The Goldstein Rotation Rule and When Valid PhL 7.14.12

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1. Derivation of the Rule

(a) Description of frames S and S' and the meaning of angular velocity **ω**.

We have two coordinate systems S and S' which have a common origin.

System S' rotates about its origin relative to system S at some instantaneous vector angular velocity **ω**. This vector **ω** has its tail on the common origin of the two systems, so that rotation of S' is occurring about that common origin. That is to say, the origin point of frame S' is the only point which is not moving relative to frame S as the frame S' rotates according to **ω**.

In general **ω** is time dependent so **ω**(t) and so both the direction and magnitude can vary with time.

The two systems have fixed Cartesian basis vectors $\mathbf{e}_n$ and $\mathbf{e}'_n$.

Let R(**θ**) be the rotation such that $\mathbf{e}'_n = R(\boldsymbol{\theta})\, \mathbf{e}_n$. Then R(**θ**) describes how system S' is oriented relative to system S at time t, so we can regard **θ** = **θ**(t). Then with respect to frame S, we have $\mathbf{e}'_n = \mathbf{e}'_n(t)$.

After a small time interval dt starting at time t, we can write

$$\mathbf{e}'_n(t+dt) = R(d\boldsymbol{\varphi})\, \mathbf{e}'_n(t),$$

where d**φ** is a tiny vector rotation which relates these two sets of basis vectors. Then

$$d\mathbf{e}'_n(t) = \mathbf{e}'_n(t+dt) - \mathbf{e}'_n(t) = [R(d\boldsymbol{\varphi}) - 1]\, \mathbf{e}'_n(t) \approx [\, -id\boldsymbol{\varphi} \bullet \mathbf{J}\,]\, \mathbf{e}'_n(t)$$

It is easy to show from the above line that (Appendix A Theorem 1)

$$d\mathbf{e}'_n(t) = d\boldsymbol{\varphi} \times \mathbf{e}'_n(t)\ .$$

Then we have

$$\dot{\mathbf{e}}'_n = (d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$$

and **ω** = d**φ**/dt is then the instantaneous rotation vector of interest -- the same **ω** as mentioned above. Since all the *basis* vectors are moving in this way, $(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n$, all points in frame S' are moving

in this same manner. If $\mathbf{r}'$ is some point in frame S', then $(d\mathbf{r}'/dt) = \boldsymbol{\omega} \times \mathbf{r}'$. If we have some rigid body at rest in frame S', then this entire rigid body is rotating relative to frame S with angular velocity $\boldsymbol{\omega}$.

(b) Facts about a vector a.

Let **a** be any vector *which has its tail at the origin* of coordinate systems S and S' . Such a vector **a** can be expanded in these two ways,

$$\mathbf{a} = a_i \mathbf{e}_i = a'_i \mathbf{e}'_i \ .$$

Note that if the tail of **a** is located at some point $\mathbf{a}_0$, then we would have

$$(\mathbf{a} - \mathbf{a}_0) = (\mathbf{a} - \mathbf{a}_0)_i \mathbf{e}_i = (\mathbf{a} - \mathbf{a}_0)'_i \mathbf{e}'_i \quad ,$$

but this is not the situation we wish to consider at the present time. We restrict **a** to have its tail at the origin of frames S and S'.

From frame S, we can compute the d**a**/dt in two different ways using the two expansions above

$$\dot{\mathbf{a}} = \dot{a}_i \mathbf{e}_i$$
$$\dot{\mathbf{a}} = \dot{a}'_i \mathbf{e}'_i \ + \ a'_i \dot{\mathbf{e}}'_i$$

The second term on the second line arises because $\mathbf{e}'_i = \mathbf{e}'_i(t)$ as observed from frame S.

