Rotation rule debug PhL 7.5.12

Status: In temp3 v4 I finally got the "right angle" involved and my dΩ complicated angle went away. But the L paradox still remains, so I will ponder them here a bit.

I am concerned that even my fancy $[\mathbf{V}]_S$ notation is insufficient for what is needed here.

Example 1. Go into the Marion prime context for a while, page 341. Space is S' and body is S in this system. The vector shown in the picture as **r** is the one of interest in $d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}$. It is a vector with its tail on the body origin. All three objects in this equation can be expressed in terms of S or S' coordinates. So I could write

$$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}$$

$$[d\mathbf{r}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{r}]_S$$

$$[d\mathbf{r}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{r}]_{S'}$$

I think I understand what these last two equations mean. And this reinforces the idea that you can take a vector rotation like the first above and write it out in any coordinate system you want. Notice that the vector **r'** appearing in the picture becomes **r** if we take out the origin shift, so don't worry now about **r'**.

Example 2. How let's try this not for position **r**, but for velocity **v**. I want to write

$$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$$

$$[d\mathbf{v}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S$$

$$[d\mathbf{v}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$$

But somehow things seem less clear in this case. I could draw the vector **v** with tail on the origin if I liked, so it looks like **r** in the Marion picture. It is the orientation of v that counts, not so much where I put its tail. Except I need the tail at the origin to understand $d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$ with a cone picture. But I guess the tip of the cone could be at any location, not just the origin, and the picture looks the same. We are just translating the cone around, the equation $d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$ stays the same.

So again, we can take each of the three vectors and write them in either frame S or S' and then the two lower equations become meaningful.

But there is a problem, Houston. I intend $[\mathbf{v}]_{S'}$ to mean the vector **v** just expressed in frame S' coordinates. But if the frame S is rotating, you might mean by $[\mathbf{v}]_{S'}$ the velocity in frame S' that arises from the two usual sources. Thus, the notation $[\mathbf{v}]_{S'}$ is ambiguous, and that spells trouble. This ambiguity did no arise with the vector **r**, but it does with the vector **v**.

Example 3. Same velocity example, but I want to get the primes back the way I have been using them in all my temp3 vn documents. So

space = S body = S'

$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$

$[d\mathbf{v}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S$

$[d\mathbf{v}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$

Now our ambiguity is with $[\mathbf{v}]_S$. I might denote the simple coordinate-change version as

$[\mathbf{v}]_S^{\omega=0}$

Then the above equations become

$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$

$[d\mathbf{v}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S^{\omega=0}$

$[d\mathbf{v}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$

But let's first revert back to **r** with this notation and see what it says

$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}$

$[d\mathbf{r}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{r}]_S^{\omega=0}$

$[d\mathbf{r}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{r}]_{S'}$

Here we don't need $\boldsymbol{\omega}=0$ on $[\mathbf{r}]_S^{\omega=0}$ because the fact that the body frame is rotating or even translating does not instantaneously affect **r**. Of course we still have that $[\mathbf{r}]_S \neq [\mathbf{r}]_{S'}$.

Now back to velocity v again, where I add yet more qualifiers,

$d\mathbf{v}^{\text{justω}} = d\boldsymbol{\theta} \times \mathbf{v}$

$[d\mathbf{v}]_S^{\text{justω}} = [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S^{\omega=0}$

$[d\mathbf{v}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$

Now I need some equations for the "other part".

$d\mathbf{v}^{\text{other}}$

$[d\mathbf{v}^{\text{other}}]_S$

$[d\mathbf{v}^{\text{other}}]_{S'}$

This represents a change in **v** because **v** is moving in the body frame S'. The last two objects are as usual not equal because a vector in different systems has different components.

Then we have the additive situation

$$d\mathbf{v} = d\mathbf{v}^{other} + d\mathbf{v}^{just\omega} = d\mathbf{v}^{other} + d\boldsymbol{\theta} \times \mathbf{v}$$

$$[d\mathbf{v}]_S = [d\mathbf{v}^{other}]_S + [d\mathbf{v}^{just\omega}]_S = [d\mathbf{v}^{other}]_S + [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S^{\omega=0}$$

$$[d\mathbf{v}]_{S'} = [d\mathbf{v}^{other}]_{S'} + [d\mathbf{v}^{just\omega}]_{S'} = [d\mathbf{v}^{other}]_{S'} + [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$$

where no qualifier on something means it is the total thing, such as with $[d\mathbf{v}]_S$. I suspect the most important of these equation is this one

$$[d\mathbf{v}]_S = [d\mathbf{v}^{other}]_S + [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S^{\omega=0}$$

Goldstein might have written body instead of other. But OK. Now divide by dt

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{other}]_S/dt + [d\boldsymbol{\theta}]_S/dt \times [\mathbf{v}]_S^{\omega=0}$$

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{other}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

