The Goldstein Rotation Rule, Version 3 PhL 7.14.12

1. Derivation of the Rule **1**
2. Three Examples **3**
3. Verify the G rule for r **4**
4. Verify the G rule for v **4**
5. Verify the G rule for L **6**

1. Derivation of the Rule

(a) Description of frames S and S' and the meaning of angular velocity **ω**.

We have two coordinate systems S and S' which have a common origin.

System S' rotates about its origin relative to system S at some instantaneous vector angular velocity **ω**. This vector **ω** has its tail on the common origin of the two systems, so that rotation of S' is occurring about that common origin. That is to say, the origin point of frame S' is the only point which is not moving relative to frame S as the frame S' rotates according to **ω**.

In general **ω** is time dependent so **ω**(t) and so both the direction and magnitude can vary with time.

The two systems have fixed Cartesian basis vectors $\mathbf{e}_n$ and $\mathbf{e}'_n$.

Let R(**θ**) be the rotation such that $\mathbf{e}'_n = R(\boldsymbol{\theta})\, \mathbf{e}_n$. Then R(**θ**) describes how system S' is oriented relative to system S at time t, so we can regard **θ** = **θ**(t). Then with respect to frame S, we have $\mathbf{e}'_n = \mathbf{e}'_n(t)$.

After a small time interval dt starting at time t, we can write

$$\mathbf{e}'_n(t+dt) = R(d\boldsymbol{\varphi})\, \mathbf{e}'_n(t),$$

where d**φ** is a tiny vector rotation which relates these two sets of basis vectors. Then

$$d\mathbf{e}'_n(t) = \mathbf{e}'_n(t+dt) - \mathbf{e}'_n(t) = [R(d\boldsymbol{\varphi}) - 1]\, \mathbf{e}'_n(t) \approx [\, -id\boldsymbol{\varphi} \bullet \mathbf{J}\,]\, \mathbf{e}'_n(t)$$

It is easy to show from the above line that (Appendix A Theorem 1)

$$d\mathbf{e}'_n(t) = d\boldsymbol{\varphi} \times \mathbf{e}'_n(t)\ .$$

Then we have

$$\dot{\mathbf{e}}'_n = (d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n \qquad \text{where } \boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$$

and **ω** = d**φ**/dt is then the instantaneous rotation vector of interest -- the same **ω** as mentioned above. Since all the *basis* vectors are moving in this way, $(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n$, all points in frame S' are moving

in this same manner. If $\mathbf{r}'$ is some point in frame S', then $(d\mathbf{r}'/dt) = \boldsymbol{\omega} \times \mathbf{r}'$. If we have some rigid body at rest in frame S', then this entire rigid body is rotating relative to frame S with angular velocity $\boldsymbol{\omega}$.

(b) Facts about a vector a.

Let $\mathbf{a}$ be any vector which has its tail at the origin of coordinate systems S and S' . Such a vector $\mathbf{a}$ can be expanded in these two ways,

$$\mathbf{a} = a_i \mathbf{e}_i = a'_i \mathbf{e}'_i \ .$$

From frame S, we can compute the $d\mathbf{a}/dt$ in two different ways using the two expansions above

$$\dot{\mathbf{a}} = \dot{a}_i \mathbf{e}_i$$
$$\dot{\mathbf{a}} = \dot{a}'_i \mathbf{e}'_i + a'_i \dot{\mathbf{e}}'_i \ .$$

The second term on the second line arises because $\mathbf{e}'_i = \mathbf{e}'_i(t)$ as observed from frame S.

We only care about the second line for our purposes here. Using (**) this can be written

$$\dot{\mathbf{a}} = \dot{a}'_i \mathbf{e}'_i + a'_i \, \boldsymbol{\omega} \times \mathbf{e}'_i = \dot{a}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times (a'_i \, \mathbf{e}'_i) = \dot{a}'_i \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}$$

$$= \boldsymbol{\omega} \times \mathbf{a} + \dot{a}'_i \mathbf{e}'_i$$

An observer glued to frame S' would "measure" $\mathbf{a}$ and $\dot{\mathbf{a}}$ in the following manner:

$$\mathbf{a} = a'_i \mathbf{e}'_i$$

$$(\dot{\mathbf{a}})_{\text{measured in S'}} = \dot{a}'_i \mathbf{e}'_i \equiv (\dot{\mathbf{a}})_{S'},$$

because for such an observer, the $\mathbf{e}'_i$ are constants. No label is needed on $\mathbf{a}$, since $\mathbf{a} = a_i \mathbf{e}_i = a'_i \mathbf{e}'_i$ is unambiguous, but a label is needed on $(\dot{\mathbf{a}})_{S'}$ to distinguish it from $(\dot{\mathbf{a}})_S$. Thus we have shown that

$$\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a} + (\dot{\mathbf{a}})_{S'}$$

