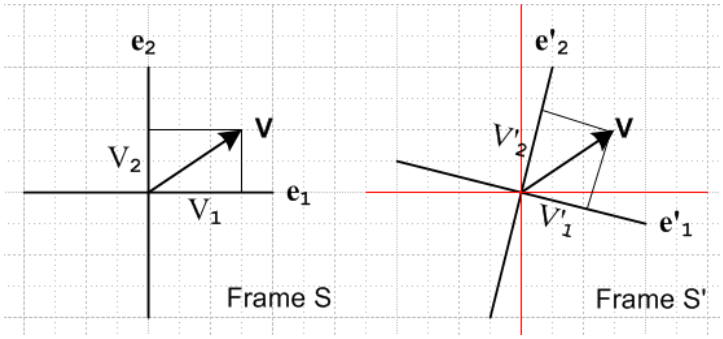
Active and Passive Vectors, Version 51 PhL 7.4.12

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1. The Passive View

We have some real-world physical vector **V** which is being observed by two observers, one in frame S and the other in frame S'. The frames have a common origin. Here is a picture



The vector **V** represents a certain specific aspect of some physical apparatus A. It could be some physical point in apparatus A, or it could be the velocity of some piece of apparatus A, etc. In this passive view, there is only one apparatus, but it is observed from two frames of reference. There is only one vector **V** and it is the same vector in both pictures. Define matrix R by

$\mathbf{e}_2 = R\,\mathbf{e}'_2$ and more generally $\mathbf{e}_n = R\,\mathbf{e}'_n$ forn = 1,2...N

In this 2D example, roughly we have $R = R_{\mathbf{e}3}(10°)$ (right hand rule).

The vector **V** has different components in the two frames. The general rule is this

$\mathbf{V} = V_n\mathbf{e}_n$ where $V_n = \mathbf{V} \bullet \mathbf{e}_n$

$\mathbf{V} = V'_n\mathbf{e}'_n$ where $V'_n = \mathbf{V} \bullet \mathbf{e}'_n$

One might wonder how the V'_n might be expressed in terms of the V_n and vice versa. Well,

$V'_n = \mathbf{V} \bullet \mathbf{e}'_n = [R\,\mathbf{V}] \bullet [R\,\mathbf{e}'_n]$ // valid for ANY rotation R

$= [R\,\mathbf{V}] \bullet \mathbf{e}_n$ // now assuming rotation shown above

$= [R\,V_m\mathbf{e}_m] \bullet \mathbf{e}_n$ // inserting first expansion above

$= V_m\ (R\mathbf{e}_m)\bullet\mathbf{e}_n$

$= V_m\,(R\mathbf{e}_m)_k(\mathbf{e}_n)_k$

$= V_m\,R_{ki}(\mathbf{e}_m)_i(\mathbf{e}_n)_k$

$= V_m\,R_{ki}\delta_{m,i}\delta_{n,k}$

$= R_{nm}\,V_m$

So we learn the that the components are related in this way

$V'_n = R_{nm}\,V_m$

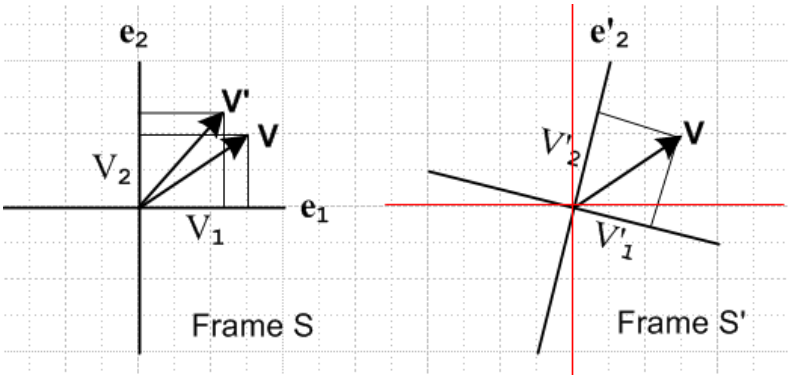
$V_n = R^{-1}{}_{nm}\,V'_m$

Now we are tempted to write the above equations in "vector notation" like so

$\mathbf{V}' = R\mathbf{V}$

$\mathbf{V} = R^{-1}\mathbf{V}'$.