We only care about the second line for our purposes here. Using (**) this can be written

$$\dot{\mathbf{a}} = \dot{a}'_i \mathbf{e}'_i \ + \ a'_i \ \boldsymbol{\omega} \times \mathbf{e}'_i \ = \dot{a}'_i \mathbf{e}'_i \ + \boldsymbol{\omega} \times (a'_i \ \mathbf{e}'_i) \ = \ \dot{a}'_i \mathbf{e}'_i \ + \boldsymbol{\omega} \times \mathbf{a}$$

$$= \boldsymbol{\omega} \times \mathbf{a} + \dot{a}'_i \mathbf{e}'_i$$

An observer glued to frame S' would measure $\dot{\mathbf{a}}$ in the following manner:

$$\mathbf{a} = a'_i \mathbf{e}'_i$$

$$(\dot{\mathbf{a}})_{\text{measured in S'}} = \dot{a}'_i \mathbf{e}'_i \ \equiv \ (\dot{\mathbf{a}})_{S'},$$

because for such an observer, the $\mathbf{e}'_i$ are constants. Thus we have shown that

$$\dot{\mathbf{a}} = \ \boldsymbol{\omega} \times \mathbf{a} + (\dot{\mathbf{a}})_{S'}$$

Since $\dot{\mathbf{a}}$ is the a quantity an observer measures in frame S (as set up above), we write

$$(\dot{\mathbf{a}})_S = \ \boldsymbol{\omega} \times \mathbf{a} + (\dot{\mathbf{a}})_{S'}$$

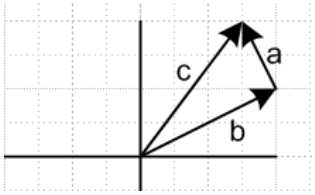
or

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'} \quad .$$

This is what I call The Goldstein Rotation Rule. Recall that it applies to any vector **a** which has its tail at the common origin of the two systems S and S'.

(c) Generalize the rule to any vector **a**.

Let **a** be any vector lying out somewhere in 3D space. Let vector **b** be a vector from the (common) origin to the tail of this vector **a**. Then vector **c** = **a**+**b** is a vector from the origin to the tip of vector **a**.



Since vector **c** has its tail at the origin, we can apply our G Rule to this vector **c**, and it then says

$$(d\mathbf{c}/dt)_S = \boldsymbol{\omega} \times \mathbf{c} + (d\mathbf{c}/dt)_{S'}$$

But we can also apply our rule to vector **b** to get

$$(d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} + (d\mathbf{b}/dt)_{S'}$$

Since **c** - **b** = **a**, Subtracting these two equations then tells us that

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$$

Thus, our G rule applies to a vector **a** even if its tail does not lie at the origin!

2. Examples of when the Rule is valid or non-valid.

Example A. Let vector **a** be **r**, the position vector of a particle with respect to the common origin of frames S and S'. Vector **r** then has its tail on the common origin, and the rule is valid, stating

$$(\dot{\mathbf{r}})_S = \boldsymbol{\omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{S'}$$

or

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'}$$

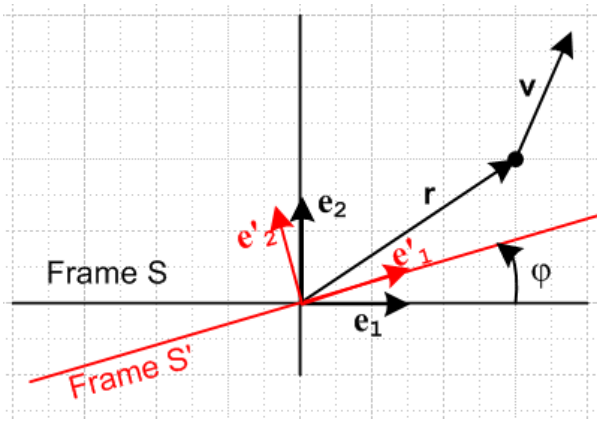
or

$$(\mathbf{v})_S = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$$

since **v** = d**r**/dt is the velocity of a particle.

