The Goldstein Rotation Rule Version 5 PhL 7.15.12

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The general context of this document is that we have some Apparatus observed from two frames of reference called S and S' . As we shall see in Figure 1 below, frame S' is rotating in a certain manner with respect to frame S. Frame S has Cartesian basis unit vectors $\mathbf{e}_n$, while Frame S' has $\mathbf{e}'_n$.

1. Preliminary Note: Notation for Differentiation d/dt.

When we say "differentiate something with respect to time t", does it make a difference if we do this in frame S or in frame S'?

Here is an example:

$d\mathbf{e}'_i/dt = \boldsymbol{\omega} \times \mathbf{e}'_i$ // in frame S, see derivation of this fact below, Eq. (4.4)
$d\mathbf{e}'_i/dt = 0$ // in frame S' (because the $\mathbf{e}'_i$ are fixed in frame S')

Therefore, it seems that the act of "applying d/dt" *can* in fact depend on which frame you do it in. Therefore, we had better add something to our notation to distinguish things. Then

$(d\mathbf{e}'_i/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_i$ // in frame S
$(d\mathbf{e}'_i/dt)_{S'} = 0$ // in frame S'

Now what about differentiating a known scalar function like $f(t) = 4t + 3t^3$. The derivative is the same in both frames for such a scalar function,

$(df(t)/dt)_S = (df(t)/dt)_{S'} = df(t)/dt = \dot{f}(t) = 4 + 9t^2$,

so for scalar functions we can dispense with frame labeling. This applies also to the component of a tensor $T_{ij}..(t)$ where we would say

$(dT_{ij..}(t)/dt)_S = (dT_{ij..}((t)/dt)_{S'} = dT_{ij..}(t)/dt$

Although $T_{ij..}(t)$ is the component of a tensor, it is a scalar function in the sense that it is just a single function. For a vector we might perhaps have $f(t) = v_i(t)$, some velocity component.

2. The G Rule for arbitrary vector a.

Note: In this and other documents we refer to the following equation

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'} \qquad (2.1)$$

as "the G Rule for vector **a**". G stands for Herbert Goldstein, the Rule appears on page 133 of his classic 1950 book *Classical Mechanics*. Perhaps another name for this rule would be "the rule which relates the derivatives of a vector taken in two frames which are rotating with respect to each other ". Since the Rule applies to any vector **a**, sometimes the vector is left out and one writes this operator equation,

$$(d/dt)_S = \boldsymbol{\omega} \times + (d/dt)_{S'} \quad . \qquad (2.2)$$

We shall now derive this Rule. Expand the vector **a** in frame S' components :

$$\mathbf{a} = a'_i \, \mathbf{e}'_i \qquad (2.2)$$

Differentiate **a** in frame S

$(d\mathbf{a}/dt)_S = (da'_i/dt)_S \, \mathbf{e}'_i + a'_i(d\mathbf{e}'_i/dt)_S$ // chain rule

$= (da'_i/dt)_S \, \mathbf{e}'_i + a'_i \, \boldsymbol{\omega} \times \mathbf{e}'_i$ // see Equ. (4.4), $(d\mathbf{e}'_i/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_i$

$= (da'_i/dt)_S \, \mathbf{e}'_i + \boldsymbol{\omega} \times a'_i \, \mathbf{e}'_i$ // move in scalar factor a'_i

$= (da'_i/dt)_S \, \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}$ // recognize expansion of **a**

$= (da'_i/dt) \, \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}$ // derivative of scalar function needs no label

so we have shown that

$$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (da'_i/dt) \, \mathbf{e}'_i \qquad (2.3)$$

Now go back to the start

$$\mathbf{a} = a'_i \, \mathbf{e}'_i \qquad \textit{(2.2)}$$

and differentiate this time in frame S' to get,

$(d\mathbf{a}/dt)_{S'} = (da'_i/dt)_{S'}\ \mathbf{e}'_i + a'_i(d\mathbf{e}'_i/dt)_{S'}$

$= (da'_i/dt)_{S'}\ \mathbf{e}'_i + a'_i * 0$ // obvious fact that $(d\mathbf{e}'_i/dt)_{S'} = 0$

$= (da'_i/dt)\ \mathbf{e}'_i$ // derivative of scalar function needs no label (2.4)

Thus our results above are

$(d\mathbf{a}/dt)_S = (da'_i/dt)\ \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{a}$ *(2.3)*

$(d\mathbf{a}/dt)_{S'} = (da'_i/dt)\ \mathbf{e}'_i$. *(2.4)*

Inserting the second into the first gives the G Rule for vector **a**,

$(d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$. *(2.1)*

This then gives the connection between the derivative of a vector **a** in S compared to in S'.

