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[This entire document is an example of : garbage in, garbage out]

(a) A Pre-calculation. Recall from above that

$\mathbf{v}_{body}(t) = R(t)\, \mathbf{v}'_{body}(t)$ [what is v_{body} ?]

[OK, let's imagine that $\mathbf{v}'_{body}$ is the velocity of our ant as measured in the rotating frame S'. Later I will call this vector $\mathbf{v}'_{S'} = \mathbf{v}' = (d\mathbf{r}'/dt)_{S'}$ with components $(v')'_i$ in the rotating frame. What then is $\mathbf{v}_{body}$ in the above equation? It cannot be $\mathbf{v}_S = \mathbf{v}$ because we know the equation is not right for that. And $\mathbf{v}'_S = (d\mathbf{r}'/dt)_S$ which from the G Rule we know is $\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{r}'$ and that doesn't seem to match our equation either. The fourth is $\mathbf{v}_{S'} = (d\mathbf{r}/dt)_{S'}$ and that seems pretty distant. So I think $\mathbf{v}_{body}$ is none of the actual four velocities in the problem. It is just an actively rotated version of $\mathbf{v}'$. So why do a calculation on something that you don't know what is ? I think I am just continuing my erroneous ways. Recall how we keep getting that "extra factor of R" in the G Rule which I though you had to then set to R = 1]

Then

$$d_t\, \mathbf{v}_{body}(t) = (d_t R)\, \mathbf{v}'_{body}(t) + R(t)\, d_t \mathbf{v}'_{body}(t)$$

or

$$\mathbf{a}_{body}(t) = (d_t R)\, \mathbf{v}'_{body}(t) + R(t)\, \mathbf{a}'_{body}(t) .$$

And we must carefully take note that we don't just have $\mathbf{a}_{body}(t) = R(t)\, \mathbf{a}'_{body}(t)$. In fact $\mathbf{a}_{body}$ is not a "vector" due to the above transformation rule with its extra piece. Using our usual method,

$$[(d_t R)\, \mathbf{v}'_{body}(t)]_i = [(d_t \exp(-i\varphi(t) J_z))\, \mathbf{v}'_{body}(t)]_i$$

$$= \{[-i\varphi'(t)\, J_z]\, [(\exp(-i\varphi(t) J_z))\, \mathbf{v}'_{body}(t)]\}_i \qquad // \quad i\,(J_k)_{ij} = \varepsilon_{kij}$$

$$= \{[-i\varphi'(t)\, J_z]\, [R\mathbf{v}'_{body}(t)]\}_i$$

$$= \{[-i\varphi'(t)\, J_z]\, [\mathbf{v}_{body}(t)]\}_i$$

$$= -i\varphi'(t)\, (J_z)_{ij}\, [\mathbf{v}_{body}(t)]_j = -\varphi'(t)\, \varepsilon_{3ij}\, [\mathbf{v}_{body}(t)]_j = -\omega_3\, \varepsilon_{3ij}\, [\mathbf{v}_{body}(t)]_j$$

$$= \varepsilon_{i3j}\, \omega_3 [\mathbf{v}_{body}(t)]_j$$

$$= \varepsilon_{ikj}\, \omega_k [\mathbf{v}_{body}(t)]_j$$

So we have shown that

$$[(d_t R)\, \mathbf{v}'_{body}(t)]_i = \varepsilon_{ikj}\, \omega_k [\mathbf{v}_{body}(t)]_j$$

$$[(d_t R)\, \mathbf{v}'_{body}(t)] = \boldsymbol{\omega} \times \mathbf{v}_{body}(t)$$

Therefore, going way back we then have

$$\mathbf{a}_{body}(t) = (d_t R)\, \mathbf{v}'_{body}(t) + R(t)\, \mathbf{a}'_{body}(t)$$

$$= R(t)\, \mathbf{a}'_{body}(t) + \boldsymbol{\omega} \times \mathbf{v}_{body}(t)$$ [and here is that fully factor of R mentioned above]

and so we have restarted our "extra term".

