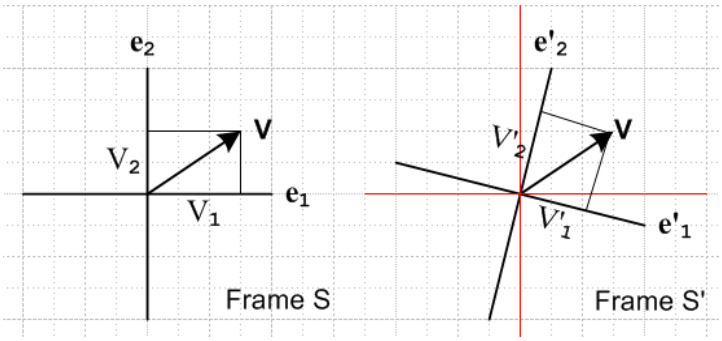
1. The Passive View **1**
2. The Active View **4**
3. Computation of dR/dt and (dR/dt)V. **4**
4. Computation of dV and dV/dt. **7**
4. Passive view : suppose frame S' (body) is moving and S (space) is at rest **11**
5. Passive View: what Newtonian Mechanics looks like in a non-inertial frame **16**
7. Comments on Goldstein's treatment of this subject. **19**
6. Comparison with Goldstein notation (p 133) TBD **20**
7. Better interpretation of dΩ and ω TBD **21**

1. The Passive View

We have some real-world physical vector **V** which is being observed by two observers, one in frame S and the other in frame S'. The frames have a common origin. Here is a picture



The vector **V** represents a certain specific aspect of some physical apparatus A. It could be some physical point in apparatus A, or it could be the velocity of some piece of apparatus A, etc. In this passive view, there is only one apparatus, but it is observed from two frames of reference. There is only one vector **V** and it is the same vector in both pictures. Define matrix R by

$\mathbf{e}_2 = R\,\mathbf{e}'_2$ and more generally $\mathbf{e}_n = R\,\mathbf{e}'_n$ forn = 1,2...N

In this 2D example, roughly we have $R = R_{\mathbf{e}3}(10°)$ (right hand rule).

The vector **V** has different components in the two frames. The general rule is this

$\mathbf{V} = V_n\mathbf{e}_n$ where $V_n = \mathbf{V} \bullet \mathbf{e}_n$
$\mathbf{V} = V'_n\mathbf{e}'_n$ where $V'_n = \mathbf{V} \bullet \mathbf{e}'_n$

One might wonder how the V'_n might be expressed in terms of the V_n and vice versa. Well,

$V'_n = \mathbf{V} \bullet \mathbf{e}'_n = [R\,\mathbf{V}] \bullet [R\,\mathbf{e}'_n]$ // valid for ANY rotation R

$= [R\,\mathbf{V}] \bullet \mathbf{e}_n$ // now assuming rotation shown above

$= [R\,V_m\mathbf{e}_m] \bullet \mathbf{e}_n$ // inserting first expansion above

$= V_m\ (R\mathbf{e}_m)\bullet\mathbf{e}_n$

$= V_m\,(R\mathbf{e}_m)_k(\mathbf{e}_n)_k$

$= V_m\,R_{ki}(\mathbf{e}_m)_i(\mathbf{e}_n)_k$

$= V_m\,R_{ki}\delta_{m,i}\delta_{n,k}$

$= R_{nm}\,V_m$

So we learn the that the components are related in this way

$V'_n = R_{nm}\,V_m$

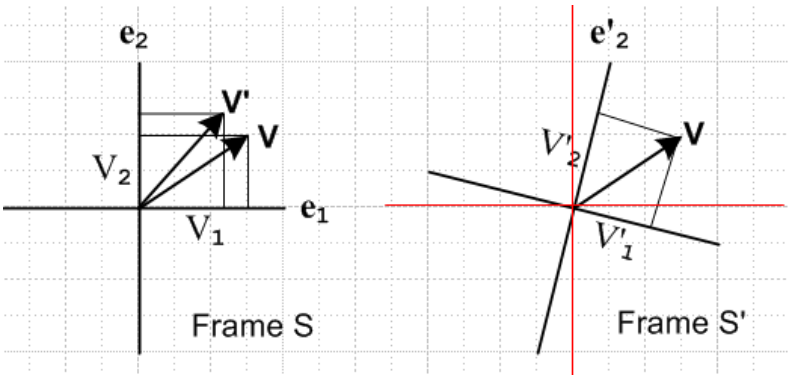
$V_n = R^{-1}_{nm}\,V'_m$

Now we are tempted to write the above equations in "vector notation" like so

$\mathbf{V}' = R\mathbf{V}$

$\mathbf{V} = R^{-1}\mathbf{V}'$.

Here are some words to go with the first equation: " If in frame S we were to rotate vector **V** into a new vector **V**' also in frame S , that new vector **V**' would have components V'_n which match those of vector **V** appearing in the S' picture above." So let's show this:

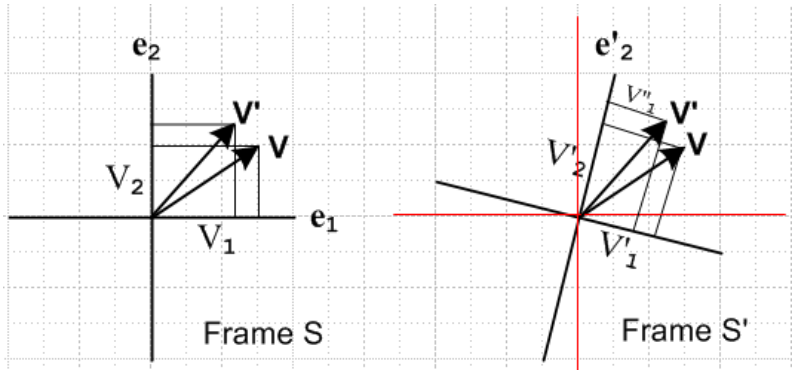


So clearly, the components of vector **V**' in frame S are (V'_1, V'_2), where (a,b,c...) indicates a column vector to save space. Now, let's write our two expansions for this new vector **V**'

$\mathbf{V}' = V'_n\mathbf{e}_n$ where $V_n = \mathbf{V}' \bullet \mathbf{e}_n = [R\mathbf{V}] \bullet [R\mathbf{e}'_n] = \mathbf{V} \bullet \mathbf{e}'_n = V'_n$

$\mathbf{V}' = V''_n\mathbf{e}'_n$ where $V''_n = \mathbf{V}' \bullet \mathbf{e}'_n = [R\mathbf{V}] \bullet [\mathbf{e}'_n] = [R^2\mathbf{V}] \bullet [R\mathbf{e}'_n] = [R^2\mathbf{V}] \bullet \mathbf{e}_n$

It is of course possible to draw this new vector **V**' in frame S' where it will have these new components which we call V''_1, V''_2 just to be able to distinguish them. Here is the above picture pair with this vector added:



The notion of vector **V'** in frame S will have a use below in our "active view", whereas the vector **V'** in frame S' will not be useful at all. But we wanted to get it drawn lest it be "mysterious".