Here are some words to go with the first equation: " If in frame S we were to rotate vector **V** into a new vector **V**' also in frame S , that new vector **V**' would have components V'_n which match those of vector **V** appearing in the S' picture above." So let's show this:

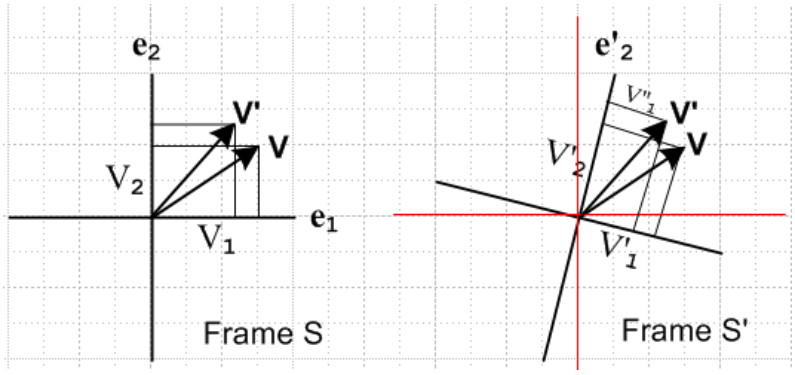


So clearly, the components of vector **V**' in frame S are (V'_1, V'_2), where (a,b,c...) indicates a column vector to save space. Now, let's write our two expansions for this new vector **V**'

$\mathbf{V}' = V'_n\mathbf{e}_n$ where $V'_n = \mathbf{V}' \bullet \mathbf{e}_n = [R\mathbf{V}] \bullet [R\mathbf{e}'_n] = \mathbf{V} \bullet \mathbf{e}'_n = V'_n$

$\mathbf{V}' = V''_n\mathbf{e}'_n$ where $V''_n = \mathbf{V}' \bullet \mathbf{e}'_n = [R\mathbf{V}] \bullet [\mathbf{e}'_n] = [R^2\mathbf{V}] \bullet [R\mathbf{e}'_n] = [R^2\mathbf{V}] \bullet \mathbf{e}_n$

It is of course possible to draw this new vector **V**' in frame S' where it will have these new components which we call V''_1, V''_2 just to be able to distinguish them. Here is the above picture pair with this vector added:



The notion of vector **V'** in frame S will have a use below in our "active view", whereas the vector **V'** in frame S' will not be useful at all. But we wanted to get it drawn lest it be "mysterious".

At this point we might introduce the following notation:

$$[\mathbf{V}]_{\mathbf{F}} \equiv \{\, \mathbf{V} \bullet \mathbf{e}^{\mathbf{F}}_{\mathbf{n}} \}$$

Then we have these four examples

$[\mathbf{V}]_{\mathbf{S}} = (V_1, V_2)$ // first term is $\mathbf{V} \bullet \mathbf{e}_1$

$[\mathbf{V}]_{\mathbf{S}'} = (V'_1, V'_2)$ // first term is $\mathbf{V} \bullet \mathbf{e}'_1$

$[\mathbf{V}']_{\mathbf{S}} = (V'_1, V'_2)$ // first term is $\mathbf{V}' \bullet \mathbf{e}_1$

$[\mathbf{V}']_{\mathbf{S}'} = (V''_1, V''_2)$ // first term is $\mathbf{V}' \bullet \mathbf{e}'_1$

When we see the symbol **V** all by itself, we usually think of $\mathbf{V} = (V_1, V_2)$. That is what we meant by it in the notation **V'** = R**V** which appears above. And similarly, we thought of $\mathbf{V}' = (V'_1, V'_2)$ in this same equation. So by **"default"** we think of **V** as meaning $[\mathbf{V}]_{\mathbf{S}}$ and **V'** as meaning $[\mathbf{V}']_{\mathbf{S}}$.

V' = R**V** means for us above $[\mathbf{V}']_{\mathbf{S}} = R[\mathbf{V}]_{\mathbf{S}}$.

However, since **V'** = R**V** is a relationship between two vectors, it should be a valid relationship when viewed from *any* observer frame. Thus we expect this also to be true

$[\mathbf{V}']_{\mathbf{S}'} = R[\mathbf{V}]_{\mathbf{S}'}$ or $(V''_1, V''_2) = R\ (V'_1, V'_2)$ or $\mathbf{V}'' = R\,\mathbf{V}'$.

Suddenly we see yet another object **V"** which we could then add to our pictures above in an obvious way, but let's NOT add it. Think of **V"** as just a shorthand for (V''_1, V''_2). So here is how we can verify that the last line above is valid:

V' = R**V** => R**V'** = R(R**V**) => **V"** = R**V'** QED

Here are some other "default" meanings (but remember that these rotate oppositely to **V**)

$\mathbf{e}_{\mathbf{n}} = [\mathbf{e}_{\mathbf{n}}]_{\mathbf{S}}$

$\mathbf{e'}_n = [\mathbf{e'}_n]_S$

$\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$ means $[\mathbf{e'}_n]_S = R^{-1}[\mathbf{e}_n]_S$

and again the equation $\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$ must also be true in frame S' so that $[\mathbf{e'}_n]_{S'} = R^{-1}[\mathbf{e}_n]_{S'}$.

