Repair of Section 1 (j) PhL 8.17.12

I decided not to put the derivations of (1.38) into frames doc since they are so unimportant and the derivations would be a very boring appendix. But I want the derivations archived in case someone finds an error or claims one.

Collection of useful facts:

$(\mathbf{e}_i \text{x } \mathbf{e}_j)_k = \varepsilon_{ijk}$

$(\omega)'_i = R_{ij}\omega_j$

$(\mathbf{e}_i \text{x } \mathbf{e}'_j)_k = \varepsilon_{kib}R_{jb}\,(\omega)'_k$

$\varepsilon_{kab}R_{ia}R_{jb} = \varepsilon_{sij}R_{sk}$

Items 1 and 2

Footnote: Here are details of the above four calculations which the reader is invited to ignore :

1. $(d\mathbf{a}/dt)_S = (d\,[(a)_i\mathbf{e}_i]/dt)_S = (d(a)_i/dt)_S\,\mathbf{e}_i + (a)_i\,(d\mathbf{e}_i/dt)_S = (d(a)_i/dt)\,\mathbf{e}_i$

where we use (1.27) . Taking the j^{th} Frame S component of the above,

$[(d\mathbf{a}/dt)_S]_j = \{(d(a)_i/dt)\,\mathbf{e}_i\} \bullet \mathbf{e}_j = (d(a)_i/dt)\,\delta_{ij} = (d(a)_j/dt)$. // **item 1**

Taking the j^{th} Frame S' component on the other hand gives this:

$[(d\mathbf{a}/dt)_S]'_j = \{(d(a)_i/dt)\,\mathbf{e}_i\} \bullet \mathbf{e}'_j = (d(a)_i/dt)\,R_{ji}$ // **item 2**

making use of the last line in (1.7).

Obscure rotation theorem from matrix Addendum : $\varepsilon_{abc}\,R_{b'b}\,R_{c'c} = \varepsilon_{a'b'c'}R_{a'a}$

2. $(d\mathbf{a}/dt)_{S'} = (d\,[(a)_i\mathbf{e}_i]/dt)_{S'} = (d(a)_i/dt)_{S'}\,\mathbf{e}_i + (a)_i\,(d\mathbf{e}_i/dt)_{S'} = (d(a)_i/dt)\,\mathbf{e}_i - (a)_i\,\boldsymbol{\omega} \text{ x } \mathbf{e}_i$

using (1.28) in the last step. Taking the j^{th} Frame S component of the above,

$[(d\mathbf{a}/dt)_{S'}]_j = \{(d(a)_i/dt)\,\mathbf{e}_i - (a)_i\,\boldsymbol{\omega} \text{ x } \mathbf{e}_i\} \bullet \mathbf{e}_j$.

But $(\boldsymbol{\omega} \text{ x } \mathbf{e}_i) \bullet \mathbf{e}_j = \boldsymbol{\omega} \bullet (\mathbf{e}_i \text{x } \mathbf{e}_j) = \omega_k\,(\mathbf{e}_i \text{x } \mathbf{e}_j)_k$. We compute,

$(\mathbf{e}_i \text{x } \mathbf{e}_j)_k = \varepsilon_{kab}(e_i)_a(e_j)_b = \varepsilon_{kab}\delta_{ia}\delta_{jb} = \varepsilon_{kij} = \varepsilon_{ijk}$ (easy to remember) $= \varepsilon_{jki}$

so that $(\boldsymbol{\omega} \times \mathbf{e}_i) \bullet \mathbf{e}_j = \omega_k \ (\mathbf{e}_i \times \mathbf{e}_j)_k = \omega_k \, \varepsilon_{jki}$. Therefore

$$[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_i/dt)\, \delta_{ij} - (a)_i \, \omega_k \, \varepsilon_{jki} = (d(a)_j/dt) - \varepsilon_{jki} \, \omega_k (a)_i$$

or

$$[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_j/dt) - \omega_k \, \varepsilon_{kij} \, (a)_i \qquad \textbf{// item 3}$$