Obtaining $\mathbf{v} = (\mathbf{v})_S$ in this manner, one can then compute $\mathbf{a} = d\mathbf{v}/dt$ to find the acceleration of the particle, and any desired higher derivative. This is the path usually followed in textbooks which treat the famous Coriolis force.

Example B. Let **v** be the velocity of a particle located at point **r** relative to the common origin. Consider this picture, where $\mathbf{v} = d\mathbf{r}/dt$,



In this particular picture, **ω** points in the e_3 direction and both **r** and **v** happen to lie in the $e_3 = 0$ plane. In general of course **r** and **v** can be arbitrary vectors.

First of all, we see the happy vector **r** used in Example A, having its tail at the origin.

Vector v is then used for Example B, and we expect this G rule to be valid

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'}$$

Example C. Using the same picture as above but with **r** and **v** in arbitrary directions, we can discuss the angular momentum vector $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. This angular momentum vector is *with respect to* the common origin of frames S and S'. That is to say, this expression for **L** gives the angular momentum with respect to the origin of the particle located at point **r** and having velocity **v**. We could define different angular momentum vectors relative to different points in space, but those are not the ones we want. For example, we could define **L** about the point **r** and that **L** would be 0.

We expect the following R rule to be valid

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$

3. Attempt to show validity for the G Rule for velocity v

We wish to verify the following G rule for the velocity vector $\mathbf{v} = d\mathbf{r}/dt$

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'}$$

I am going to show that in fact this rule is *invalid*, despite what was said above. We know from consideration of the position vector **r** that

$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$

Therefore from direct calculus calculation, we find that

$d\mathbf{v}/dt = \boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + (d\mathbf{v}/dt)_{S'} = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + + (d\mathbf{v}/dt)_{S'}$.

This will be consistent with the G rule for **v** provided the following is true:

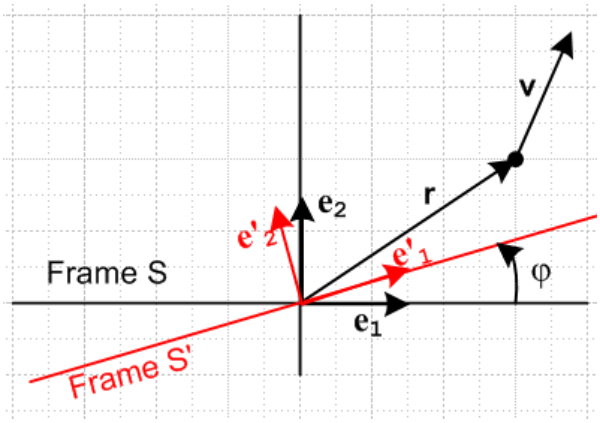
$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + (d\mathbf{v}/dt)_{S'}$

which requires that $\boldsymbol{\omega}$ = constant. But ω supposedly should be allowed to vary in time!

3. Verify validity for L

Start with (set m = 1)

$\mathbf{L} = \mathbf{r} \times \mathbf{V}$



I indicate the arbitrary picture velocity by **V** to distinguish it from the velocity $\mathbf{v} = d\mathbf{r}/dt$. These two velocities are unrelated. Then we have

$d\mathbf{L}/dt = \mathbf{r} \times (d\mathbf{V}/dt) + (d\mathbf{r}/dt) \times \mathbf{V} = \mathbf{r} \times (d\mathbf{V}/dt) + \mathbf{v} \times \mathbf{V}$

From Example A we learned that

$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$

so

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} + (\mathbf{a})_{S'}$$

Therefore we find by direct calculation that

$$d\mathbf{L}/dt = \mathbf{r} \times \mathbf{a} = \mathbf{r} \times [\dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} + (\mathbf{a})_{S'}]$$

$$= \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

Meanwhile, the G Rule for **L** states that

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$
$$= \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + (d\mathbf{L}/dt)_{S'}$$

If these two methods are to agree, we have to be able to show that

$$\boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + (d\mathbf{L}/dt)_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

STOP. This can never be true because the $(...)_{S'}$ quantities don't know about ω or $\dot{\omega}$ so there is nothing the balance off the visible **r** x ($\dot{\boldsymbol{\omega}}$ x **r**) term. Something is not right.