Example: Suppose $\mathbf{a} = \mathbf{e}'_i$. Then our rule (2.1) says

$(d\mathbf{e}'_i/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_i + (d\mathbf{e}'_i/dt)_{S'}$ (2.5)

But we noted earlier that $(d\mathbf{e}'_i/dt)_{S'} = 0$ and that $(d\mathbf{e}'_i/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_i$ so "the rule holds". We could have established (2.5) from the start "by inspection", and then derived (2.1) from it as follows. Rewrite (2.5),

$(d\mathbf{e}'_i/dt)_S - \boldsymbol{\omega} \times \mathbf{e}'_i - (d\mathbf{e}'_i/dt)_{S'} = 0$ (2.5a)

Then from the expansion $\mathbf{a} = a'_i\ \mathbf{e}'_i$ we have

$(d\mathbf{a}/dt)_S = a'_i(d\mathbf{e}'_i/dt)_S$ $\qquad \boldsymbol{\omega} \times \mathbf{a} = a'_i\ \boldsymbol{\omega} \times \mathbf{e}'_i$ $\qquad (d\mathbf{a}/dt)_{S'} = a'_i\ (d\mathbf{e}'_i/dt)_{S'}$

$(d\mathbf{a}/dt)_S - \boldsymbol{\omega} \times \mathbf{a} - (d\mathbf{a}/dt)_{S'} = a'_i(d\mathbf{e}'_i/dt)_S - a'_i\ \boldsymbol{\omega} \times \mathbf{e}'_i - a'_i\ (d\mathbf{e}'_i/dt)_{S'}$

$= a'_i\ [(d\mathbf{e}'_i/dt)_S - \boldsymbol{\omega} \times \mathbf{e}'_i - (d\mathbf{e}'_i/dt)_{S'}] = a'_i\ [0] = 0 \quad \Rightarrow (d\mathbf{a}/dt)_S = \boldsymbol{\omega} \times \mathbf{a} + (d\mathbf{a}/dt)_{S'}$

Comment: Are there any restrictions on the vector **a** for which this Rule applies? The only fact used above about **a** is that **a** can be expanded on Cartesian basis vectors as $\mathbf{a} = a'_i\ \mathbf{e}'_i$ and that the component fields $a'_i(t)$ are differentiable. Since the unit vectors $\mathbf{e}'_i$ are true tensorial vectors with respect to rotations, we conclude that **a** must also be a true tensorial vector under rotations. If the tail of vector **a** does not lie at the origin of frame S', we just translate **a** such that this is the case. The conclusion then is that this Rule applies to any vector **a** which transforms as a vector under rotations. Examples: **r**, **v**, **a**, **p**, **L** (position, velocity, acceleration, linear momentum, angular momentum).

3. The Apparatus and its Observer at Rest in Frame S'.

Imagine that we have some Apparatus sitting in frame S' which contains a Particle which undergoes some motion. An Observer also sitting in frame S' has some measurement equipment, can see the axes of frame S' of course, and does certain measurements on the Particle *when frame S' is rotating.* Here then are some of the actions of the Observer in Frame S' :

• He takes note of the location of the Particle at time t and writes it down. This location we are going to denote by the vector symbol $\mathbf{r}_{\mathbf{body}}$. Our Observer will take note of the components of this vector by using the expansion

$$\mathbf{r}_{\mathbf{body}}(t) = (\mathbf{r}_{\mathbf{body}})'_i \ \mathbf{e}'_i \qquad\qquad (\mathbf{r}_{\mathbf{body}})'_i = \mathbf{r}_{\mathbf{body}} \bullet \mathbf{e}'_i$$

The prime indicates that we have components with respect to frame S', not some other frame. The vectors $\mathbf{e}'_i$ are of course the basis vectors for frame S'.

