Rotation Rule Debug v2 PhL 7.5.12

1. Playing with the velocity. Start here

$$d\mathbf{v} = d\boldsymbol{\theta} \times \mathbf{v}$$

and consider this body frame S' evaluation,

$$[d\mathbf{v}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$$

Is there ambiguity right here? At the current instant, there is some $[\mathbf{v}]_{S'}$ which could be a function of time, but that seems unambiguous. But $[d\mathbf{v}]_{S'}$ seems very ambiguous. It does not, for example, represent a body movement of the velocity vector in the body frame. Instead, it is the vector that results from the rotating body frame, but written in those body frame coordinates. So maybe write

$$[d\mathbf{v}^{rot}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{v}]_{S'}$$

to indicate that this $[d\mathbf{v}]_{S'}$ is coming from the "rotation effect". We might at the same time have some

$$[d\mathbf{v}^{body}]_{S'} \quad \text{and} \quad [\mathbf{v}^{body}]_{S'} = [\mathbf{v}]_{S'}$$

which represent the intrinsic "body motion" and its differential version. If we wanted to evaluate these two objects in frame S coordinates, I would write

$$[d\mathbf{v}^{body}]_{S} \quad \text{and} \quad [\mathbf{v}^{body}]_{S}$$

and avoid $[\mathbf{v}]_S$ since this is ambiguous as described below.

Now take a look in the S frame

$$[d\mathbf{v}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}]_S \ .$$

So yes, now $[\mathbf{v}]_S$ is ambiguous. Here we intend it to just be the instantaneous body velocity converted to frame S coordinates, but one might alternatively consider it to be the entire $[\mathbf{v}]_S$ in frame S. Try this as a first repair

$$[d\mathbf{v}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}^{body}]_S$$

where $[\mathbf{v}^{body}]_S$ means it is the body velocity in frame S' just being converted to frame S coordinates.

OK, now $[d\mathbf{v}]_S$ is *also* ambiguous. Let's repair that by saying

$$[d\mathbf{v}^{rot}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{v}^{body}]_S$$

to indicate that this is the part of $[d\mathbf{v}]_S$ which is caused by "the rotation effect", body frame rotating.

What then would be the total $[d\mathbf{v}]_S$? I think it would be this

$$[d\mathbf{v}]_S = [d\mathbf{v}^{body}]_S + [d\mathbf{v}^{rot}]_S$$

where we use $[\mathbf{v}^{body}]_S$ as noted above. We can write out the second term and then have

$$[d\mathbf{v}]_S = [d\mathbf{v}^{body}]_S + [d\boldsymbol{\theta}]_S \times [\mathbf{v}^{body}]_S$$

Now divide by dt to get

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}^{body}]_S$$

Before doing anything rash, let's pause here and go off and deal with the position vector r. But just suppose $\mathbf{v}^{body} = 0$ so it is zero in either S or S' evaluation. And suppose also that $[d\mathbf{v}^{body}]_{S'}/dt = 0$ because our point is not accelerating either in frame S'. Does that mean $[d\mathbf{v}^{body}]_S/dt = 0$???

Maybe we have to think about the distinction between

$[d\mathbf{v}^{body}]_{S'}/dt$ and $[d\mathbf{v}^{body}/dt]_{S'}$??

Or maybe we have to somehow go compute

$[d\mathbf{v}^{body}]_S/dt$ or $[d\mathbf{v}^{body}/dt]_S$
left right

I am reminded of Goldstein's warning on page 134 that I still don't understand. In the current situation, it seems that the thing on the left = 0 but the thing on the right ≠0.

, and so evaluated in either frame gives 0. Then we get

$$[d\mathbf{v}]_S/dt = 0$$

But suppose we have $\dot{\boldsymbol{\omega}} \neq 0$. You would think that even though things are at rest in the body frame, this would cause a point to accelerate in the S frame, so 0 seems the wrong answer. Contradiction.

Somehow this "divide by dt" causes this acceleration source to be lost. Where is it being lost?

