The Goldstein Angular Momentum Paradox PhL 7.5.12

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See large red review comment below. Review complete. This is only a 6 page doc.

1. Development of the Paradox.

Angular momentum? Is it a vector [with respect to what transformation?] when R is time-dependent. Since **r** is and **v** is not, it seems that it won't be a vector. Let's investigate:

$\mathbf{L} = m\,\mathbf{r} \times \mathbf{v}$

$[\mathbf{L}]_S = m\,[\mathbf{r}]_S \times [\mathbf{v}]_S$

$\mathbf{r} = R\mathbf{r}'$ [not true since $\mathbf{r} = \mathbf{r}' + \mathbf{b}$]

$[\mathbf{r}]_S = R\,[\mathbf{r}]_{S'}$

$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S$

$[\mathbf{L}]_S = m\,(R\,[\mathbf{r}]_{S'}) \times (R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S)$

$= m\,(R\,[\mathbf{r}]_{S'}) \times (R\,[\mathbf{v}]_{S'}) + m\,(R\,[\mathbf{r}]_{S'}) \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$

A vector theorem shows that, when R is a rotation

$\mathbf{c} = \mathbf{a} \times \mathbf{b} \quad \Rightarrow \quad R\,\mathbf{c} = (R\,\mathbf{a}) \times (R\,\mathbf{b})$

as seems reasonable since we are just looking at the cross product from a rotated frame. Therefore

$[\mathbf{L}]_S = m\,(R\,[\mathbf{r}]_{S'}) \times (R\,[\mathbf{v}]_{S'}) + m\,(R\,[\mathbf{r}]_{S'}) \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$

$= R\,\{\, m[\mathbf{r}]_{S'}) \times ([\mathbf{v}]_{S'})\,\} + m\,(R\,[\mathbf{r}]_{S'}) \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$

$= R[\mathbf{L}]_{S'} + m\,(R\,[\mathbf{r}]_{S'}) \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$

$= R[\mathbf{L}]_{S'} + m\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$

Thus, angular momentum **L** does *not* transform as a true vector when R is time-dependent, and we *cannot* use **G** = **L** in Goldstein's page 133 equations. *Yet,* on page 158 Goldstein says that

$(d\mathbf{L}/dt)_{\mathbf{space}} = (d\mathbf{L}/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{L}$

How can I justify these seemingly conflicting results?

Review comment: At the time this doc was written, it was not clear to me that the G Rule must be true for all vectors, so I was wondering if the G Rule "worked" or not for vector **L**. I thought I could show yes or no. I am still using my bogus bracket notation that I think is non-meaningful. Above I show that "**L** does not transform as a vector" because there is some sort of extra term. In my more modern result frames doc (11.5), it is *true* that you find **L** = **L**' + other terms. So although I did this incorrectly above, the conclusion is correct that there are "other terms. So it is true that **L**' ≠ R**L**. I was thinking that this then disqualified the G Rule because then "**L** was not a vector". In fact the G Rule is valid for any vector that transforms as a vector under normal rotations where rotating frames are not involved, and this includes **L**, and includes any vector you can expand on the basis vectors $\mathbf{e}_n$ as shown in my G Rule proof.

So below I attempt to check whether or not the G Rule works for **L** by computing both sides of the Rule and it appears that my attempt to verify the G Rule for L fails and this is the **Paradox** of the doc title. But since I am using the meaningless notation and since I don't at this time even know what (d/dt) means, it is not surprising that things don't work right.

What I have shown above is this

$$[\mathbf{L}]_S = R[\mathbf{L}]_{S'} + m\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$$

All we can do is compute the derivative and see what happens. Consider

$$d/dt\,([\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S))$$

$$= [\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + [\mathbf{r}]_S \times (\dot{\boldsymbol{\omega}}[\mathbf{r}]_S) + [\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S))$$

For general vectors we know that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{B} \times \mathbf{A})$$

$$= (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} + (\mathbf{C}\bullet\mathbf{A})\mathbf{B} - (\mathbf{C}\bullet\mathbf{B})\mathbf{A}$$

$$= 2\,(\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} - (\mathbf{C}\bullet\mathbf{B})\mathbf{A}$$

and the main point is that it does not vanish. So we just keep all these terms for now. We know that

$$(dR/dt)\mathbf{V} = \boldsymbol{\omega} \times (R\mathbf{V})$$

so

$$(dR/dt)\,[\mathbf{L}]_{S'} = \boldsymbol{\omega} \times (R\,[\mathbf{L}]_{S'})$$

