Rotation debug 4 PhL 7.7.12

Review Comments: This is just more flailing due to my bad G Rule understanding. I compute **v** and **a** directly by differentiating **r** in Frame S and my results disagree with my interpretation of the G Rule. Probably my differentiation math is all OK. It turns out that there is a LOT more to this stuff, as now shown in frames doc, than meets the original eye.

1. Derivation of the Goldstein Rule:

Suppose we have some vector $\mathbf{V}^{\mathbf{F}}$ being viewed in some frame F of interest. In some arbitrary frame f this vector will have coordinates $[\mathbf{V}^{\mathbf{F}}]_{\mathbf{f}}$, and we could select f = F if we wanted.

If we apply to this vector in frame F a small rotation R(d**φ**) within frame F, then we get

$$d\mathbf{V}^{\mathbf{F}} = [\, -i\, d\boldsymbol{\varphi}\bullet\mathbf{J}\,]\, \mathbf{V}^{\mathbf{F}} \;=\; d\boldsymbol{\varphi} \times \mathbf{V}^{\mathbf{F}} .$$

Now suppose the frame in which **V** lies happens to be "the body frame" S' which is instantaneously rotating by d**φ** in time dt relative to the space frame. What those words mean is that the basis vectors of the body frame, as viewed in the space frame, move a little according to

$$\mathbf{e'}_{\mathbf{n}}(t+dt) = R(d\boldsymbol{\varphi})\, \mathbf{e'}_{\mathbf{n}}(t)$$

If we identify frame F as S', then our equation above would read

$$d\mathbf{V}^{\mathbf{S'}} = d\boldsymbol{\varphi} \times \mathbf{V}^{\mathbf{S'}}$$

Then if the vector **V** is static in the body frame, it has a change in the space frame given by

$$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V}$$

But how to we indicate that the **V** appearing above is in the body frame? [endless confusion]

Problem 1: An ant is glued to a phono disk at some radius r with θ = 0 at t=0. The disk starts rotating at time t=0 with some ω(t) that varies in time. Describe everything that happens and do it in polar coordinates.

$d\hat{\boldsymbol{\theta}}/dt = -\omega\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt = \omega\hat{\boldsymbol{\theta}}$

$\omega = d\theta/dt$ $\quad \theta(t) = \int_0^t \omega(t)dt$ = known. $\quad \hat{\mathbf{r}} = r\cos\theta(t)\,\hat{\mathbf{x}} + r\sin\theta(t)\,\hat{\mathbf{y}}$

$\mathbf{r}(t) = r\hat{\mathbf{r}}$ $\quad \hat{\boldsymbol{\theta}} = -r\sin\theta(t)\,\hat{\mathbf{x}} + r\cos\theta(t)\,\hat{\mathbf{y}}$

$\mathbf{v}(t) = d\mathbf{r}/dt = r\, d\hat{\mathbf{r}}/dt = r\omega\hat{\boldsymbol{\theta}}$

$\mathbf{a}(t) = d\mathbf{v}/dt = r\dot{\omega}\hat{\boldsymbol{\theta}} + r\omega(-\omega\hat{\mathbf{r}}) = r\dot{\omega}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\mathbf{r}}$

$d\mathbf{a}/dt = r\ddot{\omega}\hat{\boldsymbol{\theta}} + r\dot{\omega}(-\omega\hat{\mathbf{r}}) - 2r\omega\dot{\omega}\hat{\mathbf{r}} - r\omega^2(\omega\hat{\boldsymbol{\theta}})$

$= -3r\omega\dot{\omega}\,\hat{\mathbf{r}} + (r\ddot{\omega} - r\omega^3)\,\hat{\boldsymbol{\theta}}$

Now, how does this relate the Goldstein's various "rules". Here they are:

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times r\hat{\mathbf{r}} = \omega r\hat{\boldsymbol{\theta}}$ // agrees with above

$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times [r\omega\hat{\boldsymbol{\theta}}] = -r\omega^2\hat{\mathbf{r}}$ // disagrees with above

$d\mathbf{a}/dt = \boldsymbol{\omega} \times \mathbf{a} = \boldsymbol{\omega} \times [r\dot{\omega}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\mathbf{r}}] = -r\omega\dot{\omega}\hat{\mathbf{r}} - r\omega^3\hat{\boldsymbol{\theta}}$ // disagrees with above

[I guess my thinking here was that since the ant is glued down, then $\mathbf{v}^{body} = 0$ and $\mathbf{a}^{body} = 0$ and then one has $d\mathbf{v}_{body}/dt = 0$ etc so that is why there is only $\boldsymbol{\omega} \times \mathbf{q}$ sitting there in the last two G Rule lines. The first line would have $d\mathbf{r}_{body}/dt = 0$ because $\mathbf{r}_{body}$ does not change. If this is done correctly, the second line would be $\partial_S\mathbf{v} = \partial_{S'}\mathbf{v} + \omega \times \mathbf{v}$ and $\partial_{S'}\mathbf{v}$ is a cross acceleration that has to be computed and is not 0. I just don't yet have the tools needed to do ant problems!]