• He notes the mentioned location $\mathbf{r}_{\mathbf{body}}(t)$, waits one tick dt of time, then notes new location $\mathbf{r}_{\mathbf{body}}(t+dt)$. From these two measurements he deduces the velocity in frame S' which we shall call $\mathbf{v}_{\mathbf{body}}$. Thus

$$\mathbf{v}_{\mathbf{body}}(t) = [\mathbf{r}_{\mathbf{body}}(t+dt) - \mathbf{r}_{\mathbf{body}}(t)]/(dt) \qquad \text{or} \qquad \mathbf{v}_{\mathbf{body}}(t) = (d\mathbf{r}_{\mathbf{body}}/dt)_{\mathbf{S}'}$$

• Next, he lets another tick go by and records $\mathbf{r}_{\mathbf{body}}(t+2dt)$ as well. Then he determines the acceleration of the Particle $\mathbf{a}_{\mathbf{body}}$ according to

$$\mathbf{v}_{\mathbf{body}}(t+dt) = [\mathbf{r}_{\mathbf{body}}(t+2dt) - \mathbf{r}_{\mathbf{body}}(t+dt)]/(dt)$$

$$\mathbf{a}_{\mathbf{body}}(t) = [\mathbf{v}_{\mathbf{body}}(t+dt) - \mathbf{v}_{\mathbf{body}}(t)]/(dt) \qquad \text{or} \qquad \mathbf{a}_{\mathbf{body}}(t) = (d\mathbf{v}_{\mathbf{body}}/dt)_{\mathbf{S}'}$$

He could of course go on to measure higher derivatives of the motion, but we will let him stop here.

• Knowing $\mathbf{r}_{\mathbf{body}}$ and $\mathbf{v}_{\mathbf{body}}$, the Observer knows the angular momentum of the Particle

$$L_{\mathbf{body}}(t) = \mathbf{r}_{\mathbf{body}}(t) \times m\mathbf{v}_{\mathbf{body}}(t)$$

• Notice that each of the three vectors $\mathbf{r}_{\mathbf{body}}$, $\mathbf{v}_{\mathbf{body}}$, $\mathbf{a}_{\mathbf{body}}$ and $L_{\mathbf{body}}$ could be expanded in terms of the basis vectors of *either* frame S' (as he did above) or frame S. For example

$$\mathbf{a}_{\mathbf{body}} = (\mathbf{a}_{\mathbf{body}})'_i \ \mathbf{e}'_i \qquad\qquad (\mathbf{a}_{\mathbf{body}})'_i = \mathbf{a}_{\mathbf{body}} \bullet \mathbf{e}'_i$$
$$\mathbf{a}_{\mathbf{body}} = (\mathbf{a}_{\mathbf{body}})_i \ \mathbf{e}_i \qquad\qquad (\mathbf{a}_{\mathbf{body}})_i = \mathbf{a}_{\mathbf{body}} \bullet \mathbf{e}_i$$

Example: Suppose the entire Apparatus consists of a single Particle just floating at rest in inertial Frame S. Our Observer in rotating frame S' will duly observe that this Particle follows a curved path in frame S' with some $\mathbf{r}_{\mathbf{body}}(t)$, and from that he will deduce $\mathbf{v}_{\mathbf{body}}(t)$ and $\mathbf{a}_{\mathbf{body}}(t)$ and $L_{\mathbf{body}}(t)$.

Other Examples: The Particle might be a mass on one or more springs, or it might be Particle in ballistic flight, or it might be a Particle of matter in a gear wheel which is turning in some complicated machine, or it might be a Particle of a fluid or of an elastic solid.