At this point we might introduce the following notation:

$$[\mathbf{V}]_{\mathbf{F}} \equiv \{\, \mathbf{V} \bullet \mathbf{e}^{\mathbf{F}}_{\mathbf{n}} \}$$

Then we have these four examples

$[\mathbf{V}]_{\mathbf{S}} = (V_1, V_2)$ // first term is $\mathbf{V} \bullet \mathbf{e}_1$

$[\mathbf{V}]_{\mathbf{S}'} = (V'_1, V'_2)$ // first term is $\mathbf{V} \bullet \mathbf{e}'_1$

$[\mathbf{V}']_{\mathbf{S}} = (V'_1, V'_2)$ // first term is $\mathbf{V}' \bullet \mathbf{e}_1$

$[\mathbf{V}']_{\mathbf{S}'} = (V''_1, V''_2)$ // first term is $\mathbf{V}' \bullet \mathbf{e}'_1$

When we see the symbol **V** all by itself, we usually think of $\mathbf{V} = (V_1, V_2)$. That is what we meant by it in the notation **V'** = R**V** which appears above. And similarly, we thought of $\mathbf{V}' = (V'_1, V'_2)$ in this same equation. So by **"default"** we think of **V** as meaning $[\mathbf{V}]_{\mathbf{S}}$ and **V'** as meaning $[\mathbf{V}']_{\mathbf{S}}$.

V' = R**V** means for us above $[\mathbf{V}']_{\mathbf{S}} = R[\mathbf{V}]_{\mathbf{S}}$.

However, since **V'** = R**V** is a relationship between two vectors, it should be a valid relationship when viewed from *any* observer frame. Thus we expect this to also be true

$[\mathbf{V}']_{\mathbf{S}'} = R[\mathbf{V}]_{\mathbf{S}'}$ or $(V''_1, V''_2) = R\ (V'_1, V'_2)$ or $\mathbf{V}'' = R\,\mathbf{V}'$.

Suddenly we see yet another object **V"** which we could then add to our pictures above in an obvious way, but let's NOT add it. Think of **V"** as just a shorthand for (V''_1, V''_2). So here is how we can verify that the last line above is valid:

V' = R**V** => R**V'** = R(R**V**) => **V"** = R**V'** QED

Here are some other "default" meanings (but remember that these rotate oppositely to **V**)

$\mathbf{e}_{\mathbf{n}} = [\mathbf{e}_{\mathbf{n}}]_{\mathbf{S}}$

$\mathbf{e'}_n = [\mathbf{e'}_n]_S$

$\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$ means $[\mathbf{e'}_n]_S = R^{-1}[\mathbf{e}_n]_S$

and again the equation $\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$ must also be true in frame S' so that $[\mathbf{e'}_n]_{S'} = R^{-1}[\mathbf{e}_n]_{S'}$.

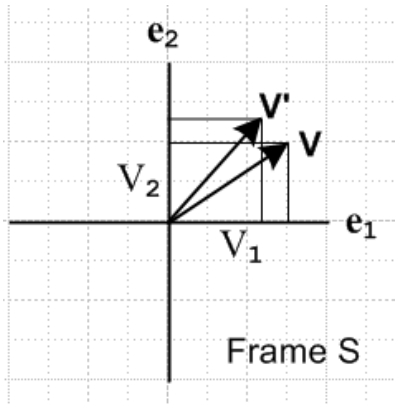
Comment:

(1) Notice that we don't put brackets around the rotation matrix R to get $[R]_F$ for some frame F. This matrix R is a set of numbers which relates **V'** to **V** in the equation **V'** = R**V.**

(2) What would tensor doc say? It would say $V'_a = R_{ab}V_b$ (dev notation) where R is the linearized transformation matrix. Recall that R lives neither in x-space nor in x'-space, but straddles the two spaces. There is never an object R'_{ab}. The equation $V'_a = R_{ab}V_b$ is not a "covariant equation" since it is not in one space and since R is not a tensor. So tensor doc agrees with comment (1).

2. The Active View

Instead of having two observers and one apparatus (which has a representative vector **V** in it), where the two observer frames are related by a rotation, now we have one observer frame S and we have two apparatuses which we call A and A'. This situation is basically that of the left picture above, which we repeat here:



In the passive view, the *frame* S' was rotated by amount R^{-1} relative to frame S, since $\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$.
In the active view, the *apparatus* A' is rotated by amount R relative to apparatus A.

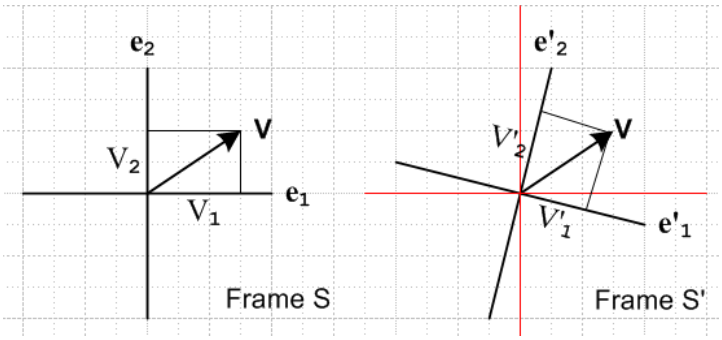
3. Computation of dR/dt and (dR/dt)V.

These results will be needed below, so we want to get them out of the way here so we don't clutter up issues which will be arising in the next section.

Right here we make the change $R \rightarrow R^{-1}$ for all R which appear in Sections 1 and 2. This means that now we have

$\mathbf{V} = R\mathbf{V}'$ and $\mathbf{e}'_n = R\,\mathbf{e}_n$.

So the axes of frame S' are now rotated by R relative to the axes of frame S. We want to think now of these axes of frame S' as being *in motion* relative to those of frame S.



We shall now derive our results two different ways.

Method 1. We can parameterize R in this way, where $\boldsymbol{\theta} = \boldsymbol{\theta}(t)$,

$$R = R(\boldsymbol{\theta}) = \exp(-i\boldsymbol{\theta}\bullet\mathbf{J}) = \exp(-i\theta_k J_k) \qquad \text{where} = i\,(J_k)_{ij} = \varepsilon_{kij} \ , \qquad (1)$$

Here the J_k are the three standard rotation generators in 3 dimensional space and the θ_k are the corresponding rotation parameters. One can regard this as a rotation by amount θ about axis $\hat{\boldsymbol{\theta}}$.

Now consider frame S' to be tumbling around as seen from frame S. At time t, the unit vectors of frame S' have reached this position,

$$\mathbf{e}'_n(t) = R(\boldsymbol{\theta}(t))\,\mathbf{e}_n \ .$$

During the next interval dt of time, the following small tumble of these axes occurs

$$\mathbf{e}'_n(t+dt) = R(d\boldsymbol{\varphi})\,\mathbf{e}'_n(t) \ .$$

Therefore at time t the vector $d\boldsymbol{\varphi}$ describes the "instantaneous axis of rotation" of the $\mathbf{e}'_n(t)$. We then have

$$\mathbf{e}'_n(t+dt) = R(d\boldsymbol{\varphi})\,R(\boldsymbol{\theta}(t))\,\mathbf{e}_n \equiv R(t+dt)\,\mathbf{e}_n \ ,$$

where we define R(t+dt) to get the total rotation as shown. Then consider

$$dR = R(t+dt) - R(t) = R(d\boldsymbol{\varphi})R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}) = [\,R(d\boldsymbol{\varphi}) - 1\,]\,R(\boldsymbol{\theta}) \approx [\,-i\,d\boldsymbol{\varphi}\bullet\mathbf{J}\,]\,R$$

Then divide by dt to get

$$dR/dt = [\,-i\,\boldsymbol{\omega}\bullet\mathbf{J}\,]\,R(\boldsymbol{\theta}) \qquad \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt$$

If we apply this to some arbitrary vector **V**, we get

$(dR/dt)\mathbf{V} = [-i\,\boldsymbol{\omega} \bullet \mathbf{J}\,]\,R\mathbf{V} = [-i\,\omega_k J_k\,]\,R\mathbf{V}$ // on next line we will use $i\,(J_k)_{ij} = \varepsilon_{kij}$

$[(dR/dt)\mathbf{V}]_i = [-i\,\omega_k (J_k)_{ij}\,]\,[R\mathbf{V}]_j = -\,\omega_k \varepsilon_{kij}\,[R\mathbf{V}]_j = \varepsilon_{ikj}\,\omega_k [R\mathbf{V}]_j$

so

$(dR/dt)\mathbf{V} = \boldsymbol{\omega} \times (R\mathbf{V})$ (1)

Method 2. Start with,

$\mathbf{V}(t) = R(t)\mathbf{V}'(t)$.