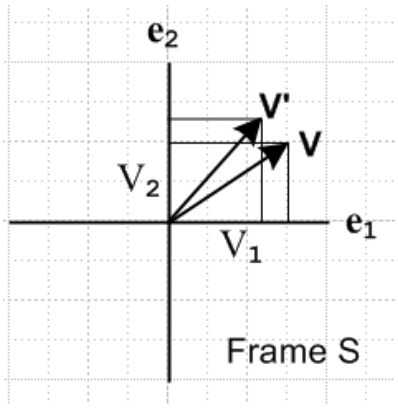
Comment:

(1) Notice that we don't put brackets around the rotation matrix R to get $[R]_F$ for some frame F. This matrix R is a set of numbers which relates **V'** to **V** in the equation **V'** = R**V.**

(2**)** What would tensor doc say? It would say $V'_a = R_{ab}V_b$ (dev notation) where R is the linearized transformation matrix. Recall that R lives neither in x-space nor in x'-space, but straddles the two spaces. There is never an object R'_{ab}. The equation $V'_a = R_{ab}V_b$ is not a "covariant equation" since it is not in one space and since R is not a tensor. So tensor doc agrees with comment (1).

2. The Active View

Instead of having two observers and one apparatus (which has a representative vector **V** in it), where the two observer frames are related by a rotation, now we have one observer frame S and we have two apparatuses which we call A and A'. This situation is basically that of the left picture above, which we repeat here:



In the passive view, the *frame* S' was rotated by amount R^{-1} relative to frame S, since $\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$.
In the active view, the *apparatus* A' is rotated by amount R relative to apparatus A.

3. Computation of dR/dt and (dR/dt)V.

These results will be needed below, so we want to get it out of the way here so we don't clutter up various complicated issues which will be arising in the next section.

We can parameterize R in this way, where $\boldsymbol{\theta} = \boldsymbol{\theta}(t)$,

$$R = R(\boldsymbol{\theta}) = \exp(-i\boldsymbol{\theta}\bullet\mathbf{J}) = \exp(-i\theta_k J_k) \qquad \text{where} = i\,(J_k)_{ij} = \varepsilon_{kij} \qquad (1)$$

(a) Here is the *wrong way* to do this computation:

$R(\boldsymbol{\theta}) = \exp(-i\theta_k J_k)$	1	ok
$dR/dt = (dR/d\theta_k)\ (d\theta_k/dt)$	2	ok
$(dR/d\theta_k) = \exp(-i\theta_j J_j)\,(-iJ_k) = R\,(-iJ_k)$	3	wrong
$dR/dt = (dR/d\theta_k)\ (d\theta_k/dt)$	5	ok
$= R\,(-iJ_k)\,(d\theta_k/dt) = R\,(-i\,\mathbf{J}\bullet[d\boldsymbol{\theta}/\mathbf{dt}]$	6	bad

The problem is that step 3 is incorrect because the J matrices don't commute! The other steps are OK but then the final line is bad because it uses step 3. To see why step 3 is wrong, consider a simpler case where **a** is a vector of time-dependent parameters and A_i are a set of constant but non-commuting matrices:

$$S = \exp(\mathbf{a}\bullet\mathbf{A}) \equiv \Sigma_{n=0}^{\infty}\,(\mathbf{a}\bullet\mathbf{A})^n/n!$$

$$\partial S/\partial a_i = \Sigma_{n=0}^{\infty}(1/n!)\,\partial(\mathbf{a}\bullet\mathbf{A})^n/\partial a_i$$

But

$$\partial(\mathbf{a}\bullet\mathbf{A})^n/\partial a_i = \partial_i(\mathbf{a}\bullet\mathbf{A})^n = \partial_i\,[(\mathbf{a}\bullet\mathbf{A})\,(\mathbf{a}\bullet\mathbf{A})\,(\mathbf{a}\bullet\mathbf{A})\ldots\,(\mathbf{a}\bullet\mathbf{A})\,]$$

$$= A_i(\mathbf{a}\bullet\mathbf{A})\,(\mathbf{a}\bullet\mathbf{A})\ldots\,(\mathbf{a}\bullet\mathbf{A}) + (\mathbf{a}\bullet\mathbf{A})\,A_i(\mathbf{a}\bullet\mathbf{A})\ldots\,(\mathbf{a}\bullet\mathbf{A}) + (\mathbf{a}\bullet\mathbf{A})\,(\mathbf{a}\bullet\mathbf{A})A_i\ldots\,(\mathbf{a}\bullet\mathbf{A}) + \ldots$$