Now to understand why, consider these facts:

$\mathbf{r} = r\hat{\mathbf{r}}$	$d\mathbf{r}/dt = r\omega\hat{\boldsymbol{\theta}}$	$\mathbf{r} \bullet d\mathbf{r}/dt = 0$
$\mathbf{v} = r\omega\hat{\boldsymbol{\theta}}$	$d\mathbf{v}/dt = r\dot{\omega}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\mathbf{r}}$	$\mathbf{v} \bullet d\mathbf{v}/dt \neq 0$
$\mathbf{a} = r\dot{\omega}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\mathbf{r}}$	$d\mathbf{a}/dt = -3r\omega\dot{\omega}\,\hat{\mathbf{r}} + (r\ddot{\omega} - r\omega^3)\,\hat{\boldsymbol{\theta}}$	$\mathbf{a} \bullet d\mathbf{a}/dt \neq 0$

When we say $\mathbf{r}$ is "instantaneously going in a circle", we mean that $\mathbf{r} \bullet d\mathbf{r}/dt = 0$, the change in $\mathbf{r}$ is exactly perpendicular to $\mathbf{r}$. So you can see that when ω is a function of time, neither $\mathbf{v}$ nor $\mathbf{a}$ goes in a circle in this instantaneous sense. This is why the Goldstein rules give wrong results in those two cases. This is the first time I have realized this simple fact.

Problem 2: This time the ant moves in radius with some r(t). Resolve the problem.

$d\hat{\boldsymbol{\theta}}/dt = -\omega\hat{\mathbf{r}}$ and $d\hat{\mathbf{r}}/dt = \omega\hat{\boldsymbol{\theta}}$

$\omega = d\theta/dt$ $\theta(t) = \int_0^t \omega(t)dt$ = known. $\hat{\mathbf{r}} = r\cos\theta(t)\,\hat{\mathbf{x}} + r\sin\theta(t)\,\hat{\mathbf{y}}$

$\mathbf{r}(t) = r\hat{\mathbf{r}}$ $\qquad\qquad$ $\hat{\boldsymbol{\theta}} = -r\sin\theta(t)\,\hat{\mathbf{x}} + r\cos\theta(t)\,\hat{\mathbf{y}}$

$\mathbf{r}(t) = r\hat{\mathbf{r}}$

$\mathbf{v}(t) = d\mathbf{r}/dt = r\,d\hat{\mathbf{r}}/dt + \dot{r}\,\hat{\mathbf{r}} = r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$\mathbf{a}(t) = d\mathbf{v}/dt = r\dot{\omega}\hat{\boldsymbol{\theta}} + r\omega(-\omega\hat{\mathbf{r}}) + \ddot{r}\hat{\mathbf{r}} + \dot{r}\,\omega\hat{\boldsymbol{\theta}} = (r\dot{\omega} + \dot{r}\,\omega)\,\hat{\boldsymbol{\theta}} + (\ddot{r} - r\omega^2)\,\hat{\mathbf{r}}$

Now we have

$\mathbf{r} = r\hat{\mathbf{r}}$ $\qquad$ $d\mathbf{r}/dt = r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$ $\qquad$ $\mathbf{r} \bullet d\mathbf{r}/dt \neq 0$

$\mathbf{v} = r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$ $\qquad$ $d\mathbf{v}/dt = (r\dot{\omega} + \dot{r}\,\omega)\,\hat{\boldsymbol{\theta}} + (\ddot{r} - r\omega^2)\,\hat{\mathbf{r}}$ $\qquad$ $\mathbf{v} \bullet d\mathbf{v}/dt \neq 0$

Therefore, when $\dot{r} \neq 0$, even the simplest Goldstein equation will fail. As a check

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times r\hat{\mathbf{r}} = \omega r\hat{\boldsymbol{\theta}}$ $\qquad$ // not valid since missing the $\dot{r}\,\hat{\mathbf{r}}$ term.

Problem 3: Solve Problem 2 using the rotation plus extra piece idea.

$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + \dot{r}\,\hat{\mathbf{r}}$ $\qquad$ $\dot{r} = v_{\mathbf{r}}$ "in the body frame"

$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r} + dr\,\hat{\mathbf{r}}$ $\qquad$ // same thing, note extra piece.