This equation tells us (using the default interpretation mentioned above) the S space components of a body frame vector **V'**, and we allow that this body frame vector may be moving within S', hence **V'**(t) .

Given some **V'**(t), we want to apply an extra small rotation $R(d\boldsymbol{\varphi})$ so that we get

$\mathbf{V}'(t+dt) = R(d\boldsymbol{\varphi})\,\mathbf{V}'(t) = R(d\boldsymbol{\varphi})\,R(t)\,\mathbf{V} \equiv R(t+dt)\,\mathbf{V}$.

We argue that in going from **V'**(t) to **V'**(t+dt), we are rotating **V'**(t) about an "instantaneous axis of rotation" $d\boldsymbol{\varphi}$. Then we find that

$dR = R(t+dt) - R(t) = R(d\boldsymbol{\varphi})R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}) = [\,R(d\boldsymbol{\varphi}) - 1]\,R(\boldsymbol{\theta}) \approx [\,-i\,d\boldsymbol{\varphi} \bullet \mathbf{J}\,]\,R$.

This is the same dR as shown in Method 1, and we arrive by the same path at the same conclusion,

$(dR/dt)\mathbf{V} = \boldsymbol{\omega} \times (R\mathbf{V})$. (1)

Comment: The main idea here is this:

$R(t+dt) = R(d\boldsymbol{\varphi})R(\boldsymbol{\theta}(t))$

where we then interpret $d\boldsymbol{\varphi}$ as the instantaneous axis of rotation at time t of $\mathbf{e'}_n(t)$ or of **V'**(t).

One might well wonder about this alternative approach,

$R_1(t+dt) = R(\boldsymbol{\theta} + d\boldsymbol{\theta})$.

In this case one can consider the product

$R(\boldsymbol{\theta} + d\boldsymbol{\theta})R^{-1}(\boldsymbol{\theta})$.

Since this is the product of two rotations, it must be a rotation. And since the product approaches 1 as $d\theta \rightarrow 0$, it must be a small rotation. Write it then as follows:

$R(\boldsymbol{\theta} + d\boldsymbol{\theta})R^{-1}(\boldsymbol{\theta}) = R(d\boldsymbol{\Omega})$.

In working out the details, one will find that $d\boldsymbol{\Omega} = \mathbf{f}(\hat{\boldsymbol{\theta}},\hat{d\boldsymbol{\theta}})\, d\theta$ which says $d\Omega$ is proportional to $d\theta$, but the direction of vector $d\boldsymbol{\Omega}$ is in general some complicated function of the directions of $\boldsymbol{\theta}$ and $d\boldsymbol{\theta}$ (unless they happen to point in the same direction). Nevertheless, since the above says $R(\boldsymbol{\theta} + d\boldsymbol{\theta}) = R(d\boldsymbol{\Omega})R(\boldsymbol{\theta})$ we can compute

$$dR = R_1(t+dt) - R(t) = R(d\boldsymbol{\Omega})R(\boldsymbol{\theta}) - R(\boldsymbol{\theta}) = [\, R(d\boldsymbol{\Omega}) - 1]\, R(\boldsymbol{\theta}) \approx [\, -i\, d\boldsymbol{\Omega}\bullet\mathbf{J}\,]\, R\ .$$

and then by the same path as above we would find that

$$dR/dt = [\, -i\, \boldsymbol{\omega}_1\bullet\mathbf{J}\,]\, R(\boldsymbol{\theta}) \qquad\qquad \boldsymbol{\omega}_1 \equiv d\boldsymbol{\theta}/dt$$

$$(dR/dt)\mathbf{V} = \boldsymbol{\omega}_1 \times (R\mathbf{V})$$

Although these results are valid, the vectors $d\boldsymbol{\theta}$ and $\boldsymbol{\omega}_1$ do not represent the instantaneous axis of rotation in the sense described above, and for this reason these results are not very useful.

4. Computation of dV and dV/dt.

In the first line of Section 3 Method 2 above we had this statement,

$$\mathbf{V}(t) = R(t)\mathbf{V}'(t)\ .$$

(a) The case $\mathbf{V}'(t) = \mathbf{V}'$. If we momentarily regard vector $\mathbf{V}'(t)$ as *stationary* in frame S', so that $\mathbf{V}'(t) = \mathbf{V}'$, then

$$\mathbf{V}(t) = R(t)\mathbf{V}'$$

$$\mathbf{V}(t+dt) = R(t+dt)\mathbf{V}'$$

$$d\mathbf{V} = \mathbf{V}(t+dt) - \mathbf{V}(t) = [\, R(t+dt) - R(t)]\, \mathbf{V}' = dR\, \mathbf{V}' = [\, -i\, d\boldsymbol{\varphi}\bullet\mathbf{J}\,]\, R\, \mathbf{V}' = d\boldsymbol{\varphi} \times [R\mathbf{V}'] = d\boldsymbol{\varphi} \times \mathbf{V},$$

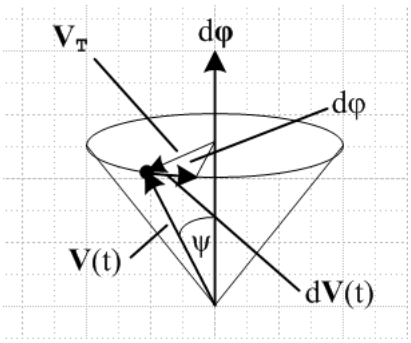
where we are just following the steps used above with regard to $i\,(J_k)_{ij} = \varepsilon_{kij}$. That is to say, if $\mathbf{V}'(t) = \mathbf{V}'$ is stationary in frame S', then we have

$$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V} \qquad\qquad (*)$$

and

$$d\mathbf{V}/dt = \boldsymbol{\omega} \times \mathbf{V} \qquad \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt\ .$$

In this case, since frame S' is instantaneously rotating by $d\boldsymbol{\varphi}$ (or $\boldsymbol{\omega}$) relative to frame S, and since $\mathbf{V}'$ is motionless in frame S', the vector $\mathbf{V}$ seen in frame S is instantaneously traveling in a circle in frame S. Here is the usual picture associated with this idea ,



The tip of vector **V** is instantaneously rotating in circular motion where the circle has radius Vsinψ. In fact, the vector $\mathbf{V_T}$ (Transverse) from the rotation axis out to the tip of vector **V** is this

$$\mathbf{V_T} = \mathbf{V} - V\cos\psi\, \hat{d\boldsymbol{\varphi}} \ .$$

This vector $\mathbf{V_T}$ is the radius vector of the circular motion. The motion is instantaneously circular at time t if the following is true

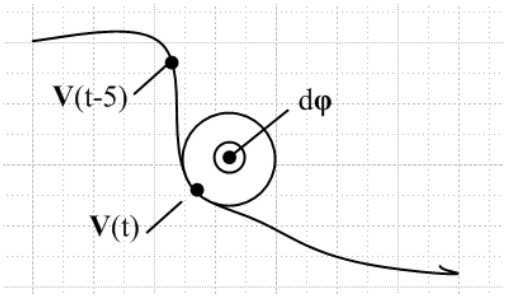
$$d\mathbf{V} \bullet \mathbf{V_T} = 0 \ .$$

We can verify that motion (*) fulfills this requirement,

$$d\mathbf{V} \bullet \mathbf{V_T} = (d\boldsymbol{\varphi} \times \mathbf{V}) \bullet (\mathbf{V} - V\cos\psi\, \hat{d\boldsymbol{\varphi}}) = 0 - 0 = 0$$

and this is why $d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V}$ describes instantaneous circular motion.