Now taking the j^{th} Frame S' component of our first line we get

$$[(d\mathbf{a}/dt)_{S'}]'_j = \{(d(a)_i/dt)\, \mathbf{e}_i - (a)_i \, \boldsymbol{\omega} \times \mathbf{e}_i \} \bullet \mathbf{e}'_j \ .$$

But $(\boldsymbol{\omega} \times \mathbf{e}_i) \bullet \mathbf{e}'_j = \boldsymbol{\omega} \bullet (\mathbf{e}_i \times \mathbf{e}'_j) = \omega_k \ (\mathbf{e}_i \times \mathbf{e}'_j)_k$. We compute, using the last line of (1.7),

$$(\mathbf{e}_i \times \mathbf{e}'_j)_k = \varepsilon_{kab} (e_i)_a (e'_j)_b = \varepsilon_{kab} \delta_{ia} R_{jb} = \varepsilon_{kib} R_{jb}$$

so that $(\boldsymbol{\omega} \times \mathbf{e}_i) \bullet \mathbf{e}'_j = \omega_k \ (\mathbf{e}_i \times \mathbf{e}'_j)_k = \omega_k \, \varepsilon_{kib} R_{jb}$. Therefore, further using (1.7),

$$[(d\mathbf{a}/dt)_{S'}]'_j = (d(a)_i/dt)\, R_{ji} - \omega_k \, \varepsilon_{kib} (a)_i R_{jb} \qquad \textbf{// item 4}$$

We can summarize our results to this point:

$$[(d\mathbf{a}/dt)_S]_j = (d(a)_j/dt)$$

$$[(d\mathbf{a}/dt)_S]'_j = (d(a)_i/dt)\, R_{ji}$$

$$[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_j/dt) - \varepsilon_{jki} \, \omega_k (a)_i$$

$$[(d\mathbf{a}/dt)_{S'}]'_j = (d(a)_i/dt)\, R_{ji} - \omega_k \, \varepsilon_{kib} (a)_i \, R_{jb}$$

Now *what happens* if we redo all of this using the other expansion for **a** ?

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1. $(d\mathbf{a}/dt)_S = (d\,[(a)'_i \mathbf{e}'_i]/dt)_S = (d(a)'_i/dt)_S \, \mathbf{e}'_i + (a)'_i \, (d\mathbf{e}'_i/dt)_S = (d(a)'_i/dt)_S \, \mathbf{e}'_i + (a)'_i \, \boldsymbol{\omega} \times \mathbf{e}'_i$

where we use (1.25) . Taking the j^{th} Frame S component of the above,

$$[(d\mathbf{a}/dt)_S]_j = \{ (d(a)'_i/dt)_S \, \mathbf{e}'_i + (a)'_i \, \boldsymbol{\omega} \times \mathbf{e}'_i \} \bullet \mathbf{e}_j$$

$$= (d(a)'_i/dt)_S \, \mathbf{e}'_i \bullet \mathbf{e}_j + (a)'_i \ \boldsymbol{\omega} \bullet (\mathbf{e}'_i \times \mathbf{e}_j)$$

$$= (d(a)'_i/dt)_S \, R_{ij} - (a)'_i \ \boldsymbol{\omega} \bullet (\mathbf{e}_j \times \mathbf{e}'_i)$$

$= (d(\mathbf{a})'_i/dt)_S\, R_{ij} - (\mathbf{a})'_i\ \omega_k\, (\mathbf{e}_j \times \mathbf{e}'_i)_k$

$= (d(\mathbf{a})'_i/dt)_S\, R_{ij} - (\mathbf{a})'_i\, \omega_k\, \varepsilon_{kjb} R_{ib}$

We can write this another way as follows, using $R_{ks}\omega_s = (\omega)'_k$ so $R_{sk}\,(\omega)'_s = \omega_k$,

$\omega_k\, \varepsilon_{kjb} R_{ib} = R_{sk}\,(\omega)'_s\, \varepsilon_{kjb} R_{ib} = (\omega)'_s\, [\varepsilon_{kjb}\, R_{sk} R_{ib}]$

$= (\omega)'_s\, [\varepsilon_{jbk}\, R_{ib} R_{sk}] = (\omega)'_s\, [\varepsilon_{ais}\, R_{aj}]$