Since $\dot{\mathbf{a}}$ is the a quantity an observer measures in frame S (as treated above), we write

$$(\dot{\mathbf{a}})_S = \boldsymbol{\omega} \times \mathbf{a} + (\dot{\mathbf{a}})_{S'}$$

or

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$$

or

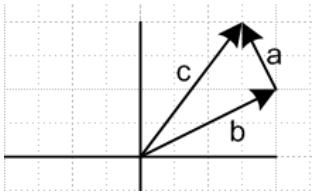
$$d\mathbf{a}/dt = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$$

where in the third case an object with no label is implied to be an S frame measured object.

This is what I call The Goldstein Rotation Rule. Recall that it applies to any vector **a** which has its tail at the common origin of the two systems S and S', a restriction we shall remove in the next section.

(c) Generalize the rule to any vector **a**.

Let **a** be any vector lying out somewhere in 3D space. Let vector **b** be a vector from the (common) origin to the tail of this vector **a**. Then vector **c** = **a**+**b** is a vector from the origin to the tip of vector **a**.



Since vector **c** has its tail at the origin, we can apply our G Rule to this vector **c** to get,

$$(d\mathbf{c}/dt)_S = \boldsymbol{\omega} \times \mathbf{c} + (d\mathbf{c}/dt)_{S'},$$

But we can also apply our rule to vector **b** to get

$$(d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} + (d\mathbf{b}/dt)_{S'},$$

Since **c** - **b** = **a**, Subtracting these two equations then tells us that

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'},$$

Thus, our G rule applies to a vector **a** even if its tail does not lie at the origin. Generally just translating a vector does not change things, but here we just want to make sure. Earlier I thought the tail at origin business was a genuine restriction, but now I realize that it is not so.

2. Three Examples

Let **r** be the position of a particle, **v** be its velocity, and **L** be its angular momentum. Then we have the following three G Rules:

$$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'},$$

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'},$$

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'},$$

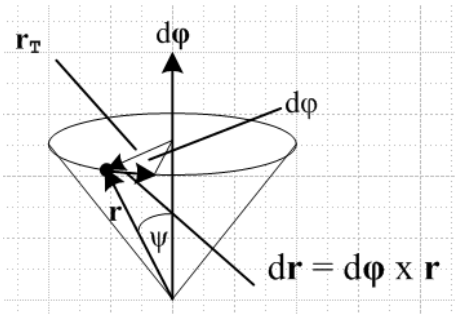
At one time I doubted that all three of these rules were valid for the very general conditions stated in Section 1 (a) above (variable $\boldsymbol{\omega}(t)$, etc)), but I am now a believer.

3. Verify the G rule for r

If **r** is the position vector of some particle, then we can write

$$d\mathbf{r} = d\boldsymbol{\varphi} \times \mathbf{r} + (d\mathbf{r})_{S'} \qquad \boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$$

where the first term gives the change in **r** due to the fact that S' is rotating small amount d**φ** and the second term is the change in **r** that is measured in frame S'. The first term is illustrated by this picture,



Rotation about d**φ** causes the d**r** shown in the picture. The magnitude is $r\sin\psi d\varphi = r_T d\varphi$, and the direction is given by the right hand rule.

We just add these two differential amounts and there is really nothing to do. Divide by dt and the G rule for **r** follows at once.

$$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'}$$

or

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$$

There is nothing we have to be "consistent with" here.

4. Verify the G rule for v

Before considering the G rule for **v**, let's take the **v** obtained at the end of = Section 3 and differentiate it to find d**v**/dt. We do this in the space frame so there are no labels.

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$$

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$$

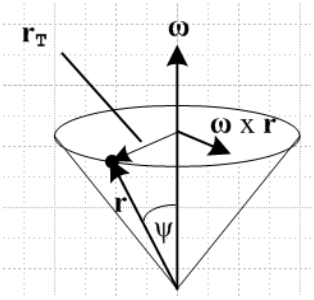
$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (\mathbf{v})_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$$

We want now to pause to interpret each of these four terms.