********** details refining the equality that is not true **********

Now I presume that

$$(\mathbf{L})_{S'} = (\mathbf{r})_{S'} \times (\mathbf{v})_{S'}$$

$$(d\mathbf{L}/dt)_{S'} = (\mathbf{v})_{S'} \times (\mathbf{v})_{S'} + (\mathbf{r})_{S'} \times (\mathbf{v})_{S'} = (\mathbf{r})_{S'} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\mathbf{v})_{S'}$$

Then we have to show that

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

Now consider this identity **A** x (**B** x **C**) + cyclic = 0 or

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

$$-\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$$

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) = \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v})$$

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega})$$

Thus we have to show that

$$\mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

$$[\boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}] \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

$$(\mathbf{v})_{S'} \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

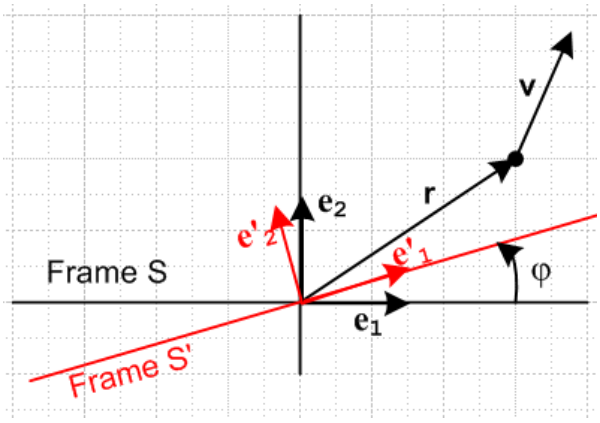
which is about as simple as I can make it. But this can never be true for the following reason. All the $(....)_{S'}$ factors cannot depend on ω or on $\dot{\omega}$. Only one term has $\dot{\omega}$ but this term does not vanish so we cannot possibly balance the $\dot{\omega}$ terms, so my claim is somehow false!!

********** scraps **********

3. Verify validity for L

Start with (set m = 1)

$$\mathbf{L} = \mathbf{r} \times \mathbf{V}$$



I indicate the arbitrary picture velocity by **V** to distinguish it from the velocity $\mathbf{v} = d\mathbf{r}/dt$. These two velocities are unrelated. Then we have

$$d\mathbf{L}/dt = \mathbf{r} \times (d\mathbf{V}/dt) + (d\mathbf{r}/dt) \times \mathbf{V} = \mathbf{r} \times (d\mathbf{V}/dt) + \mathbf{v} \times \mathbf{V} .$$

While we're at it, we may state that

$$(\mathbf{L})_{S'} = \mathbf{r} \times (\mathbf{V})_{S'}$$

$$(d\mathbf{L}/dt)_{S'} = \mathbf{r} \times (d\mathbf{V}/dt)_{S'} + (d\mathbf{r}/dt)_{S'} \times \mathbf{V} = \mathbf{r} \times (d\mathbf{V}/dt)_{S'} + (\mathbf{v})_{S'} \times \mathbf{V} .$$

Now here is what the G Rule for **L** claims:

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$

or

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{V}) + \mathbf{r} \times (d\mathbf{V}/dt)_{S'} + (\mathbf{v})_{S'} \times \mathbf{V} .$$

Comparing our two calculations, we need the following to be true

$$\mathbf{r} \times (d\mathbf{V}/dt) + \mathbf{v} \times \mathbf{V} = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{V}) + \mathbf{r} \times (d\mathbf{V}/dt)_{S'} + (\mathbf{v})_{S'} \times \mathbf{V}$$