Now $d\boldsymbol{\varphi}(t)$ and $\boldsymbol{\omega}(t)$ may have some arbitrary time dependence, as determined by the mechanism that is driving the "platform" which is frame S'. These vectors may be changing in both magnitude and direction as time progresses. In this case, the actual trajectory of **V**(t) in frame S won't be a circular path, but will in general be some complicated three dimensional curve (all this time **V**' = constant in frame S'). At any instant in time, however, a small piece of this curve near the point **V**(t) will be planar and will have some curvature in that plane, and **V**(t) will be doing instantaneous circular motion relative to $d\boldsymbol{\varphi}$ at that instant,



This notion can be understood in terms of an iterative program of advancement where the first few steps of this program starting at time t would be

$\mathbf{V}(t)$

$d\mathbf{V}(t) = d\boldsymbol{\varphi}(t) \times \mathbf{V}(t)$
$\mathbf{V}(t+dt) = \mathbf{V}(t) + d\mathbf{V}(t)$

$d\mathbf{V}(t+dt) = d\boldsymbol{\varphi}(t+dt) \times \mathbf{V}(t+dt)$
$\mathbf{V}(t+2dt) = \mathbf{V}(t+dt) + d\mathbf{V}(t+dt)$ etc.

In this manner the trajectory $\mathbf{V}(t)$ is generated, and at any time one maintains the instantaneous circular motion indicated by

$d\mathbf{V}(t) = d\boldsymbol{\varphi}(t) \times \mathbf{V}(t)$

(b) The case $\mathbf{V'}(t) = \mathbf{V'}(t)$. We just redo the steps done above, adding changes as needed:

$\mathbf{V}(t) = R(t)\mathbf{V'}(t)$

$\mathbf{V}(t+dt) = R(t+dt)\mathbf{V'}(t+dt)$

$d\mathbf{V'} = \mathbf{V'}(t+dt) - \mathbf{V'}(t)$

$d\mathbf{V} = \mathbf{V}(t+dt) - \mathbf{V}(t)$

$= R(t+dt)\mathbf{V'}(t+dt) - R(t)\mathbf{V'}(t)$

$= R(t+dt)[\mathbf{V'}(t) + d\mathbf{V'}] - R(t)\mathbf{V'}(t)$

$= [R(t+dt) - R(t)]\,\mathbf{V'}(t) + R(t+dt)\,d\mathbf{V'}$.

In the last term we will use

$R(t+dt) = R(d\boldsymbol{\varphi})R(\boldsymbol{\theta}(t)) = [\,1 - i\,d\boldsymbol{\varphi}\bullet\mathbf{J}\,]\,R$

$R(t+dt)\,d\mathbf{V'} = [1 - i\,d\boldsymbol{\varphi}\bullet\mathbf{J}\,]\,R\,d\mathbf{V'} = R\,d\mathbf{V'} - i\,d\boldsymbol{\varphi}\bullet\mathbf{J}\,R\,d\mathbf{V'} = Rd\mathbf{V'} + d\boldsymbol{\varphi} \times (RdV')$

For the first two terms we will use this fact shown earlier

$[R(t+dt) - R(t)]\,\mathbf{V'}(t) = d\boldsymbol{\varphi} \times \mathbf{V}$

Thus we end up with

$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V} + Rd\mathbf{V'} + d\boldsymbol{\varphi} \times (Rd\mathbf{V'})$

Here, the change $d\mathbf{V'}$ arises because $\mathbf{V'}$ is moving within frame S'. At this point, we need some way to identify the change of $\mathbf{V'}$ within frame S', so let's just define

$d\mathbf{V}'_{\mathbf{body}} \equiv d\mathbf{V}'(t)$.

Then we can define $d\mathbf{V}_{\mathbf{body}}$ (no prime) to be this same change, but expressed in frame S components,

$$d\mathbf{V}_{\mathbf{body}} \equiv R\, d\mathbf{V}'_{\mathbf{body}} = R\, d\mathbf{V}'(t) \quad .$$

Therefore we have shown that

$$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V} + d\mathbf{V}_{\mathbf{body}} + d\boldsymbol{\varphi} \times d\mathbf{V}_{\mathbf{body}}$$

$$d\mathbf{V}/dt = \boldsymbol{\omega} \times \mathbf{V} + d\mathbf{V}_{\mathbf{body}}/dt + \boldsymbol{\omega} \times (d\mathbf{V}_{\mathbf{body}}/dt)dt$$

and we then drop the last term in the limit $dt \rightarrow 0$ and get these famous results

$$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V} + d\mathbf{V}_{\mathbf{body}}$$

$$d\mathbf{V}/dt = \boldsymbol{\omega} \times \mathbf{V} + d\mathbf{V}_{\mathbf{body}}/dt$$

Had we not added the body label, the two $d\mathbf{V}$ objects in the first line above would be indistinguishable.

How might we isolate the two terms in the last equation above?

The first term is all there is if it happens that $d\mathbf{V}_{\mathbf{body}}/dt = 0$ which means $d\mathbf{V}'_{\mathbf{body}}/dt = 0$ in terms of frame S' components. This in turn could happen if $\mathbf{V}'_{\mathbf{body}} = \mathbf{V}'$, a constant, or it could happen just because we happen to have $d\mathbf{V}'_{\mathbf{body}}/dt = 0$ at some particular time t. Note that we could have $\mathbf{V}'_{\mathbf{body}} = 0$ while at the same time $d\mathbf{V}'_{\mathbf{body}}/dt \neq 0$.

The objects $d\mathbf{V}'_{\mathbf{body}}/dt$ and $d\mathbf{V}_{\mathbf{body}}/dt$ could in theory be functions of ω, since we allow an arbitrary body motion. For this reason, just setting $\omega = 0$ to kill off the $\boldsymbol{\omega} \times \mathbf{V}$ term won't necessarily expose all of the $d\mathbf{V}_{\mathbf{body}}/dt$ term, since $\omega = 0$ might kill off some part of $d\mathbf{V}_{\mathbf{body}}/dt$. Therefore we cannot say that the second term is all there is if we set $\omega = 0$. It is of course all there is if we set $\boldsymbol{\omega} \times \mathbf{V} = 0$.

When both terms are present, we loose the simple interpretation of vector $\mathbf{V}$ instantaneously rotating about the axis $\hat{\boldsymbol{\omega}}$.