If it happens that the particle in frame S' is at always at rest in S', so that $(\mathbf{v})_{S'} \equiv 0$, then we have just the first and third terms shown above,

$d\mathbf{v}/dt = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}$ if $(\mathbf{v})_{S'} \equiv 0$.

The first term is the centripetal acceleration due to the instantaneously circular motion. To see this, look at the above picture redrawn, where we now show the vector $\boldsymbol{\omega} \times \mathbf{r}$:



Using the right hand rule, one can see then that $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector pointing in the opposite direction of the vector labeled $\mathbf{r}_T$. From the location of the particle, this vector points toward the center of the circle on which the particle is instantaneously rotating at time t, and this is the correct direction for centripetal acceleration. The magnitude of the vector is $|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = \omega \, | \, \boldsymbol{\omega} \times \mathbf{r} \, | = \omega^2 r \sin\psi = \omega^2 r_T$ and this is the correct magnitude for the centripetal acceleration.

The second term $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ in the last equation above is a tangential acceleration that the particle experiences if the rotation rate is varying with time $\boldsymbol{\omega}(t)$.

Now, if we allow $(\mathbf{v})_{S'} \neq 0$ (an ant crawls around on the phonograph record as it spins), we pick up two more terms (shown in red) in the total acceleration $d\mathbf{v}/dt$ seen in frame S.

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (\mathbf{v})_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$$

The last term $d(\mathbf{v})_{S'}/dt$ is just the acceleration the particle would have if frame S' were at permanent rest relative to frame S. The second term is a Coriolis-like acceleration which can be explained as follows: Suppose the ant on the phono record walks directly away from the spindle. Since he is not sliding relative to the record, and since he is moving to a larger radius, his tangential speed must be increasing, so he has a tangential acceleration in the $\boldsymbol{\omega} \times (\mathbf{v})_{S'}$ direction. On the other hand, if the ant walks tangentially at $(\mathbf{v})_{S'}$ in the direction of the rotation, his centripetal acceleration is larger than the first term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ by the amount $\boldsymbol{\omega} \times (\mathbf{v})_{S'}$ which then points toward the spindle.

Now here is a statement of the G Rule for $\mathbf{v}$, (alternate notation on the right)

$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'}$ $\qquad$ $\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} + (\mathbf{a})_{S'}$ $\qquad$ $(\mathbf{a})_{S'} = a'_i \mathbf{e}'_i$

Comparison with the result derived above,

$$d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (\mathbf{v})_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$$

shows that the following must be true :

$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$. $\quad (\mathbf{a})_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$

Suppose once again that $(\mathbf{v})_{S'} \equiv 0$ for our particle. Then the last term is 0 and we get

$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r}$ $\quad (\mathbf{a})_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r}$

The particle in frame S', although not moving in this frame, still experiences ("measures", "observes") tangential acceleration $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ due to varying $\boldsymbol{\omega}(t)$. The particle of course also experiences centripetal acceleration toward the rotation axis, but as we have seen above, that effect is totally accounted for by the $\boldsymbol{\omega} \times \mathbf{v}$ term.

So in terms of the value of $(d\mathbf{v}/dt)_{S'} = (\mathbf{a})_{S'}$, one might say that the particle is unaware of $\boldsymbol{\omega}$ (since frame S' is not rotating when viewed from frame S'), but is aware of $\dot{\boldsymbol{\omega}}$ and higher derivatives of $\boldsymbol{\omega}$.

The notation is a bit subtle here. The derivative $d(\mathbf{v})_{S'}/dt$ refers to the intrinsic or explicit change in $(\mathbf{v})_{S'}$ and could perhaps be written as $(\partial\mathbf{v}/\partial t)_{S'}$. The derivative $(d\mathbf{v}/dt)_{S'}$ means the acceleration experienced by the particle in frame S', excluding the $\boldsymbol{\omega} \times \mathbf{v}$ sources. Then we could write

$$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + (\partial\mathbf{v}/\partial t)_{S'}$$

5. Verify the G rule for L

Again, before considering the G Rule for L, let's compute $d\mathbf{L}/dt$ in frame S by "brute force". We set mass m = 1 to reduce symbol clutter. In the end it can be reinstalled by replacing all **L** by **L**/m. So

$$\mathbf{L} = \mathbf{r} \times \mathbf{v}$$

$$d\mathbf{L}/dt = \dot{\mathbf{r}} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}} = \mathbf{r} \times \dot{\mathbf{v}} \ .$$