********** **scraps** **********

From Example A we learned that

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$$

so

$$\mathbf{a} = \dot{\mathbf{v}} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} + (\mathbf{a})_{S'}$$

Therefore we find by direct calculation that

$$d\mathbf{L}/dt = \mathbf{r} \times \mathbf{a} = \mathbf{r} \times [\dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} + (\mathbf{a})_{S'}]$$

$$= \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

Meanwhile, the G Rule for **L** states that

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$
$$= \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + (d\mathbf{L}/dt)_{S'}$$

If these two methods are to agree, we have to be able to show that

$$\boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + (d\mathbf{L}/dt)_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

STOP. This can never be true because the $(...)_{S'}$ quantities don't know about ω or $\dot{\omega}$ so there is nothing the balance off the visible $\mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r})$ term. Something is not right.

********** details refining the equality that is not true **********

Now I presume that

$$(\mathbf{L})_{S'} = (\mathbf{r})_{S'} \times (\mathbf{v})_{S'}$$

$$(d\mathbf{L}/dt)_{S'} = (\mathbf{v})_{S'} \times (\mathbf{v})_{S'} + (\mathbf{r})_{S'} \times (\mathbf{v})_{S'} = (\mathbf{r})_{S'} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\mathbf{v})_{S'}$$

Then we have to show that

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

or

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

Now consider this identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ + cyclic = 0 or

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$

$$-\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) - \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$$

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) = \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v})$$

$$-\boldsymbol{\omega} \times (\mathbf{v} \times \mathbf{r}) - \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega})$$

Thus we have to show that

$$\mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

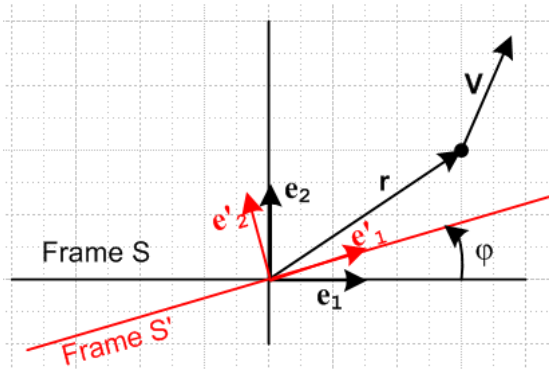
$$[\boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}] \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

$$(\mathbf{v})_{S'} \times (\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\mathbf{v})_{S'} = \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\mathbf{a})_{S'}$$

which is about as simple as I can make it. But this can never be true for the following reason. All the $(....)_{S'}$ factors cannot depend on ω or on $\dot{\omega}$. Only one term has $\dot{\omega}$ but this term does not vanish so we cannot possibly balance the $\dot{\omega}$ terms, so my claim is somehow false!!

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Example B Revisited.



The velocity vector in the picture is **V** to distinguish it from **v** = d**r**/dt. We should be able to translate our vector **V** so its tail lies at the origin, so it is then an OK vector for our G Rule consideration. The rule would say that

$$(d\mathbf{V}/dt)_S = \boldsymbol{\omega} \times \mathbf{V} + (d\mathbf{V}/dt)_{S'}$$

But this is just not right. The vector **V** really lies out there where I show it in the picture. Suppose our rotation $\boldsymbol{\omega}$ were about the $\mathbf{e}_2$ axis. For constant $\boldsymbol{\omega}$, the vector **V** traces out a conic frustum, not a cone. The d**V** is then not what the cone model says. But it should be fairly easy to get the right model.

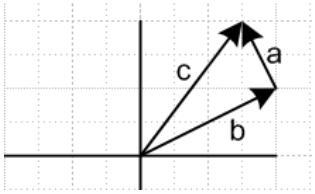
10. Derivation of the Generalized G Rule

(a) Description of frames S and S' and the meaning of angular velocity $\boldsymbol{\omega}$.