2. Playing with the position vector r. Just mimic the above: Start here

$$d\mathbf{r} = d\boldsymbol{\theta} \times \mathbf{r}$$

and consider this body frame S' evaluation,

$$[d\mathbf{r}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{r}]_{S'}$$

Is there ambiguity right here? At the current instant, there is some $[\mathbf{r}]_{S'}$ which could be a function of time, but that seems unambiguous. But $[d\mathbf{r}]_{S'}$ seems very ambiguous. It does not, for example, represent

a body movement of the posityion vector in the body frame. Instead, it is the vector that results from the rotating body frame, but written in those body frame coordinates. So maybe write

$$[d\mathbf{r}^{rot}]_{S'} = [d\boldsymbol{\theta}]_{S'} \times [\mathbf{r}]_{S'}$$

to indicate that this $[d\mathbf{r}]_{S'}$ is coming from the "rotation effect". We might at the same time have some

$$[d\mathbf{r}^{body}]_{S'} \qquad \text{and} \qquad [\mathbf{r}^{body}]_{S'} = [\mathbf{r}]_{S'}$$

which represent the intrinsic "body motion" and its differential version. If we wanted to evaluate these two objects in frame S coordinates, I would write

$$[d\mathbf{r}^{body}]_{S} \qquad \text{and} \qquad [\mathbf{r}^{body}]_{S}$$

Now take a look in the S frame

$$[d\mathbf{r}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{r}^{body}]_S \ .$$

As done above, I would make this clearer by writing

$$[d\mathbf{r}^{rot}]_S = [d\boldsymbol{\theta}]_S \times [\mathbf{r}^{body}]_S$$

What then would be the total $[d\mathbf{r}]_S$? I think it would be this

$$[d\mathbf{r}]_S = [d\mathbf{r}^{body}]_S + [d\mathbf{r}^{rot}]_S$$

where we use $[\mathbf{v}^{body}]_S$ as noted above. We can write out the second term and then have

$$[d\mathbf{r}]_S = [d\mathbf{r}^{body}]_S + [d\boldsymbol{\theta}]_S \times [\mathbf{r}^{body}]_S$$

Now divide by dt to get

$$[d\mathbf{r}]_S/dt = [d\mathbf{r}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S$$

I think I can write this as

$$[\mathbf{v}]_S = [\mathbf{v}^{body}]_S + [\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S$$

Notice that $[\mathbf{v}^{body}]_S$ refers to the body intrinsic velocity, but expressed in frame S coordinates. This perhaps lines up with Goldstein's equation p 135 which says

$$\mathbf{v}_S = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$

3. Resume with velocity. At the end of section 1 we had

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}^{body}]_S$$

But at the end of section 2 we have

$$[\mathbf{v}^{body}]_S = [\mathbf{v}]_S - [\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S$$

Suppose we insert this in both places in our previous equation :

$$[d\mathbf{v}]_S/dt = [d\mathbf{v}^{body}]_S/dt + [\boldsymbol{\omega}]_S \times [\mathbf{v}^{body}]_S$$

$$= [\, d\{[\mathbf{v}]_S - [\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S\}/dt + [\boldsymbol{\omega}]_S \times \{[\mathbf{v}]_S - [\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S\}$$

$$= [d\mathbf{v}]_S/dt - [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}^{body}]_S - [\boldsymbol{\omega}]_S \times d[\mathbf{r}^{body}]_S/dt$$

$$+ [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S - [\boldsymbol{\omega}]_S \times ([\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S)$$

This does not look very good to me. It says that

$$0 = - [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}^{body}]_S - [\boldsymbol{\omega}]_S \times [\mathbf{v}^{body}]_S + [\boldsymbol{\omega}]_S \times [\mathbf{v}]_S - [\boldsymbol{\omega}]_S \times ([\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S)$$

$$= - [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}^{body}]_S + [\boldsymbol{\omega}]_S \times \{[\mathbf{v}]_S - [\mathbf{v}^{body}]_S\} - [\boldsymbol{\omega}]_S \times ([\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S)$$

$$= - [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}^{body}]_S + [\boldsymbol{\omega}]_S \times \{[\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S\} - [\boldsymbol{\omega}]_S \times ([\boldsymbol{\omega}]_S \times [\mathbf{r}^{body}]_S)$$

$$= - [\dot{\boldsymbol{\omega}}]_S \times [\mathbf{r}^{body}]_S$$

which seems to say that $[\dot{\boldsymbol{\omega}}]_S = 0$! So I think we have our first contraction.