Because the A_i matrices don't commute, we cannot just shove all the A_i factors to the left or to the right. If we *could* shove them all to the right, then we would get

$$\partial(\mathbf{a}\bullet\mathbf{A})^n/\partial a_i = n\,(\mathbf{a}\bullet\mathbf{A})^{n-1}\,A_i$$

and then we find

$$\partial S/\partial a_i = \Sigma_{n=0}^{\infty}(1/n!)\,\partial(\mathbf{a}\bullet\mathbf{A})^n/\partial a_i = \Sigma_{n=0}^{\infty}(n/n!)\,(\mathbf{a}\bullet\mathbf{A})^{n-1}\,A_i = \Sigma_{n=1}^{\infty}(n/n!)\,(\mathbf{a}\bullet\mathbf{A})^{n-1}\,A_i$$

$$= \Sigma_{n=1}^{\infty}(1/[n-1]!)\,(\mathbf{a}\bullet\mathbf{A})^{n-1}\,A_i = \Sigma_{m=0}^{\infty}(1/m!)\,(\mathbf{a}\bullet\mathbf{A})^m\,A_i = \exp(\mathbf{a}\bullet\mathbf{A})\,A_i$$

This then agrees with step 3 which in our case here is this

$$\partial/\partial a_i(\exp(\mathbf{a}\bullet\mathbf{A})) = \exp(\mathbf{a}\bullet\mathbf{A})\,A_i$$

If all the matrices in the set $\{ A_i \}$ commute, then this would be true. But we know that the angular momentum generators J_i do NOT commute since $[J_i,J_j] = i\varepsilon_{ijk}J_k \neq 0$.

(b) Here is the *right way* to do this computation. We note that some small rotation $R(d\boldsymbol{\Omega})$ must exist so that

$$R(\boldsymbol{\theta}+d\boldsymbol{\theta})\, R(\boldsymbol{\theta})^{-1} = R(d\boldsymbol{\Omega}) \qquad // \Rightarrow \qquad R(\boldsymbol{\theta}+d\boldsymbol{\theta}) = R(d\boldsymbol{\Omega})\, R(\boldsymbol{\theta}) \qquad (2)$$

This $R(d\boldsymbol{\Omega})$ has to exist because rotations form a group which means the product of any two rotations is itself a rotation. And the argument $d\boldsymbol{\Omega}$ is small because as $d\boldsymbol{\theta} \rightarrow 0$ we can see that $d\boldsymbol{\Omega} \rightarrow 0$. To first order in smallness one will find that $|d\boldsymbol{\Omega}|$ is proportional to $|\mathbf{d}\theta|$. In general, however, $d\Omega$ is a complicated function of $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$ except in the case that $d\boldsymbol{\theta}$ happens to be in the same direction as $\boldsymbol{\theta}$. This subject is addressed in "a rotation problem 2012" Section 5. Basically, all we need to know here is that if $d\boldsymbol{\theta} = d\varphi\hat{\mathbf{m}}$ and $\boldsymbol{\theta} = \boldsymbol{\theta}\,\hat{\mathbf{n}}$, we will find that $d\boldsymbol{\Omega} = d\Omega\,\hat{\mathbf{s}}$ where $d\Omega = f(\theta, \hat{\mathbf{n}}, \hat{\mathbf{m}})\, d\varphi$ and $\hat{\mathbf{s}} = \hat{\mathbf{s}}(\theta, \hat{\mathbf{n}}, \hat{\mathbf{m}})$.