This section is unchanged from that appearing in the regular R rule discussion.

(b) Facts about a vector a.

Let **a** be any vector lying out somewhere in 3D space. Let vector **b** be a vector from the (common) origin to the tail of this vector **a**. Then vector **c** = **a**+**b** is a vector from the origin to the tip of vector **a**.



Since vector **c** has its tail at the origin, we can apply our G Rule to this vector **c**, and it then says

$$(d\mathbf{c}/dt)_S = \boldsymbol{\omega} \times \mathbf{c} + (d\mathbf{c}/dt)_{S'}$$

But we can also apply our rule directly to **b** to get

$$(d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} + (d\mathbf{b}/dt)_{S'}$$

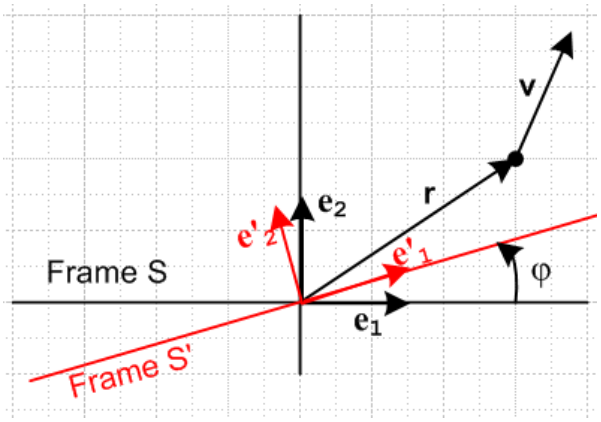
Adding these two equations then tells us that

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$$

Thus, our G rule applies to a vector **a** even if its tail does NOT lie at the origin!

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Example B. Let **v** be the velocity of a particle located at point **r** relative to the common origin. Consider this picture :



In this particular picture, $\boldsymbol{\omega}$ points in the e_3 direction and both **r** and **v** happen to lie in the $e_3 = 0$ plane. In general of course **r** and **v** can be arbitrary vectors, but the tail of **v** always lies at the end of **r**.

First of all, we see the happy vector **r** used in Example A, having its tail at the origin.

Unless $\mathbf{r} = 0$, however, the tail of **v** does not lie at the common origin of S and S', and therefore for such a vector **v**, the Goldstein Rule is invalid! That is to say,

$$(d\mathbf{v}/dt)_S \neq \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'} \qquad \text{// except when } \mathbf{r} = 0$$

In other words, an evaluation of $\boldsymbol{\omega}$ x $\mathbf{v}$ + $(d\mathbf{v}/dt)_{S'}$ will not give the result $\mathbf{a}(t)$ found at the end of the previous Example.

Example C. Using the same picture as above but with $\mathbf{r}$ and $\mathbf{v}$ in arbitrary directions, we can discuss the angular momentum vector $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$. This angular momentum vector is *with respect to* the common origin of frames S and S'. That is to say, this expression for $\mathbf{L}$ gives the angular momentum with respect to the origin of the particle located at point $\mathbf{r}$ and having velocity $\mathbf{v}$. We can thus regard $\mathbf{L}$ as a vector having its tail at the origin. We could define different angular momentum vectors relative to different points in space, but those are not the ones we want. For example, we could define $\mathbf{L}$ about the point $\mathbf{r}$ and that $\mathbf{L}$ would be 0.

Since the tail of this L is at the origin, we expect the Goldstein Rule to be valid

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$

Comments: These examples explain why it is that Goldstein, at the start of Section 4-89 on page 132, mentions the position vector and angular momentum as candidate $\mathbf{G}$ vectors (our $\mathbf{a}$), but does *not* mention the velocity vector. The Rule for $\mathbf{L}$ as stated above appears on page 158. For Goldstein, S = space and S' = body.