Now what about

$$
\begin{aligned}
d\mathbf{v}/dt &= \boldsymbol{\omega} \times \mathbf{v} + \text{extra} \\
&= \boldsymbol{\omega} \times [r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}] + \text{extra} \\
&= r\omega^2(-\hat{\mathbf{r}}) + \omega\,\dot{r}\,\hat{\boldsymbol{\theta}} + \text{extra} \\
&= -r\omega^2\hat{\mathbf{r}} + \omega\,\dot{r}\,\hat{\boldsymbol{\theta}} + \{\,\ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\omega\hat{\boldsymbol{\theta}}\,\} \qquad \text{extra} = \{\,\ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\omega\hat{\boldsymbol{\theta}}\,\}
\end{aligned}
$$

Here we are able to "prop up" the $d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r}$ equation with a simple extra term. Can we somehow interpret this extra term in this way Goldstein would like:

$(d\mathbf{v}/dt)_{\mathbf{body}} = \{\,\ddot{r}\,\hat{\mathbf{r}} + \dot{r}\,\omega\hat{\boldsymbol{\theta}}\,\}$

Well, maybe we look back at

$\mathbf{v} = d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + \dot{r}\,\hat{\mathbf{r}} = \mathbf{v}^{\mathbf{rot}} + \mathbf{v}^{\mathbf{body}}$ $\qquad$ $\mathbf{v}^{\mathbf{body}} = \dot{r}\,\hat{\mathbf{r}}$

It seems to work!

Problem 4: Solve the problem with ant doing some *arbitrary* body motion.

$\mathbf{r}_{body} = r\hat{\mathbf{r}}$

$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r} + d\mathbf{r}_{body}$

$\mathbf{v} = d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}_{body}/dt) = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} = \omega r\hat{\boldsymbol{\theta}} + \mathbf{v}_{body}$

$\mathbf{v}_{body} = (d\mathbf{r}_{body}/dt) = d(r\hat{\mathbf{r}})/dt = r\,\omega\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$\mathbf{v} = d\mathbf{r}/dt = \omega r\hat{\boldsymbol{\theta}} + [r\,\omega\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}] = 2r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$\mathbf{a} = d\mathbf{v}/dt = d/dt(2r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}) = d_t(2r\omega\hat{\boldsymbol{\theta}}) + d_t(\dot{r}\,\hat{\mathbf{r}})$

$d\mathbf{a}/dt = d^2_t(2r\omega\hat{\boldsymbol{\theta}}) + d^2_t(\dot{r}\,\hat{\mathbf{r}})$

I have failed to provide the ant with some general trajectory on the disk. Needs more work. I think problem 4 is one worth studying a lot.

We used the first basic Goldstein extra thing to get **v** and then we just diff'ed to get **a** and d**a**/dt.

Now try the next level Goldstein extra rule:

$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v} + d\mathbf{v}_{body}$
$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}_{body}/dt)$

$\mathbf{v}_{body} = d_t(r\hat{\mathbf{r}}) = r\,\omega\,\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}$

$(d\mathbf{v}_{body}/dt) = d_t(r\,\omega\,\hat{\boldsymbol{\theta}}) + d_t(\dot{r}\,\hat{\mathbf{r}})$

Now let's see if it works:

$\mathbf{a} = d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}_{body}/dt)$

$= \boldsymbol{\omega} \times [2r\omega\hat{\boldsymbol{\theta}} + \dot{r}\,\hat{\mathbf{r}}] + \quad d_t(r\,\omega\,\hat{\boldsymbol{\theta}}) + d_t(\dot{r}\,\hat{\mathbf{r}})$

$$= 2r\omega^2(-\hat{\mathbf{r}}) + \omega\dot{r}\,\hat{\boldsymbol{\theta}} + d_t(r\,\omega\,\hat{\boldsymbol{\theta}}) + d_t(\dot{r}\,\hat{\mathbf{r}})$$

In order for this to "work" (ie, match **a** above) we would need to have this be true

$$2r\omega^2(-\hat{\mathbf{r}}) + \omega\dot{r}\,\hat{\boldsymbol{\theta}} = d_t(r\,\omega\,\hat{\boldsymbol{\theta}})$$

Well

$$\text{RHS} = \dot{r}\,\omega\,\hat{\boldsymbol{\theta}} + r\,\dot{\omega}\hat{\boldsymbol{\theta}} + r\omega\,(-\omega\hat{\mathbf{r}})$$

and we see it does NOT work because RHS has a $\dot{\omega}$.

So *why* does this not work?