(b) The Goldstein equation for v . That equation says this

$$d\mathbf{v} = \mathbf{d\varphi} \times \mathbf{v} + d\mathbf{v}_{body}$$

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + d\mathbf{v}_{body}/dt$$ [OK, if correctly interpreted: $d\mathbf{v}/dt = \partial_s \mathbf{v}$ and $d\mathbf{v}_{body}/dt = \partial_{s'} \mathbf{v}$]

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} + a_{body}$$

Installing our pre-calculation result, this says

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} + R(t)\, \mathbf{a}'_{body}(t) + \boldsymbol{\omega} \times \mathbf{v}_{body}(t)$$

We may regard our first equation as a first order PDE in variable $\mathbf{v}(t)$

$$d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v} = R(t)\, \mathbf{a}'_{body}(t) + \boldsymbol{\omega} \times \mathbf{v}_{body}(t)$$

We think we know everything on the RHS, but let's make sure. For the first term we get,

$$\mathbf{a}'_{body} = d\mathbf{v}'_{body}/dt = d_t[\dot{r}'\, \hat{\mathbf{r}}' + r'\, \dot{\theta}'\, \hat{\boldsymbol{\theta}}'] = \text{five terms}$$

$$= \ddot{r}'\, \hat{\mathbf{r}}' + \dot{r}'\, \dot{\theta}'\, \hat{\boldsymbol{\theta}}' + \dot{r}'\, \dot{\theta}'\, \hat{\boldsymbol{\theta}}' + r'\ddot{\theta}'\, \hat{\boldsymbol{\theta}}' + r'\, \dot{\theta}' [-\dot{\theta}'\, \hat{\mathbf{r}}']$$

$$= (\ddot{r}' - r'\dot{\theta}'^2)\, \hat{\mathbf{r}}' + (r'\ddot{\theta}' + 2\, \dot{r}'\, \dot{\theta}')\, \hat{\boldsymbol{\theta}}'$$

and then (this removes the primes from the unit vectors)

$$R(t)\, \mathbf{a}'_{body}(t) = (\ddot{r}' - r'\dot{\theta}'^2)\, \hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\, \dot{r}'\, \dot{\theta}')\, \hat{\boldsymbol{\theta}}$$

For the second term we have,

$$\mathbf{v}_{body} = \dot{r}'\, \hat{\mathbf{r}} + r'\, \dot{\theta}'\, \hat{\boldsymbol{\theta}}$$ // from Section 3

$$\boldsymbol{\omega} \times \mathbf{v}_{\mathbf{body}}(t) = [\omega\hat{\mathbf{z}}] \times [\dot{r}'\,\hat{\mathbf{r}} + r'\,\dot{\theta}'\,\hat{\boldsymbol{\theta}}] = \omega\dot{r}'\hat{\boldsymbol{\theta}} - \omega\, r'\,\dot{\theta}'\hat{\mathbf{r}}$$

The combined RHS is then

$$(\ddot{r}' - r'\dot{\theta}'^2)\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}')\,\hat{\boldsymbol{\theta}} + \omega\,\dot{r}'\,\hat{\boldsymbol{\theta}} - \omega\, r'\,\dot{\theta}'\hat{\mathbf{r}}$$

$$= (\ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}')\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}')\,\hat{\boldsymbol{\theta}}$$

Therefore, our Goldstein equation is this

$$d\mathbf{v}/dt - \boldsymbol{\omega} \times \mathbf{v} = (\ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}')\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}')\,\hat{\boldsymbol{\theta}}$$

Now we want to get the components for the terms on the LHS:

$$\mathbf{v} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} \qquad \boldsymbol{\omega} = \omega\hat{\mathbf{z}} \qquad \Rightarrow \qquad \boldsymbol{\omega} \times \mathbf{v} = [\omega\hat{\mathbf{z}}] \times [v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}] = \omega\, v_r\,\hat{\boldsymbol{\theta}} - \omega\, v_\theta\hat{\mathbf{r}}$$

$$d\mathbf{v}/dt = \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{v}_\theta\hat{\boldsymbol{\theta}} + v_\theta[-\dot{\theta}\,\hat{\mathbf{r}}] = (\dot{v}_r - v_\theta\dot{\theta})\hat{\mathbf{r}} + (v_r\dot{\theta} + \dot{v}_\theta)\,\hat{\boldsymbol{\theta}}$$

So here is our grand equation

$$(\dot{v}_r - v_\theta\dot{\theta})\hat{\mathbf{r}} + (v_r\dot{\theta} + \dot{v}_\theta)\,\hat{\boldsymbol{\theta}} - \omega\, v_r\,\hat{\boldsymbol{\theta}} + \omega\, v_\theta\hat{\mathbf{r}}$$

$$= (\ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}')\,\hat{\mathbf{r}} + (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}')\,\hat{\boldsymbol{\theta}}$$

