}}_B)_i$$

Conjecture: I think all of the equations above are unchanged if you write them in ^ type orthogonal coordinates. The equations are all rotationally covariant! For example,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_S, \qquad (6.6c)$$
$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_A + \dot{\mathbf{b}}_B$$

But then maybe we could just stick with our original notation and write

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_S, \qquad (6.6c)$$
$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r} + \dot{\boldsymbol{b}}_B \qquad (6.6a)$$

Let's play with this a bit for the turntable in cylindricals. We have

$\boldsymbol{\omega} = \omega\hat{\mathbf{z}} = \omega\hat{\mathbf{z}} = \boldsymbol{\omega}$ $\qquad$ $\omega = \omega$

$\boldsymbol{b} = b\hat{\mathbf{r}} = -b\,\hat{\mathbf{r}}$ $\qquad$ $b = -\,b$

$\dot{\boldsymbol{b}}_B = 0 \qquad \ddot{\boldsymbol{b}}_B = 0$

$\boldsymbol{r} = r_r \hat{\mathbf{r}} + r_\theta \hat{\boldsymbol{\theta}}$

$\boldsymbol{\omega} \times \boldsymbol{r} = [\omega \hat{\mathbf{z}}] \times [r_r \hat{\mathbf{r}} + r_\theta \hat{\boldsymbol{\theta}}] = \omega r_r \hat{\boldsymbol{\theta}} - \omega r_\theta \hat{\mathbf{r}}$

In components the above reads

$$v_i = v'_i + [\boldsymbol{\omega} \times \boldsymbol{r}]_i + (\dot{b}_B)_i \qquad (6.6a)$$

and therefore

$$v_r = v'_r + [\boldsymbol{\omega} \times \boldsymbol{r}]_r + (\dot{b}_B)_r$$
$$v_\theta = v'_\theta + [\boldsymbol{\omega} \times \boldsymbol{r}]_\theta + (\dot{b}_B)_\theta$$

or

$$v_r = v'_r - \omega r_\theta + (\dot{b}_B)_r$$
$$v_\theta = v'_\theta + \omega r_r + (\dot{b}_B)_\theta$$

Apply this to the turntable scenario to get

$$v_r = v'_r - \omega r_\theta$$
$$v_\theta = v'_\theta + \omega r_r$$

Can this solve any useful problems???

Problem 1. Ant walks radially so $v'_\theta = 0$. Equations above read

$$v_r = v'_r - \omega r_\theta$$
$$v_\theta = \omega r_r$$