Now we are ready to compute dR/dt:

$$dR = [R(\boldsymbol{\theta}+d\boldsymbol{\theta}) - R(\boldsymbol{\theta})] = R(d\boldsymbol{\Omega})\, R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}) = (R(d\boldsymbol{\Omega}) - 1)\, R(\boldsymbol{\theta}) \approx [-i\, d\boldsymbol{\Omega} \bullet \mathbf{J}\,]\, R(\boldsymbol{\theta})$$

$$\Rightarrow \qquad dR/dt = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R(\boldsymbol{\theta}) \qquad \text{where } \boldsymbol{\omega} \equiv d\boldsymbol{\Omega}/dt. \qquad (3)$$

(c) Now let's compute this combination object

$$(dR/dt)\mathbf{V}$$

Since R is a matrix, dR/dt is a matrix, $\mathbf{V}$ is a vector, so this object is a vector. We have

$$(dR/dt)\mathbf{V} = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R\mathbf{V} = [-i\, \omega_k J_k\,]\, R\mathbf{V} \qquad // \text{ on next line we will use } i\,(J_k)_{ij} = \varepsilon_{kij}$$

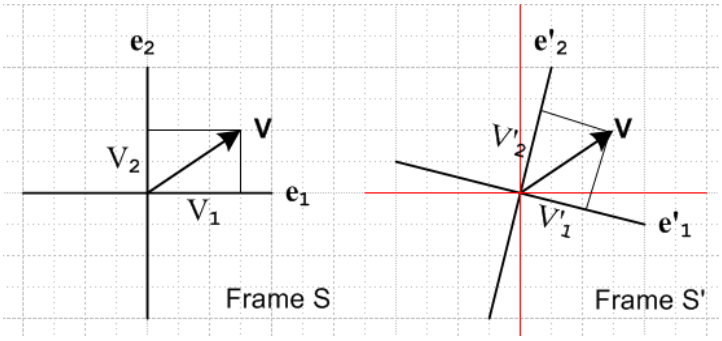
$$[(dR/dt)\mathbf{V}]_i = [-i\, \omega_k (J_k)_{ij}\,]\, [R\mathbf{V}]_j = -\, \omega_k \varepsilon_{kij}\, [R\mathbf{V}]_j = \varepsilon_{ikj}\, \omega_k [R\mathbf{V}]_j$$

Therefore we find that

$$(dR/dt)\mathbf{V} = \boldsymbol{\omega} \times (R\mathbf{V}) \qquad (4)$$

As developed above, we are talking about N = 3 dimensions only for our vectors.

4. Passive view : suppose frame S' (body) is moving and S (space) is at rest .



space frame body frame

Right here we small make the convenience change $R \rightarrow R^{-1}$ for all R which appear in Sections 1 and 2 (but not Section 3). After this change, we have from Section 1 above,

$$\mathbf{e'}_n(t) = R(t)\,\mathbf{e}_n \qquad \mathbf{e}_n = R^{-1}(t)\,\mathbf{e'}_n(t) \tag{1}$$

Now consider some vector **V** lying in the body S' frame. We know again from Section 1 (with change),

$$\mathbf{V}(t) = R(t)\mathbf{V'}(t) \tag{2}$$

which we can interpret in this hybrid manner (see end of Section 1 above for this notation),

$$[\mathbf{V}(t)]_S = R(t)\,[\mathbf{V}(t)]_{S'} \quad . \tag{3}$$

The object $[\mathbf{V}(t)]_{S'}$ is vector **V** in the body frame S', and this vector may be moving in frame S'. Viewed from the space frame, what one sees is $[\mathbf{V}(t)]_S$. From now on, we shall suppress the (t) arguments to reduce clutter.

Comment: Notice that, in reference to equation (2), we said that **V** was some "vector". By definition, a vector with respect to transformation R is something that "transforms as a vector according to (2)". When R is time-dependent, things that one normally thinks of as being "vectors" are in fact not vectors because their components do not satisfy (2). For example, the position vector **r** is a vector under R(t), but as we shall see, the velocity vector **v** is not a vector under R(t), though it is a vector if R is not time-dependent.

Now some time dt goes by and we get some changes:

$$d[\mathbf{V}]_S = R\,d[\mathbf{V}]_{S'} + dR\,[\mathbf{V}]_{S'} \quad . \tag{4}$$

The *first* term arises because there was a change *within* the body S' frame in our vector **V**, and that change then gets mapped to the space frame S by rotation R. The *second* term arises because even if $[\mathbf{V}]_{S'}$ were not changing within the body frame, the body frame is moving, which causes a change of $[\mathbf{V}]_S$ in the space frame.

Now divide by dt to get

$$d[\mathbf{V}]_S/dt = R\,(d[\mathbf{V}]_{S'}/dt) + (dR/dt)\,[\mathbf{V}]_{S'} \quad . \tag{5}$$

Then use result 3.(4) for (dR/dt) acting on a vector to get

$$(d[\mathbf{V}]_S/dt) = R\,(d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times (R\,[\mathbf{V}]_{S'}) \quad . \tag{6}$$

Then use result (3) in the last term of (6) to get

$$(d[\mathbf{V}]_S/dt) = R\,(d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}]_S \quad . \tag{7}$$

We could apply d/dt to equation (7) and in so doing obtain four terms. Rather than do that here, we will do it in the special application of **V** = **r** of the next section.