For $\dot{\mathbf{v}}$ we use the exact same form shown in the previous section

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$$

$$\dot{\mathbf{v}} = d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt$$

Thus we find

$$d\mathbf{L}/dt = \mathbf{r} \times [\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + d(\mathbf{v})_{S'}/dt\ ]$$

$$= \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (\partial\mathbf{v}/\partial t)_{S'}$$

Next, we use the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \text{cyclic} = 0$,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$$
$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = -\mathbf{B} \times (\mathbf{C} \times \mathbf{A}) - \mathbf{C} \times (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \times (\mathbf{A} \times \mathbf{C}) + \mathbf{C} \times (\mathbf{B} \times \mathbf{A})$$

so

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) + \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad = \boldsymbol{\omega} \times \mathbf{L} + \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Then we have

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (d(\mathbf{v})_{S'}/dt) \ .$$

But we can expand the second term as follows, using $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}$ from Section 3,

$$\mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r}) = [\boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}] \times (\boldsymbol{\omega} \times \mathbf{r}) = (\boldsymbol{\omega} \times \mathbf{r}) \times (\boldsymbol{\omega} \times \mathbf{r}) + (\mathbf{v})_{S'} \times (\boldsymbol{\omega} \times \mathbf{r}) = (\mathbf{v})_{S'} \times (\boldsymbol{\omega} \times \mathbf{r})$$

so then

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (\mathbf{v})_{S'} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (d(\mathbf{v})_{S'}/dt)$$

Now here is a statement of the G Rule for **L**,

$$d\mathbf{L}/dt = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'}$$

Comparison with the result above shows that we must have

$$(d\mathbf{L}/dt)_{S'} = \mathbf{v} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (d(\mathbf{v})_{S'}/dt)$$

$$(d\mathbf{L}/dt)_{S'} = (\mathbf{v})_{S'} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (d(\mathbf{v})_{S'}/dt)$$

Now in frame S' one has

$$(\mathbf{L})_{S'} = \mathbf{r} \times (\mathbf{v})_{S'}$$

$$d(\mathbf{L})_{S'}/dt = \mathbf{r} \times (d(\mathbf{v})_{S'}/dt) + (d\mathbf{r}/dt)_{S'} \times (\mathbf{v})_{S'}$$

From Section 3 recall that

$$d\mathbf{r}/dt = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'}$$

so therefore

$$d(\mathbf{L})_{S'}/dt = \mathbf{r} \times (d(\mathbf{v})_{S'}/dt) + [d\mathbf{r}/dt - \boldsymbol{\omega} \times \mathbf{r}] \times (\mathbf{v})_{S'}$$

$$= \mathbf{r} \times (d(\mathbf{v})_{S'}/dt) + \mathbf{v} \times (\mathbf{v})_{S'} - (\boldsymbol{\omega} \times \mathbf{r}) \times (\mathbf{v})_{S'}$$

Finally we can write

$\mathbf{v} \times (\mathbf{v})_{S'} = [\boldsymbol{\omega} \times \mathbf{r} + (\mathbf{v})_{S'}] \times (\mathbf{v})_{S'} = (\boldsymbol{\omega} \times \mathbf{r}) \times (\mathbf{v})_{S'}$

and then we find that

$d(\mathbf{L})_{S'}/dt = \mathbf{r} \times (d(\mathbf{v})_{S'}/dt)$

which is the last term in our expression above for $(d\mathbf{L}/dt)_{S'}$.

In any event, we showed above that this must be true,

$(d\mathbf{L}/dt)_{S'} = (\mathbf{v})_{S'} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r}) + \mathbf{r} \times (d(\mathbf{v})_{S'}/dt)$.

I will try to provide arm-waving interpretations of these terms.

First term: If the ant on our phono record is walking out to larger radius at speed $(\mathbf{v})_{S'}$, the first term is 0. But if the ant is walking tangentially at speed $(\mathbf{v})_{S'}$ in the direction of phono platter rotation, then he is increasing the overall r x v by r x $(\mathbf{v})_{S'}$. But this would not seem to be a $(d\mathbf{L}/dt)_{S'}$, it would be more a constant increase in L. This effect is really what the last term is about. I am at a complete loss on this first term, maybe it is wrong.