Notice that we could define $\mathbf{V}_{\mathbf{body}} = R(t)\, \mathbf{V}'(t)$, but this is in fact $\mathbf{V}(t)$. So we have this situation

$$\mathbf{V} = \mathbf{V}_{\mathbf{body}} \qquad // = R(t)\mathbf{V}'(t)$$

but

$$d\mathbf{V} \neq d\mathbf{V}_{\mathbf{body}} \quad .$$

In fact, we just showed that

$$d\mathbf{V} = d\boldsymbol{\varphi} \times \mathbf{V} + d\mathbf{V}_{\mathbf{body}} \quad .$$

As *defined* above, $d\mathbf{V}_{\mathbf{body}}$ is the "intrinsic" change in **V** within frame S' (but converted to S components) as if there were no frame rotation, whereas d**V** *includes* the effect of this rotation and gives the correct total change in **V** in frame S.

It is convenient to make this notational definition,

$$(d\mathbf{V}/dt)_{\mathbf{body}} = (d/dt)_{\mathbf{body}}\mathbf{V} \equiv (d\mathbf{V}_{\mathbf{body}}/dt) = (d/dt)\,\mathbf{V}_{\mathbf{body}}$$

because then we can write (*) as

$$(d/dt)\,\mathbf{V} = \boldsymbol{\omega} \times \mathbf{V} + (d/dt)_{\mathbf{body}}\mathbf{V}$$

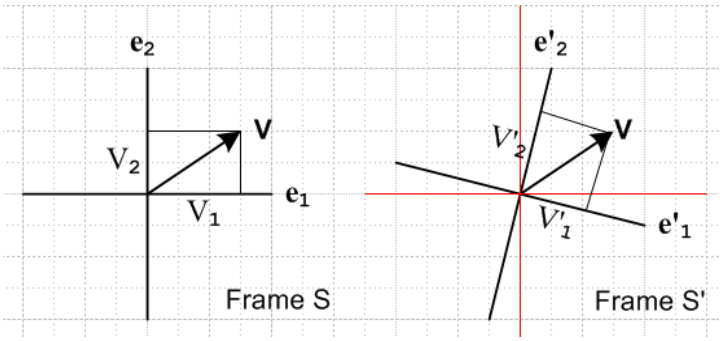
and since this applies to any vector **V**, we can write it in the following operator notation,

$$(d/dt) = (d/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times$$

See for example equations 4-99 through 4-102 of Goldstein (page 133). Note that $d\boldsymbol{\Omega}_{\mathbf{Goldstein}} = d\boldsymbol{\varphi}$.

We are now ready to see some Examples.

4. Passive view : suppose frame S' (body) is moving and S (space) is at rest .



space frame body frame

Right here we small make the convenience change $R \rightarrow R^{-1}$ for all R which appear in Sections 1 and 2 (but not Section 3). After this change, we have from Section 1 above,

$$\mathbf{e'}_n = R(\boldsymbol{\theta})\,\mathbf{e}_n \qquad \mathbf{e}_n = R^{-1}(\boldsymbol{\theta})\,\mathbf{e'}_n \tag{1}$$

where $R(\boldsymbol{\theta})$ describes the current position (t = t, say) of the S' frame relative to the S frame. Now consider some vector **V** lying in the body S' frame. We know again from Section 1 (with change),

$$\mathbf{V}(t) = R(\boldsymbol{\theta})\mathbf{V'}(t) \tag{2}$$

which we can interpret in this hybrid manner (see end of Section 1 above for this notation),

$$[\mathbf{V}(t)]_{S} = R(\boldsymbol{\theta})\,[\mathbf{V}(t)]_{S'} \quad . \tag{3}$$

The object $[\mathbf{V}(t)]_{S'}$ is a vector $\mathbf{V}$ in the body frame S', and this vector may be moving in frame S'. Viewed from the space frame, what one sees is $[\mathbf{V}(t)]_{S}$.

We now assume that frame S' is "instantaneously rotating" by some vector amount $d\boldsymbol{\theta}$ in time dt. If we momentarily assume that $[\mathbf{V}(t)]_{S'}$ is not moving within frame S', we would find that there is still a change in frame S, namely

$$d[\mathbf{V}(t)]_{S} = R(d\boldsymbol{\theta})\,[\mathbf{V}(t)]_{S'} - [\mathbf{V}(t)]_{S'} \tag{4}$$

Now a rotation can be written in terms of the rotation generator matrices in this manner

$$R(\boldsymbol{\theta}) = \exp(-i\,\boldsymbol{\theta} \bullet \mathbf{J}) \tag{5}$$

where

$$i\,(J_k)_{ij} = \varepsilon_{kij} \; . \tag{6}$$

Therefore, keeping only the first two terms in a Taylor expansion about $\boldsymbol{0}$,

$$R(d\boldsymbol{\theta}) = \exp(-i\,d\boldsymbol{\theta} \bullet \mathbf{J}) \approx 1 - i\,d\boldsymbol{\theta} \bullet \mathbf{J} \; . \tag{7}$$

Using this in (4) gives

$$d[\mathbf{V}(t)]_{S} = R(d\boldsymbol{\theta})\,[\mathbf{V}(t)]_{S'} - [\mathbf{V}(t)]_{S'} = (1 - i\,d\boldsymbol{\theta} \bullet \mathbf{J})\,[\mathbf{V}(t)]_{S'} - [\mathbf{V}(t)]_{S'}$$

$$= - i\,(d\boldsymbol{\theta} \bullet \mathbf{J})\,[\mathbf{V}(t)]_{S'}$$

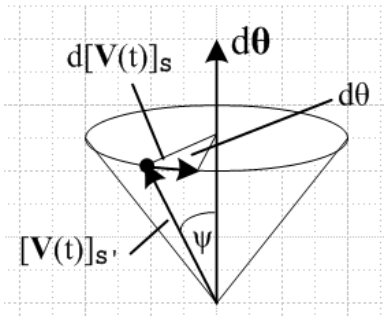
Writing this in components gives

$$(d[\mathbf{V}(t)]_{S})_{i} = - i\,d\theta_{k}\,(J_k)_{ij}\,([\mathbf{V}(t)]_{S'})_{j} = -i\,\varepsilon_{kij}\,d\theta_{k}\,([\mathbf{V}(t)]_{S'})_{j} = + i\,\varepsilon_{ikj}\,d\theta_{k}([\mathbf{V}(t)]_{S'})_{j}$$

which when written back in vector notation says

$$d[\mathbf{V}(t)]_{S} = d\boldsymbol{\theta} \times [\mathbf{V}(t)]_{S'} \tag{11}$$

Usually one simply starts with equation (11) with the following justification: If we rotate $[\mathbf{V}(t)]_{S'}$ by amount $d\theta$ about an axis in the $d\boldsymbol{\theta}$ direction, then $d[\mathbf{V}(t)]_{S}$ is as shown in this picture,



and then the radius out from the axis is the length of the vector $[\mathbf{V}(t)]_{S'}$ times the sine of the cone half angle ψ, and the direction of $d[\mathbf{V}(t)]_S$ is perpendicular to both $d\boldsymbol{\theta}$ and $[\mathbf{V}(t)]_S$ and that justifies (11) since the cross product involves sinψ. Our reason for not just starting with (11) will appear below.

In the development above, we assumed that $[\mathbf{V}(t)]_{S'}$ was not changing within frame S', but we now want to relax that constraint and allow it to move. From this movement alone we get

$$d[\mathbf{V}(t)]_S = R(\theta)\, d[\mathbf{V}(t)]_{S'} \quad . \qquad (12)$$

Notice the presence of matrix R(θ) which maps the change $d[\mathbf{V}(t)]_{S'}$ expressed in frame S' coordinates to the coordinates of frame S.