So now differentiate (*) using the above pieces,

$$[\mathbf{L}]_S = R[\mathbf{L}]_{S'} + m\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$$

$$d[\mathbf{L}]_S/dt = (dR/dt)[\mathbf{L}]_{S'} + R\,(d[\mathbf{L}]_{S'}/dt) + m\,d/dt\,[[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)]$$

$$= \boldsymbol{\omega} \times (R\,[\mathbf{L}]_{S'}) + R\,(d[\mathbf{L}]_{S'}/dt) + \\ + m[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\dot{\boldsymbol{\omega}}[\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S))$$

or

$$(d[\mathbf{L}]_S/dt) = R\,(d[\mathbf{L}]_{S'}/dt) + \boldsymbol{\omega} \times (R\,[\mathbf{L}]_{S'}) \\ + m[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\dot{\boldsymbol{\omega}}[\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S))$$

Now use () to write

$$\boldsymbol{\omega} \times (R\,[\mathbf{L}]_{S'}) = \boldsymbol{\omega} \times \{ [\mathbf{L}]_S - m\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \}$$

$$= \omega \times [\mathbf{L}]_S - m\,\boldsymbol{\omega} \times [\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)\,]$$

We then have

$$(d[\mathbf{L}]_S/dt) = R\,(d[\mathbf{L}]_{S'}/dt) + \omega \times [\mathbf{L}]_S$$

$$+ m[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\dot{\boldsymbol{\omega}}[\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S))$$

$$- m\,\boldsymbol{\omega} \times [\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)\,]$$

In order to verify Goldstein's p 158 claim, we have to show that the last two lines vanish! Let's start by assuming the $\boldsymbol{\omega}$ is a constant, which he assumed earlier, so we have to show that

$$(d[\mathbf{L}]_S/dt) = R\,(d[\mathbf{L}]_{S'}/dt) + \omega \times [\mathbf{L}]_S$$

$$+ m[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + m[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S)) \quad - m\,\boldsymbol{\omega} \times (\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)\,)$$

So we have to show that

$$[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + [\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S)) = \boldsymbol{\omega} \times (\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)\,)$$

This seems pretty impossible to me because there is no $[\mathbf{v}]_S$ on the RHS. For example, suppose it happens that $[\mathbf{v}]_S = 0$ at some instant in time. The above would then require that

$$0 = \boldsymbol{\omega} \times (\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)\,)$$

This is some function of $\boldsymbol{\omega}$ and $[\mathbf{r}]_S$ which in general is not zero. If it were, then we would have a contradiction when $[\mathbf{v}]_S = 0$.

We do know that

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S$$

but that is not going to help. I can do the triple cross product like this

$$\boldsymbol{\omega} \times ([\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S))$$

$$= \mathbf{A} \times [\, \mathbf{B} \times (\mathbf{C} \times \mathbf{D}) \,] = \mathbf{B}\,[\mathbf{A},\mathbf{C},\mathbf{D}] - (\mathbf{A}\bullet\mathbf{B})\,(\mathbf{C} \times \mathbf{D})$$

$$= \mathbf{A} \times [\, \mathbf{B} \times (\mathbf{A} \times \mathbf{B}) \,] = \mathbf{B}\,[\mathbf{A},\mathbf{A},\mathbf{B}] - (\mathbf{A}\bullet\mathbf{B})\,(\mathbf{A} \times \mathbf{B})$$

Now $[\mathbf{A},\mathbf{A},\mathbf{B}] = \mathbf{A} \bullet (\mathbf{A} \times \mathbf{B}) = 0$ so we have

$$= - (\mathbf{A}\bullet\mathbf{B})\,(\mathbf{A} \times \mathbf{D}) = - (\boldsymbol{\omega}\bullet [\mathbf{r}]_S)\,(\boldsymbol{\omega} \times [\mathbf{r}]_S)$$