Then we have

$[(d\mathbf{a}/dt)_S]_j = (d(\mathbf{a})'_i/dt)_S\, R_{ij} - (\mathbf{a})'_i\, (\omega)'_s\, [\varepsilon_{ais}\, R_{aj}]$

or

$[(d\mathbf{a}/dt)_S]_j = (d(\mathbf{a})'_i/dt)_S\, R_{ij} - (\mathbf{a})'_i\, (\omega)'_k\, [\varepsilon_{mik}\, R_{mj}]$

or

$[(d\mathbf{a}/dt)_S]_j = (d(\mathbf{a})'_i/dt)_S\, R_{ij} + (\mathbf{a})'_i\, (\omega)'_k\, [\varepsilon_{kim}\, R_{mj}]$ **// item 8**

Resuming, taking the j^{th} Frame S' component on the other hand gives this:

$[(d\mathbf{a}/dt)_S]'_j = \{(d(\mathbf{a})'_i/dt)_S\, \mathbf{e}'_i + (\mathbf{a})'_i\, \boldsymbol{\omega} \times \mathbf{e}'_i\} \bullet \mathbf{e}'_j =$

$= (d(\mathbf{a})'_i/dt)_S\, \mathbf{e}'_i \bullet \mathbf{e}'_j + (\mathbf{a})'_i\, (\boldsymbol{\omega} \times \mathbf{e}'_i) \bullet \mathbf{e}'_j$

$= (d(\mathbf{a})'_i/dt)_S\, \delta_{ij} + (\mathbf{a})'_i\, \boldsymbol{\omega} \bullet (\mathbf{e}'_i \times \mathbf{e}'_j) = (d(\mathbf{a})'_j/dt)_S + (\mathbf{a})'_i\, \omega_k\, (\mathbf{e}'_i \times \mathbf{e}'_j)_k$

We compute,

$(\mathbf{e}'_i \times \mathbf{e}'_j)_k = \varepsilon_{kab} (e'_i)_a (e'_j)_b = \varepsilon_{kab} R_{ia} R_{jb} = \varepsilon_{sij} R_{sk}$

where the last equality follows from an obscure theorem about rotation matrices. Therefore

$[(d\mathbf{a}/dt)_S]'_j = (d(\mathbf{a})'_j/dt)_S + (\mathbf{a})'_i\, \omega_k\, \varepsilon_{sij} R_{sk}$

But $(\omega)'_s = R_{sk}\, \omega_k$ so we can write the last result as

$[(d\mathbf{a}/dt)_S]'_j = (d(\mathbf{a})'_j/dt)_S + (\mathbf{a})'_i\, (\omega)'_s\, \varepsilon_{sij}$

or

$[(d\mathbf{a}/dt)_S]'_j = (d(\mathbf{a})'_j/dt)_S + (\mathbf{a})'_i\, (\omega)'_k\, \varepsilon_{kij}$ **// item 7**

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2. $(d\mathbf{a}/dt)_{S'} = (d\,[\,(\mathbf{a})'_i \mathbf{e}'_i\,]/dt)_{S'} = (d(\mathbf{a})'_i/dt)_{S'}\, \mathbf{e}'_i + (\mathbf{a})'_i\, (d\mathbf{e}'_i/dt)_{S'} = (d(\mathbf{a})'_i/dt)\, \mathbf{e}'_i$

using (1.26) in the last step. Taking the j^{th} Frame S component of the above,

$[(d\mathbf{a}/dt)_{S'}]_j = \{ (d(a)'_i/dt)\ \mathbf{e}'_i \} \bullet \mathbf{e}_j = (d(a)'_i/dt)\ R_{ij}$ // **item 6**

Now taking the j^{th} Frame S' component of our first line we get

$[(d\mathbf{a}/dt)_{S'}]'_j = \{(d(a)'_i/dt)\ \mathbf{e}'_i \} \bullet \mathbf{e}'_j = (d(a)'_i/dt)\ \delta_{ij} = (d(a)'_j/dt)$ // **item 5**

The rules to get from first group to the second are

S↔S' on both labels and component index types

$\omega \rightarrow -\omega$

$R \rightarrow R^{-1} = R^T$