We can then add both the two motion effects just reviewed to get

$$d[\mathbf{V}(t)]_S = R(\theta)\, d[\mathbf{V}(t)]_{S'} + d\boldsymbol{\theta} \times [\mathbf{V}(t)]_{S'} \qquad (13)$$

If we now divide all three terms by dt, we get

$$(d[\mathbf{V}(t)]_S/dt) = R(\theta)\, (d[\mathbf{V}(t)]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}(t)]_{S'} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt \qquad (14)$$

If we now assume that at our time t of interest, frames S and S' are "lined up", then $\boldsymbol{\theta} = 0$ and $R(\boldsymbol{\theta}) = 1$ and we have

$$(d[\mathbf{V}(t)]_S/dt) = (d[\mathbf{V}(t)]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}(t)]_{S'} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt \text{ ; lined up} \qquad (15)$$

But if the frames are lined up, then $[\mathbf{V}(t)]_{S'} = [\mathbf{V}(t)]_S$ and we might just call it $\mathbf{V}(t)$. And we slightly change the location of the dt's in the derivatives, so (15) becomes,

$$(d/dt\, [\mathbf{V}(t)]_S) = (d/dt\, [\mathbf{V}(t)]_{S'}) + \boldsymbol{\omega} \times \mathbf{V}(t) \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt \text{ ; lined up} \qquad (16)$$

Now we make the following definition where F is a frame (either S or S')

$$(d/dt)_F\, [\mathbf{V}(t)] \equiv (d/dt\, [\mathbf{V}(t)]_F)$$

Then (16) can be written as

$$(d/dt)_S\,[\mathbf{V}(t)] = (d/dt)_{S'}\,[\mathbf{V}(t)] + \boldsymbol{\omega} \times \mathbf{V}(t) \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt\,; \quad \text{lined up} \qquad (17)$$

Since this is true for any vector **V**, we can write it as an operator equation

$$(d/dt)_S = (d/dt)_{S'} + \boldsymbol{\omega} \times \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt\,; \quad \text{lined up} \qquad (18)$$

Example #1. Suppose **V** = **r**, the position vector. Then $[\mathbf{r}]_{S'}$ is the position of some point in frame S', and $[\mathbf{r}]_S$ is the corresponding position in frame S, and we have

$$[\mathbf{r}]_S = R(\boldsymbol{\theta})\,[\mathbf{r}]_{S'} = [\mathbf{r}]_{S'} \text{ if "lined up"} = \mathbf{r}$$

Then (16) reads

$$(d/dt\,[\mathbf{r}]_S) = (d/dt\,[\mathbf{r}]_{S'}) + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt\,; \quad \text{lined up} \qquad (19)$$

We then make the obvious definitions

$$[\mathbf{v}]_S = (d[\mathbf{r}]_S/dt) = \text{velocity of point } \mathbf{r} \text{ in the space frame S} \qquad (20a)$$

$$[\mathbf{v}]_{S'} = (d[\mathbf{r}]_{S'}/dt) = \text{velocity of point } \mathbf{r} \text{ in the body frame S'} \qquad (20b)$$

and then (19) becomes

$$[\mathbf{v}]_S = [\mathbf{v}]_{S'} + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt\,; \quad \text{lined up} \qquad (21)$$

STOP for PARADOX. Above equation (16) we said that $[\mathbf{V}(t)]_{S'} = [\mathbf{V}(t)]_S$ when the two frames are lined up, so applying that to **V** = **v**, we ought to have $[\mathbf{v}]_S = [\mathbf{v}]_{S'}$, but this disagrees with (21).

Example #2. Suppose **V** = **v**, the velocity vector. Then $[\mathbf{v}]_{S'}$ is the velocity of some point in frame S', and $[\mathbf{v}]_S$ is the corresponding velocity in frame S. Then (16) reads

$$(d/dt\,[\mathbf{v}]_S) = (d/dt\,[\mathbf{v}]_{S'}) + \boldsymbol{\omega} \times \mathbf{v} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt\,; \quad \text{lined up} \qquad (22)$$

We then make the obvious definitions

$$[\mathbf{a}]_S = (d[\mathbf{v}]_S/dt) = \text{acceleration of point } \mathbf{r} \text{ in the space frame S} \qquad (23a)$$

$$[\mathbf{a}]_{S'} = (d[\mathbf{v}]_{S'}/dt) = \text{acceleration of point } \mathbf{r} \text{ in the body frame S'} \qquad (23b)$$

and then (22) becomes

$$[\mathbf{a}]_S = [\mathbf{a}]_{S'} + \boldsymbol{\omega} \times \mathbf{v}$$

We see that this is the wrong result! The reason is this: Look back at (14)

$$(d[\mathbf{V}(t)]_S/dt) = R(\theta)\,(d[\mathbf{V}(t)]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}(t)]_{S'} \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt$$

and apply this with $\mathbf{V} = \mathbf{r}$ and use the above velocity definitions to get

$$[\mathbf{v}(t)]_S = R(\boldsymbol{\theta})\,[\mathbf{v}(t)]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\theta}/dt$$

What we learn here is that velocity **v** is not really a "vector" because a vector must transform according to this equation,

$$[\mathbf{V}(t)]_S = R(\boldsymbol{\theta})\,[\mathbf{V}(t)]_{S'} \quad . \qquad (3)$$

Since **v** is not a true vector under the transformation from frame S' to frame S, we cannot make use of (22).

Example #3. Suppose $\mathbf{V} = \mathbf{L} = \mathbf{r} \times p = m\,\mathbf{r} \times \mathbf{v}$, the usual angular momentum vector. Then

$$[\mathbf{L}]_S = m\,[\mathbf{r}]_S \times [\mathbf{v}]_S$$

I show in separate notes that

$$[\mathbf{L}]_S = R[\mathbf{L}]_{S'} + m\,[\mathbf{r}]_S \times (\boldsymbol{\omega} \times [\mathbf{r}]_S)$$

and therefore **L** is not a true vector and again (22) is not allowed. But both Goldstein and Marion say it is allowed! So my paradox remains.

We could have alternately described this same change in a different manner,

$$[\mathbf{V}(t+dt)]_S = R(\boldsymbol{\theta} + d\boldsymbol{\psi})\,[\mathbf{V}(t)]_{S'}$$

That means that $[\mathbf{V}(t+dt)]_{S'} = R(d\boldsymbol{\theta})\,[\mathbf{V}(t)]_{S'}$

At this point, we wish to apply a small rotation $R(d\boldsymbol{\theta})$ to $[\mathbf{V}(t)]_{S'}$ and see what change that causes in $[\mathbf{V}]_S$

Comment: Notice that, in reference to equation (2), we said that **V** was some "vector". By definition, a vector with respect to transformation R is something that "transforms as a vector according to (2)". When R is time-dependent, things that one normally thinks of as being "vectors" are in fact not vectors because their components do not satisfy (2). For example, the position vector **r** is a vector under R(t), but as we shall see, the velocity vector **v** is not a vector under R(t), though it is a vector if R is not time-dependent.

Now some time dt goes by and we get some changes:

$$d[\mathbf{V}]_S = R\, d[\mathbf{V}]_{S'} + dR\, [\mathbf{V}]_{S'} \quad . \tag{4}$$

The *first* term arises because there was a change *within* the body S' frame in our vector **V**, and that change then gets mapped to the space frame S by rotation R. The *second* term arises because even if $[\mathbf{V}]_{S'}$ were not changing within the body frame, the body frame is moving, which causes a change of $[\mathbf{V}]_S$ in the space frame.