We can also compute

$$[\mathbf{v}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) + [\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{v}]_S))$$

$$= \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{B} \times \mathbf{A})$$

$$= (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} + (\mathbf{C}\bullet\mathbf{A})\mathbf{B} - (\mathbf{C}\bullet\mathbf{B})\mathbf{A}$$

$$= 2\,(\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} - (\mathbf{C}\bullet\mathbf{B})\mathbf{A}$$

$$= 2\,([\mathbf{v}]_S \bullet [\mathbf{r}]_S)\,\boldsymbol{\omega} - ([\mathbf{v}]_S \bullet \boldsymbol{\omega})\,[\mathbf{r}]_S - ([\mathbf{r}]_S \bullet \boldsymbol{\omega})\,[\mathbf{v}]_S$$

and then we have to show that

$$2\,([\mathbf{v}]_S \bullet [\mathbf{r}]_S)\,\boldsymbol{\omega} - ([\mathbf{v}]_S \bullet \boldsymbol{\omega})\,[\mathbf{r}]_S - ([\mathbf{r}]_S \bullet \boldsymbol{\omega})\,[\mathbf{v}]_S = - (\boldsymbol{\omega}\bullet [\mathbf{r}]_S)\,(\boldsymbol{\omega} \times [\mathbf{r}]_S)$$

which is just another way to state what I can see is not true.

So now we have a **Big Paradox** to deal with!! The whole foundation of my document on this stuff is now completely destroyed by Goldstein's claim that his rule works for **L**. [this is what always happens here on the Sisyphus Ranch] . [it was a shabby foundation!]

2. What does Marion have to say?

Goldstein mentions position **r** and "total angular momentum" as candidate **G**'s, and later p 158 explicitly applies it to **L** = **r** x **p**, but he does not mention **v**. Would he apply his rule to **v**? He did not include it in his list, that is all I know.

Marion does address this subject on page 341. His use of primes is exactly the reverse of mine. He develops the idea for vector **r**, then claims it is true for any "vector" **Q**. He then says he is going to use this **Q** equation 11.7 for velocity **v**, but that is *not* what he does. He only uses 11.6 which is for **r**. But then later on page 388 he does that same quote that Goldstein does.

Marion does have some differences, however. In his page 341 picture he does NOT show the two systems aligned with each other [for a long time I thought this might be a condition of validity for the G Rule. Of course it is not.] . He also, incidentally, includes a translation vector which I can set to zero for now. So with the systems not aligned, he still comes up with what I would write as

$$(d[\mathbf{r}]_S/dt) = R\,(d[\mathbf{r}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{r}]_S$$

but he writes as

$$(d\mathbf{r}/dt)_{\mathtt{fixed}} = (d\mathbf{r}/dt)_{\mathtt{rotating}} + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{p 342 11.6}$$

Maybe there is some issue with the axis of rotation, [no, that is not the main problem!] and maybe mine is not right. Here is what I say in my section (4),

$$d[\mathbf{r}]_S = R\,d[\mathbf{r}]_{S'} + dR\,[\mathbf{r}]_{S'} \quad . \qquad (4)$$

$$d[\mathbf{r}]_S/dt = R\,(d[\mathbf{r}]_{S'}/dt) + (dR/dt)\,[\mathbf{r}]_{S'} \quad . \qquad (5)$$

But here is what he says (translated to my notation). He is thinking that the rotating system at some instant is rotating *about its origin* by little amount $d\boldsymbol{\theta}$ and then he says, assuming that $[\mathbf{r}]_{S'}$ = constant (is at rest in the rotating frame),

$$d[\mathbf{r}]_S = d\boldsymbol{\theta} \times [\mathbf{r}]_{S'} \qquad (11.2)$$

In my section 3 I have that

$$dR = [-i\,d\boldsymbol{\Omega} \bullet \mathbf{J}\,]\,R$$

$$dR\,[\mathbf{r}]_{S'} = [-i\,d\boldsymbol{\Omega} \bullet \mathbf{J}\,]\,R\,[\mathbf{r}]_{S'} = d\boldsymbol{\Omega} \times (R\,[\mathbf{r}]_{S'})$$

So let's compare our two expressions of $d[\mathbf{r}]_S$:

$$[\mathbf{r}]_S = R\,[\mathbf{r}]_{S'}$$

$$d[\mathbf{r}]_S = R\,d[\mathbf{r}]_{S'} + dR\,d[\mathbf{r}]_{S'} = dR\,d[\mathbf{r}]_{S'} \qquad \text{// } \mathbf{r} \text{ not moving in rotating frame}$$

$$= d\boldsymbol{\Omega} \times (R\,[\mathbf{r}]_{S'}) = d\boldsymbol{\Omega} \times [\mathbf{r}]_S \qquad \text{// me}$$

$$d[\mathbf{r}]_S = d\boldsymbol{\theta} \times [\mathbf{r}]_{S'} \qquad \text{// Marion}$$

In his form, $[\mathbf{r}]_{S'}$ is the correct thing to have because we are rotating around the origin of the rotating frame. So his seems right and mine seems wrong.