5. Passive View: what Newtonian Mechanics looks like in a non-inertial frame

Now as a particular application of 4.(7) above, let **V** = **r**, the position vector :

$$(d[\mathbf{r}]_S/dt) = R\,(d[\mathbf{r}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{r}]_S \tag{1}$$

We next identify

$$[\mathbf{v}]_S = (d[\mathbf{r}]_S/dt) = \text{velocity of point } \mathbf{r} \text{ in the space frame S} \tag{2a}$$

$$[\mathbf{v}]_{S'} = (d[\mathbf{r}]_{S'}/dt) = \text{velocity of point } \mathbf{r} \text{ in the body frame S'} \tag{2b}$$

so (1) can be rewritten as

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S \quad . \tag{3}$$

Next, apply d/dt to equation (3) to get

$$(d[\mathbf{v}]_S/dt) = (dR/dt)\,[\mathbf{v}]_{S'} + R\,(d[\mathbf{v}]_{S'}/dt) + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (d[\mathbf{r}]_S/dt) \quad . \tag{4}$$

We now identify these accelerations,

$$[\mathbf{a}]_S = (d[\mathbf{v}]_S/dt) = \text{acceleration of point } \mathbf{r} \text{ in the space frame S} \tag{5a}$$

$$[\mathbf{a}]_{S'} = (d[\mathbf{v}]_{S'}/dt) = \text{acceleration of point } \mathbf{r} \text{ in the body frame S'} \tag{5b}$$

so that, along with (2a) in the last term, we get

$$[\mathbf{a}]_S = (dR/dt)\,[\mathbf{v}]_{S'} + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times [\mathbf{v}]_S \quad . \tag{6}$$

Then use 3.(4) for (dR/dt) acting on a vector to get

$$[\mathbf{a}]_S = \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times [\mathbf{v}]_S \quad . \tag{7}$$

Finally, use (3) to replace $[\mathbf{v}]_S$ in the last term :

$$[\mathbf{a}]_S = \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \quad . \tag{8}$$

We see that two $\boldsymbol{\omega}$ x terms are the same, so we add them, and then put $R\,[\mathbf{a}]_{S'}$ first to get

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \tag{9}$$

This then is the final and very general result relating the acceleration in the space frame S to that in the body frame S'.

Obviously, if $\boldsymbol{\omega}$ is constant in time (such as the rotation of the earth) the above simplifies to

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \tag{10}$$

If *in addition* it happens that the space and body frames are exactly aligned at the instant t which is of interest to us, we may replace R = 1 in both places in (10). Then of course $[\mathbf{r}]_S = [\mathbf{r}]_{S'}$ and we might as well just call it $\mathbf{r}$. Then we get at this instant in time,

$$[\mathbf{a}]_S = [\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \tag{11}$$

In the space frame S (inertial since at rest) we have Newton's law being valid,

$$[\mathbf{F}]_S = m\,[\mathbf{a}]_S \, . \tag{12}$$

Using (11) we can say that

$$[\mathbf{F}_{\mathbf{eff}}]_{S'} \equiv m\,[\mathbf{a}]_{S'} = m\,[\mathbf{a}]_S - 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= [\mathbf{F}]_S - 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \tag{13}$$

This says that we can pretend there is a Newton's law in the (non-inertial) rotating frame,

$$[\mathbf{F}_{\mathbf{eff}}]_{S'} \equiv m\,[\mathbf{a}]_{S'}$$

as long as we are *willing to include* the two "fictitious forces" shown in (13). The second fictitious force is just the usual centrifugal force, pointing exactly away from the rotation axis. The first term is the new object of interest : the Coriolis force $-2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'}$.

6. A Paradox and its Resolution

This small section hopefully illuminates the Comment made above equation 4.(4). Note that several equations in this section are wrong so please don't use them!