Now divide by dt to get

$$d[\mathbf{V}]_S/dt = R\, (d[\mathbf{V}]_{S'}/dt) + (dR/dt)\, [\mathbf{V}]_{S'} \quad . \tag{5}$$

Then use result 3.(4) for (dR/dt) acting on a vector to get

$$(d[\mathbf{V}]_S/dt) = R\, (d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times (R\, [\mathbf{V}]_{S'}) \quad . \tag{6}$$

Then use result (3) in the last term of (6) to get

$$(d[\mathbf{V}]_S/dt) = R\, (d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}]_S \quad . \tag{7}$$

We could apply d/dt to equation (7) and in so doing obtain four terms. Rather than do that here, we will do it in the special application of **V** = **r** of the next section.

5. Passive View: what Newtonian Mechanics looks like in a non-inertial frame

Now as a particular application of 4.(7) above, let **V** = **r**, the position vector :

$$(d[\mathbf{r}]_S/dt) = R\, (d[\mathbf{r}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{r}]_S \tag{1}$$

We next identify

$$[\mathbf{v}]_S = (d[\mathbf{r}]_S/dt) = \text{velocity of point } \mathbf{r} \text{ in the space frame S} \tag{2a}$$

$$[\mathbf{v}]_{S'} = (d[\mathbf{r}]_{S'}/dt) = \text{velocity of point } \mathbf{r} \text{ in the body frame S'} \tag{2b}$$

so (1) can be rewritten as

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S \quad . \tag{3}$$

Next, apply d/dt to equation (3) to get

$$(d[\mathbf{v}]_S/dt) = (dR/dt)\,[\mathbf{v}]_{S'} + R\,(d[\mathbf{v}]_{S'}/dt) + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (d[\mathbf{r}]_S/dt) \quad . \tag{4}$$

We now identify these accelerations,

$$[\mathbf{a}]_S = (d[\mathbf{v}]_S/dt) = \text{acceleration of point } \mathbf{r} \text{ in the space frame S} \tag{5a}$$

$$[\mathbf{a}]_{S'} = (d[\mathbf{v}]_{S'}/dt) = \text{acceleration of point } \mathbf{r} \text{ in the body frame S'} \tag{5b}$$

so that, along with (2a) in the last term, we get

$$[\mathbf{a}]_S = (dR/dt)\,[\mathbf{v}]_{S'} + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times [\mathbf{v}]_S \quad . \tag{6}$$

Then use 3.(4) for (dR/dt) acting on a vector to get

$$[\mathbf{a}]_S = \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times [\mathbf{v}]_S \quad . \tag{7}$$

Finally, use (3) to replace $[\mathbf{v}]_S$ in the last term :

$$[\mathbf{a}]_S = \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + R\,[\mathbf{a}]_{S'} + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \quad . \tag{8}$$

We see that two $\boldsymbol{\omega}$ x terms are the same, so we add them, and then put $R\,[\mathbf{a}]_{S'}$ first to get

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + (d\boldsymbol{\omega}/dt) \times [\mathbf{r}]_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \tag{9}$$

This then is the final and very general result relating the acceleration in the space frame S to that in the body frame S'.

Obviously, if $\boldsymbol{\omega}$ is constant in time (such as the rotation of the earth) the above simplifies to

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times (R[\mathbf{v}]_{S'}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}]_S) \tag{10}$$

If *in addition* it happens that the space and body frames are exactly aligned at the instant t which is of interest to us, we may replace R = 1 in both places in (10). Then of course $[\mathbf{r}]_S = [\mathbf{r}]_{S'}$ and we might as well just call it $\mathbf{r}$. Then we get at this instant in time,

$$[\mathbf{a}]_S = [\mathbf{a}]_{S'} + 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \tag{11}$$

In the space frame S (inertial since at rest) we have Newton's law being valid,

$$[\mathbf{F}]_S = m\,[\mathbf{a}]_S \,. \tag{12}$$

Using (11) we can say that

$$[\mathbf{F_{eff}}]_{S'} \equiv m\,[\mathbf{a}]_{S'} = m\,[\mathbf{a}]_S - 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= [\mathbf{F}]_S - 2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad (13)$$

This says that we can pretend there is a Newton's law in the (non-inertial) rotating frame,

$$[\mathbf{F_{eff}}]_{S'} \equiv m\,[\mathbf{a}]_{S'}$$

as long as we are *willing to include* the two "fictitious forces" shown in (13). The second fictitious force is just the usual centrifugal force, pointing exactly away from the rotation axis. The first term is the new object of interest : the Coriolis force $-2\,\boldsymbol{\omega} \times [\mathbf{v}]_{S'}$.

6. A Paradox and its Resolution

This small section hopefully illuminates the Comment made above equation 4.(4). Note that several equations in this section are wrong so please don't use them!

Equation 4.(7) is supposedly valid for any vector **V**. What happens if we take **V** = **v**, the velocity of a particle. It then says,

$$(d[\mathbf{V}]_S/dt) = R\,(d[\mathbf{V}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{V}]_S \qquad \textit{4.(7)}$$

$$(d[\mathbf{v}]_S/dt) = R\,(d[\mathbf{v}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{v}]_S \qquad \textit{4.(7) with } \boldsymbol{V} = \boldsymbol{v}$$

We identify accelerations as in 5.(5) and this then says

$$[\mathbf{a}]_S = R\,[\mathbf{a}]_{S'} + \boldsymbol{\omega} \times [\mathbf{v}]_S \; .$$

Comparison with 5.(6) shows that this last equation is wrong because we are missing two terms! So what is going on here?? If we trace backwards from 4.(7) we get to 4.(3) which says, for **V** = **v**,

$$\mathbf{v} = R\,\mathbf{v}'$$

which we could be written, according to the end of Section 1,

$$[\mathbf{v}]_S = R\;[\mathbf{v}]_{S'} \quad .$$

But this is not true for **V** = **v** ! In fact, what is true is 5.(3)

$$[\mathbf{v}]_S = R\,[\mathbf{v}]_{S'} + \boldsymbol{\omega} \times [\mathbf{r}]_S$$

What this means is that, although **v** "looks like" a vector, it does not transform as a vector since we have this extra $\boldsymbol{\omega} \times [\mathbf{r}]_S$ term (when R is time dependent). We are only allowed to apply 4.(7) to objects that

really transform as a vector under our transformation between frames S and S'. **V** = **r** really does transform as a vector.

7. Comments on Goldstein's treatment of this subject.

The theory of transforming between rotating frames is discussed on pages 132-135 of Herbert Goldstein's famous book "Classical Mechanics" (1950, 7th printing 1965), with some support material on earlier pages and some implications on later pages. These pages form Sections 4-8 and the start of 4-9 of that book. Goldstein never draws a picture of the two frames in question, which we have called S (space) and S' (body). He seems to make an assumption (which I cannot find stated anywhere) that, at the moment of interest t, the two frames S and S' line up with each other, which means we can set R = 1 in any of our final expressions above. With this assumption, we may quote some of our results from above, replacing our generic vector **V** by generic vector **G**,

$$\mathbf{(}d[\mathbf{G}]_S/dt) = R\,(d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{G}]_S \quad . \qquad \text{4.(7)}$$

$$\mathbf{(}d[\mathbf{G}]_S/dt) = (d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times [\mathbf{G}]_S \quad . \qquad \text{4.(7) with R=1}$$

Goldstein writes the last equation above in the following manner,

$$(d\mathbf{G}/dt)_{\mathbf{space}} = (d\mathbf{G}/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{G} \qquad \text{Goldstein (4-100)}$$