Question: Why does "divide by dt" give a different result from "differentiate wrt t" ? Above you see the divide by dt result. Here I will do the differentiation result. Repeat the three equations above but for r

$$d\mathbf{r} = d\mathbf{r}^{other} + d\mathbf{r}^{just\omega} = d\mathbf{r}^{other} + d\boldsymbol{\theta} \times \mathbf{r}$$

$$[d\mathbf{r}]_S = [d\mathbf{r}^{other}]_S + [d\mathbf{r}^{just\omega}]_S = [d\mathbf{r}^{other}]_S + [d\boldsymbol{\theta}]_S \times [\mathbf{r}]_S^{\omega=0}$$

$$[d\mathbf{r}]_{S'} = [d\mathbf{r}^{other}]_{S'} + [d\mathbf{r}^{just\omega}]_{S'} = [d\mathbf{r}^{other}]_{S'} + [d\boldsymbol{\theta}]_{S'} \times [\mathbf{r}]_{S'}$$

Then do the divide by dt thing,

$$d\mathbf{r}/dt = d\mathbf{r}^{other}/dt + d\mathbf{r}^{just\omega}/dt = d\mathbf{r}^{other}/dt + d\boldsymbol{\theta}/dt \times \mathbf{r}$$

$$[d\mathbf{r}]_S/dt = [d\mathbf{r}^{other}]_S/dt + [d\mathbf{r}^{just\omega}]_S/dt = [d\mathbf{r}^{other}]_S/dt + [d\boldsymbol{\theta}]_S/dt \times [\mathbf{r}]_S^{\omega=0}$$

$$[d\mathbf{r}]_{S'}/dt = [d\mathbf{r}^{other}]_{S'}/dt + [d\mathbf{r}^{just\omega}]_{S'}/dt = [d\mathbf{r}^{other}]_{S'}/dt + [d\boldsymbol{\theta}]_{S'}/dt \times [\mathbf{r}]_{S'}$$

which says

$$\mathbf{v} = d\mathbf{r}^{other}/dt + d\mathbf{r}^{just\omega}/dt = d\mathbf{r}^{other}/dt + \boldsymbol{\omega} \times \mathbf{r}$$

$$[\mathbf{v}]_S = [d\mathbf{r}^{other}]_S/dt + [d\mathbf{r}^{just\omega}]_S/dt = [d\mathbf{r}^{other}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{r}]_S^{\omega=0}$$

$$[\mathbf{v}]_{S'} = [d\mathbf{r}^{other}]_{S'}/dt + [d\mathbf{r}^{just\omega}]_{S'}/dt = [d\mathbf{r}^{other}]_{S'}/dt + [\boldsymbol{\omega}]_{S'} \times [\mathbf{r}]_{S'}$$

and the key result here is

$$[\mathbf{v}]_S = [d\mathbf{r}^{other}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{r}]_S^{\omega=0}$$

or

$$[\mathbf{v}]_S = [\mathbf{v}^{body}]_S + [\boldsymbol{\omega}]_S \times [\mathbf{r}]_S$$

Now if we differentiate this wrt time, we get a $\dot{\boldsymbol{\omega}}$ term

$$d[\mathbf{v}]_S/dt = d[\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times (d[\mathbf{r}]_S/dt) + [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}]_S^{\omega=0}$$

or

$$d[\mathbf{v}]_S/dt = d[\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S + [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}]_S^{\omega=0}$$

which compare to the divide by dt result

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

If these are both valid, then it must be true that

$$[\boldsymbol{\omega}]_S \times [\mathbf{v}]_S + [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}]_S^{\omega=0} = [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

or

$$[\boldsymbol{\omega}]_S \times \{[\mathbf{v}]_S - [\mathbf{v}]_S^{\omega=0}\} = -[\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}]_S$$

Everything has components in frame S. Rewrite as

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

Maybe go back to the uncommitted version,

$$d\mathbf{v} = d\mathbf{v}^{body} + d\boldsymbol{\theta} \times \mathbf{v}$$

$$d\mathbf{v}/dt = d\mathbf{v}^{body}/dt + d\boldsymbol{\theta}/dt \times \mathbf{v}$$

$$\mathbf{a} = (d\mathbf{v}/dt) = (d\mathbf{v}^{body}/dt) + \boldsymbol{\omega} \times \mathbf{v}$$

Is this true? I guess so. Now can I generate the acceleration paradox? Well write as

$$[\mathbf{a}]_S = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

Suppose the velocity is at rest in the body frame. Then the first term should be 0 and then

$$[\mathbf{a}]_S = [d\mathbf{v}]_S/dt = [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S^{\omega=0}$$

But I am expecting some sort of $\dot{\boldsymbol{\omega}}$ term and I don't see it.

How do I express the acceleration in the body frame? Maybe it is this

$$[\mathbf{a}^{body}]_{S'} = [d\mathbf{v}^{body}]_{S'}/dt$$

Now maybe I can write the following vector equation,

$$d\mathbf{v} = d\mathbf{v}^{just\omega} + d\mathbf{v}^{\omega=0}$$

$$[d\mathbf{v}]_S = [d\mathbf{v}^{just\omega}]_S + [d\mathbf{v}^{\omega=0}]_S$$

$$[d\mathbf{v}]_{S'} = [d\mathbf{v}^{just\omega}]_{S'} + [d\mathbf{v}^{\omega=0}]_{S'}$$