Equation 4.(7) is supposedly valid for any vector **V**. What happens if we take **V** = **v**, the velocity of a particle. It then says,

$$(d[\mathbf{V}]_S/dt) = R\,(d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}]_S \qquad 4.(7)$$

$$(d[\mathbf{v}]_S/dt) = R\,(d[\mathbf{v}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{v}]_S \qquad 4.(7) \text{ with } \boldsymbol{V} = \boldsymbol{v}$$

We identify accelerations as in 5.(5) and this then says

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + \boldsymbol{\omega} \times [\mathbf{v}]_S \,.$$

Comparison with 5.(6) shows that this last equation is wrong because we are missing two terms! So what is going on here?? If we trace backwards from 4.(7) we get to 4.(3) which says, for **V** = **v**,

$$\mathbf{v} = R\,\mathbf{v}'$$

which we could be written, according to the end of Section 1,

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} \,.$$

But this is not true for **V** = **v** ! In fact, what is true is 5.(3)

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S$$

What this means is that, although **v** "looks like" a vector, it does not transform as a vector since we have this extra $\boldsymbol{\omega} \times [\mathbf{r}]_S$ term (when R is time dependent). We are only allowed to apply 4.(7) to objects that really transform as a vector under our transformation between frames S and S'. **V** = **r** really does transform as a vector.

7. Comments on Goldstein's treatment of this subject.

The theory of transforming between rotating frames is discussed on pages 132-135 of Herbert Goldstein's famous book "Classical Mechanics" (1950, 7th printing 1965), with some support material on earlier pages and some implications on later pages. These pages form Sections 4-8 and the start of 4-9 of that book. Goldstein never draws a picture of the two frames in question, which we have called S (space) and S' (body). He seems to make an assumption (which I cannot find stated anywhere) that, at the moment of interest t, the two frames S and S' line up with each other, which means we can set R = 1 in any of our final expressions above. With this assumption, we may quote some of our results from above, replacing our generic vector **V** by generic vector **G**,

$$(d[\mathbf{G}]_S/dt) = R\,(d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{G}]_S \quad . \qquad 4.(7)$$

$$(d[\mathbf{G}]_S/dt) = (d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{G}]_S \quad . \qquad 4.(7) \text{ with } R=1$$

Goldstein writes the last equation above in the following manner,

$(d\mathbf{G}/dt)_{\mathbf{space}} = (d\mathbf{G}/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{G}$ Goldstein (4-100)

First of all, if R = 1, then since **G** is a "vector" we know from 4.(3) that $[\mathbf{G}]_S = R\ [\mathbf{G}]_{S'} = [\mathbf{G}]_{S'}$. Since we then have $[\mathbf{G}]_S = [\mathbf{G}]_{S'}$, we might as well just call it **G**, as he does above. As an example, consider our equation 5.(3) from above, but in which we set R = 1 and so $[\mathbf{r}]_S = [\mathbf{r}]_{S'} = \mathbf{r}$,

$[\mathbf{v}]_S = [\mathbf{v}]_{S'} + \boldsymbol{\omega} \times \mathbf{r}$. 5.(3) with R=1

which we might also write as

$[\mathbf{v}]_{\mathbf{space}} = [\mathbf{v}]_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{r}$

This then is an example of Goldstein (4-100). Since (4-100) is valid for any (true!) vector **G**, we can abstract out the vector **G** and treat it as an operator equation of sorts (which can only act on true vectors)

$(d/dt)_{\mathbf{space}} = (d/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times$ Goldstein (4-102)

In our notation, this could be written several ways, removing the **G** vector acted upon,

$$\mathbf{(}d[\ \ ]_S/dt) = (d[\ \ ]_{S'}/dt) + \boldsymbol{\omega} \times$$

or

$$\mathbf{(}d/dt)[\ \ ]_S = (d/dt)[\ \ ]_{S'} + \boldsymbol{\omega} \times$$

and if we *define* operator $\mathbf{(}d/dt)_S[\ \ ] \equiv \mathbf{(}d/dt)[\ \ ]_S$, then finally

$$\mathbf{(}d/dt)_S[\ \ ] = (d/dt)_{S'}[\ \ ] + \boldsymbol{\omega} \times$$

or

$$\mathbf{(}d/dt)_S = (d/dt)_{S'} + \boldsymbol{\omega} \times$$

or

$$\mathbf{(}d/dt)_{\mathbf{space}} = (d/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times$$

and this last form then aligns Goldstein (4-102). The operator notation is nice, but it can be a little confusing what it really means, so we feel it is best to keep the vector G present and, assuming R = 1, use this form:

$\mathbf{(}d[\mathbf{G}]_S/dt) = (d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times \mathbf{G}$. our 4.(7) with R=1

$(d\mathbf{G}/dt)_{\mathbf{space}} = (d\mathbf{G}/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{G}$ Goldstein (4-100)