First of all, if R = 1, then since **G** is a "vector" we know from 4.(3) that $[\mathbf{G}]_S = R\ [\mathbf{G}]_{S'} = [\mathbf{G}]_{S'}$. Since we then have $[\mathbf{G}]_S = [\mathbf{G}]_{S'}$, we might as well just call it **G**, as he does above. As an example, consider our equation 5.(3) from above, but in which we set R = 1 and so $[\mathbf{r}]_S = [\mathbf{r}]_{S'} = \mathbf{r}$,

$$[\mathbf{v}]_S = [\mathbf{v}]_{S'} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \text{5.(3) with R=1}$$

which we might also write as

$$[\mathbf{v}]_{\mathbf{space}} = [\mathbf{v}]_{\mathbf{body}} + \boldsymbol{\omega} \times \mathbf{r}$$

This then is an example of Goldstein (4-100). Since (4-100) is valid for any (true!) vector **G**, we can abstract out the vector **G** and treat it as an operator equation of sorts (which can only act on true vectors)

$$(d/dt)_{\mathbf{space}} = (d/dt)_{\mathbf{body}} + \boldsymbol{\omega} \times \qquad \text{Goldstein (4-102)}$$

In our notation, this could be written several ways, removing the **G** vector acted upon,

$$\mathbf{(}d[\ \ ]_S/dt) = (d[\ \ ]_{S'}/dt) + \boldsymbol{\omega} \times$$

or

$$\mathbf{(}d/dt)[\ \ ]_S = (d/dt)[\ \ ]_{S'} + \boldsymbol{\omega} \times$$

and if we *define* operator $\mathbf{(}d/dt)_S[\ \ ] \equiv \mathbf{(}d/dt)[\ \ ]_S$, then finally

$$(d/dt)_S[\ \ ] = (d/dt)_{S'}[\ \ ] + \boldsymbol{\omega} \times$$

or

$$(d/dt)_S = (d/dt)_{S'} + \boldsymbol{\omega} \times$$

or

$$(d/dt)_{space} = (d/dt)_{body} + \boldsymbol{\omega} \times$$

and this last form then aligns Goldstein (4-102). The operator notation is nice, but it can be a little confusing what it really means, so we feel it is best to keep the vector G present and, assuming R = 1, use this form:

$(d[\mathbf{G}]_S/dt) = (d[\mathbf{G}]_{S'}/dt) + \boldsymbol{\omega} \times \mathbf{G}$. our 4.(7) with R=1

$(d\mathbf{G}/dt)_{space} = (d\mathbf{G}/dt)_{body} + \boldsymbol{\omega} \times \mathbf{G}$ Goldstein (4-100)

6. Comparison with Goldstein notation (p 133) **TBD**

Goldstein uses vector **G** instead of **V** for his generic vector. So I will now translate some equations above into **G** form, and then do side-by side comparison with Goldstein:

$[d\mathbf{G}]_S = [d\mathbf{G}]_S^{(body)} + [d\boldsymbol{\Omega}]_S \times [\mathbf{G}]_S$ *(23a)*

$(d\mathbf{G})_{space} = (d\mathbf{G})_{(body)} + d\boldsymbol{\Omega} \times \mathbf{G}$ p 133 (4-99)

$[d\mathbf{G}/dt]_S = [d\mathbf{G}/dt]_S^{(body)} + [\boldsymbol{\omega}]_S \times [\mathbf{G}]_S$ where $\boldsymbol{\omega} = d\boldsymbol{\Omega}/dt$ *(23b)*

$(d\mathbf{G}/dt)_{space} = (d\mathbf{G}/dt)_{(body)} + \omega \times \mathbf{G}$ p 133 (4-100)

$(d/dt) = (d/dt)_{(body)} + \boldsymbol{\omega} \times$ (21)

$(d/dt)_{space} = (d/dt)_{(body)} + \omega \times$ p 133 (4-102)

Several comments are in order:

(1) *All* of Goldstein's equations shown above have all quantities *evaluated* in frame S (space). He uses what I call the "default meanings" such that $[\mathbf{G}]_S = \mathbf{G}$, $[\boldsymbol{\omega}]_S = \boldsymbol{\omega}$.

(2) His body notation is ambiguous I feel because it suggests that the term is evaluated in the body frame when in fact it is a body change which is evaluated in the space frame. His notation does not get that point across.

7. Better interpretation of dΩ and ω **TBD**

We showed in the last section that

$$dR(t) = [-i\, d\boldsymbol{\Omega} \bullet \mathbf{J}\,]\, R$$

$$dR(t)/dt \approx [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R \qquad\qquad \omega = d\boldsymbol{\Omega}/dt$$

Now consider the basis vector equation (after the change noted at the start of Section 3)

$$\mathbf{e'}_n(t) = R(t)\, \mathbf{e}_n$$

$$d\mathbf{e'}_n(t)/dt = dR(t)/dt\, \mathbf{e}_n = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, R\, \mathbf{e}_n = [-i\, \boldsymbol{\omega} \bullet \mathbf{J}\,]\, \mathbf{e'}_n(t) = [-i\, \omega_k J_k\,]\, \mathbf{e'}_n(t) = -i\omega_k [J_k\, \mathbf{e'}_n(t)]$$

and taking a component (using $i\,(J_k)_{ij} = \varepsilon_{kij}$)

$$[d\mathbf{e'}_n(t)/dt]_i = -i\omega_k [J_k\, \mathbf{e'}_n(t)]_i = -i\omega_k (J_k)_{ij}\, [\mathbf{e'}_n(t)]_j = -\,\omega_k\, \varepsilon_{kij}\, [\mathbf{e'}_n(t)]_j = \varepsilon_{ikj}\, \omega_k\, [\mathbf{e'}_n(t)]_j$$

so that

$$d\mathbf{e'}_n(t)/dt = \boldsymbol{\omega} \times \mathbf{e'}_n(t)$$

Therefore $\boldsymbol{\omega}$ is the amount and direction of the instantaneous rate of rotation of any axis of the body frame. The entire body frame is rotating according to this vector $\boldsymbol{\omega}$. The above can be regarded as a generic equation which we would normally evaluate with all terms of the type $[..]_s$.

Perhaps here is another derivation of our basic result

$$\mathbf{V} = V'_n \mathbf{e'}_n$$

$$d\mathbf{V} = dV'_n\, \mathbf{e'}_n + V'_n\, d\mathbf{e'}_n$$

$$d\mathbf{V}/dt = (dV'_n/dt)\, \mathbf{e'}_n + V'_n\, d\mathbf{e'}_n/dt$$

$$d\mathbf{V}/dt = (d\mathbf{V}/dt)_{(body)} + V'_n\, \boldsymbol{\omega} \times \mathbf{e'}_n(t)$$

$$d\mathbf{V}/dt = (d\mathbf{V}/dt)_{(body)} + \boldsymbol{\omega} \times \mathbf{V} \qquad (6.1)$$

The key idea here is that

$(dV'_{\mathbf{n}}/dt)\ \mathbf{e}'_{\mathbf{n}}$ = rate of change of vector **V** seen in body frame = $(d\mathbf{V}/dt)_{\mathbf{(body)}}$

which I think agrees with the pictures and discussion above. We can regard (6.1) as being an equation which we can evaluate in any frame, and we usually evaluate it in frame S in which case it becomes

$$[d\mathbf{V}/dt]_{\mathbf{S}} = [(d\mathbf{V}/dt)_{\mathbf{(body)}}]_{\mathbf{S}} + [\boldsymbol{\omega}]_{\mathbf{S}} \times [\mathbf{V}]_{\mathbf{S}} \qquad \textit{(18b)}$$