Separating by components, we get these two equations

$$(\dot{v}_r - v_\theta\dot{\theta})\hat{\mathbf{r}} + \omega\, v_\theta\hat{\mathbf{r}} = (\ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}')\,\hat{\mathbf{r}}$$

$$(v_r\dot{\theta} + \dot{v}_\theta)\,\hat{\boldsymbol{\theta}} - \omega\, v_r\,\hat{\boldsymbol{\theta}} = (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}')\,\hat{\boldsymbol{\theta}}$$

or

$$(\dot{v}_r - v_\theta\dot{\theta}) + \omega\, v_\theta = (\ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}')$$

$$(v_r\dot{\theta} + \dot{v}_\theta) - \omega\, v_r = (r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}')$$

or

$$\dot{v}_r - v_\theta\dot{\theta} + \omega\, v_\theta = \ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}'$$

$$v_r\dot{\theta} + \dot{v}_\theta - \omega\, v_r = r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}'$$

or

$$\dot{v}_r - (\dot{\theta} - \omega)\, v_\theta = \ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\,\dot{\theta}'$$

$$\dot{v}_\theta + (\dot{\theta} - \omega)v_r = r'\ddot{\theta}' + 2\,\dot{r}'\,\dot{\theta}' + \omega\,\dot{r}'$$

And now the two equations are not decoupled! Things are much more complicated than before. No doubt we could solve these equations somehow, but I am not going to try. My main interest is to verify that these equations are true!!

From the previous section we found that

$$\mathbf{v}(t) = v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} \quad \text{where} \quad v_r = \dot{r} \quad v_\theta = r\dot{\theta}$$

So we can say

$$v_r = \dot{r}$$
$$\dot{v}_r = \ddot{r}$$

$$v_\theta = r\dot{\theta}$$
$$\dot{v}_\theta = r\ddot{\theta} + \dot{r}\dot{\theta}$$

Then our two coupled equations above become

$$\dot{v}_r - (\dot{\theta}-\omega)\, v_\theta = \ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\dot{\theta}'$$
$$\dot{v}_\theta + (\dot{\theta} - \omega)v_r = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}' + \omega\dot{r}'$$

or

$$\ddot{r} - (\dot{\theta}-\omega)\, r\dot{\theta} = \ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\dot{\theta}'$$
$$r\ddot{\theta} + \dot{r}\dot{\theta} + (\dot{\theta} - \omega)\dot{r} = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}' + \omega\dot{r}'$$

Let's whittle away on the first equation:

$$\ddot{r} - (\dot{\theta}-\omega)\, r\dot{\theta} = \ddot{r}' - r'\dot{\theta}'^2 - \omega\, r'\dot{\theta}'$$

$$\ddot{r} - (\dot{\theta}-\omega)\, r\dot{\theta} = \ddot{r} - r\dot{\theta}'^2 - \omega\, r\dot{\theta}' \qquad \text{// since } r = r'$$

$$-(\dot{\theta}-\omega)\, r\dot{\theta} = -r\dot{\theta}'^2 - \omega\, r\dot{\theta}'$$

$$-\dot{\theta}'\, r\dot{\theta} = -r\dot{\theta}'^2 - \omega\, r\dot{\theta}'$$

$$-\dot{\theta}'\dot{\theta} = -\dot{\theta}'^2 - \omega\dot{\theta}'$$

$$-\dot{\theta} = -\dot{\theta}' - \omega$$

$$\dot{\theta} = \dot{\theta}' + \omega$$

and this is true, so we have verified our first equation !

Now let's whittle away at the second equation

$$r\ddot{\theta} + \dot{r}\dot{\theta} + (\dot{\theta} - \omega)\dot{r} = r'\ddot{\theta}' + 2\dot{r}'\dot{\theta}' + \omega\dot{r}'$$

$$r\ddot{\theta} + \dot{r}\dot{\theta} + (\dot{\theta} - \omega)\dot{r} = r\ddot{\theta}' + 2\dot{r}\dot{\theta}' + \omega\dot{r} \qquad \text{// since } r = r'$$

$$r\ddot{\theta} + \dot{r}\dot{\theta} + \dot{\theta}\dot{r} - \omega\dot{r} = r\ddot{\theta}' + 2\dot{r}\dot{\theta}' + \omega\dot{r}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - \omega\dot{r} = r\ddot{\theta}' + 2\dot{r}\dot{\theta}' + \omega\dot{r}$$

$$r(\ddot{\theta} - \ddot{\theta}') + 2\dot{r}(\dot{\theta} - \dot{\theta}') = 2\omega\dot{r}$$

$$r\dot{\omega} + 2\dot{r}\omega = 2\omega\dot{r}$$

$$r\dot{\omega} = 0$$

Close but something is still wrong.