6. Comparison with Goldstein notation (p 133) **TBD**

Goldstein uses vector **G** instead of **V** for his generic vector. So I will now translate some equations above into **G** form, and then do side-by side comparison with Goldstein:

$$[d\mathbf{G}]_S = [d\mathbf{G}]_S^{(body)} + [d\boldsymbol{\Omega}]_S \times [\mathbf{G}]_S \qquad (23a)$$

$$(d\mathbf{G})_{space} = (d\mathbf{G})_{(body)} + d\boldsymbol{\Omega} \times \mathbf{G} \qquad \text{p 133 (4-99)}$$

$$[d\mathbf{G}/dt]_S = [d\mathbf{G}/dt]_S^{(body)} + [\boldsymbol{\omega}]_S \times [\mathbf{G}]_S \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\Omega}/dt \qquad (23b)$$

$$(d\mathbf{G}/dt)_{space} = (d\mathbf{G}/dt)_{(body)} + \omega \times \mathbf{G} \qquad \text{p 133 (4-100)}$$

$$(d/dt) = (d/dt)_{(body)} + \boldsymbol{\omega} \times \qquad (21)$$

$$(d/dt)_{space} = (d/dt)_{(body)} + \omega \times \qquad \text{p 133 (4-102)}$$

Several comments are in order:

(1) *All* of Goldstein's equations shown above have all quantities *evaluated* in frame S (space). He uses what I call the "default meanings" such that $[\mathbf{G}]_S = \mathbf{G}$, $[\boldsymbol{\omega}]_S = \boldsymbol{\omega}$.

(2) His body notation is ambiguous I feel because it suggests that the term is evaluated in the body frame when in fact it is a body change which is evaluated in the space frame. His notation does not get that point across.

7. Better interpretation of dΩ and ω **TBD**

We showed in the last section that

$$dR(t) = [-i\, d\boldsymbol{\Omega} \bullet \mathbf{J}\,]\, R$$

$$dR(t)/dt \approx [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R \qquad \omega = d\boldsymbol{\Omega}/dt$$

Now consider the basis vector equation (after the change noted at the start of Section 3)

$$\mathbf{e'}_n(t) = R(t)\, \mathbf{e}_n$$

$$d\mathbf{e'}_n(t)/dt = dR(t)/dt\, \mathbf{e}_n = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R\, \mathbf{e}_n = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, \mathbf{e'}_n(t) = [-i\, \omega_k J_k\,]\, \mathbf{e'}_n(t) = -i\omega_k [J_k\, \mathbf{e'}_n(t)]$$

and taking a component (using i $(J_k)_{ij} = \varepsilon_{kij}$)

$$[d\mathbf{e'}_n(t)/dt]_i = -i\omega_k[J_k\,\mathbf{e'}_n(t)]_i = -i\omega_k(J_k)_{ij}\,[\mathbf{e'}_n(t)]_j = -\omega_k\,\varepsilon_{kij}\,[\mathbf{e'}_n(t)]_j = \varepsilon_{ikj}\,\omega_k\,[\mathbf{e'}_n(t)]_j$$

so that

$$d\mathbf{e'}_n(t)/dt = \boldsymbol{\omega} \times \mathbf{e'}_n(t)$$

Therefore $\boldsymbol{\omega}$ is the amount and direction of the instantaneous rate of rotation of any axis of the body frame. The entire body frame is rotating according to this vector $\boldsymbol{\omega}$. The above can be regarded as a generic equation which we would normally evaluate with all terms of the type $[..]_S$.

Perhaps here is another derivation of our basic result

$$\mathbf{V} = V'_n\mathbf{e'}_n$$

$$d\mathbf{V} = dV'_n\,\mathbf{e'}_n + V'_n\,d\mathbf{e'}_n$$

$$d\mathbf{V}/dt = (dV'_n/dt)\,\mathbf{e'}_n + V'_n\,d\mathbf{e'}_n/dt$$

$$d\mathbf{V}/dt = (d\mathbf{V}/dt)_{(body)} + V'_n\,\boldsymbol{\omega} \times \mathbf{e'}_n(t)$$

$$d\mathbf{V}/dt = (d\mathbf{V}/dt)_{(body)} + \boldsymbol{\omega} \times \mathbf{V} \qquad (6.1)$$

The key idea here is that

$(dV'_n/dt)\,\mathbf{e'}_n$ = rate of change of vector $\mathbf{V}$ seen in body frame = $(d\mathbf{V}/dt)_{(body)}$

which I think agrees with the pictures and discussion above. We can regard (6.1) as being an equation which we can evaluate in any frame, and we usually evaluate it in frame S in which case it becomes

$$[d\mathbf{V}/dt]_S = [(d\mathbf{V}/dt)_{(body)}]_S + [\boldsymbol{\omega}]_S \times [\mathbf{V}]_S \qquad \textit{(18b)}$$