4. The Purpose of this Effort

Given the frame S' Observer's measured values $\mathbf{r}_{body}(t)$, $\mathbf{v}_{body}(t)$, $\mathbf{a}_{body}(t)$ we want to be able to know the values of $\mathbf{r}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ which describe the motion of the same Particle as observed from a different frame S. The relation between the two frames is given by this (simplified) picture:

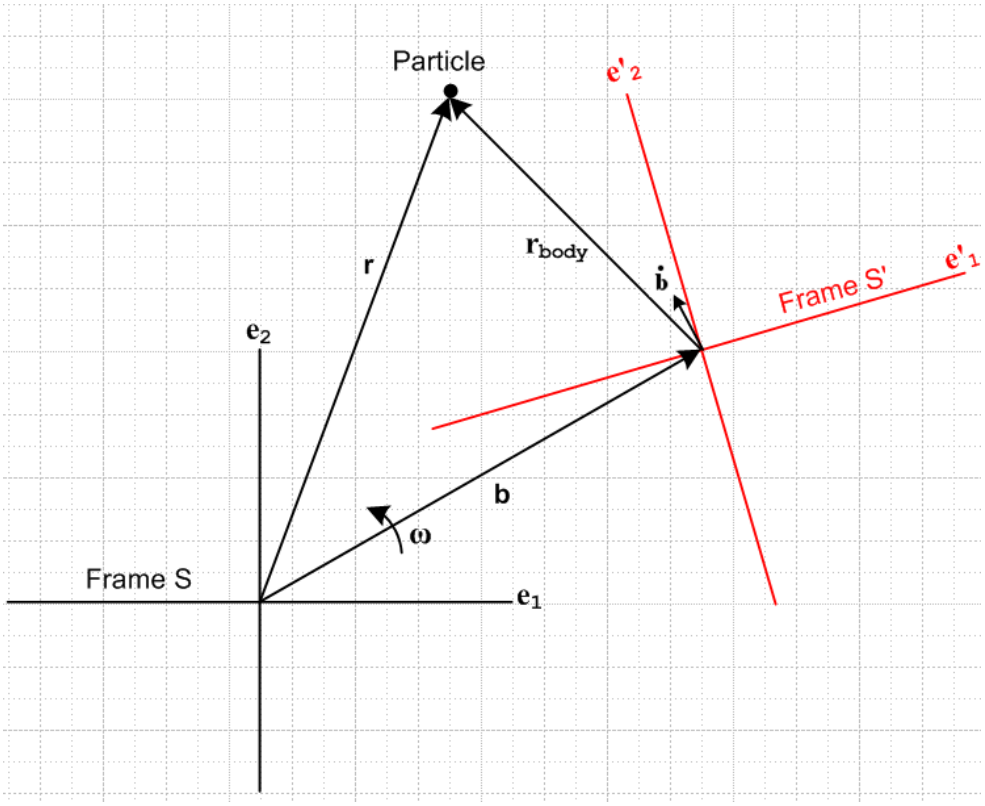


Fig 1

The picture requires some explanation. At some time t, frame S' has some particular Orientation relative to frame S which is determined by some vector of rotation parameters $\boldsymbol{\theta}$ (= $\boldsymbol{\theta}(t)$),

$$\mathbf{e'}_n(t) = R(\boldsymbol{\theta})\,\mathbf{e}_n \qquad R(\boldsymbol{\theta}) = \exp(-i\boldsymbol{\theta} \bullet \mathbf{J}) \qquad i\,(J_k)_{ij} = \varepsilon_{kij} \qquad (4.1)$$

The origin of frame S' is located at the tip of a vector called **b** whose tail lies at the origin of frame S. As time progresses, the frame S' rotates according to some vector function $\boldsymbol{\omega}(t)$ about the origin of frame S. We think of frame S' including its axes, its $\mathbf{e'}_n$ unit basis vectors , and its contained Apparatus as rigidly rotating at this rate $\boldsymbol{\omega}$. The length of vector **b** is a constant during this rotation.

This $\boldsymbol{\omega}(t)$ is the *instantaneous* angular velocity of frame S' relative to frame S. We say "instantaneous" since both $\hat{\boldsymbol{\omega}}$ and $|\boldsymbol{\omega}|$ can be changing in time. To understand the meaning of $\boldsymbol{\omega}$, consider

$\mathbf{e'_n}(t+dt) = R(d\boldsymbol{\varphi})\ \mathbf{e'_n}(t)$ where $R(d\boldsymbol{\varphi})$ = some small rotation (4.2)

$d\mathbf{e'_n} = \mathbf{e'_n}(t+dt) - \mathbf{e'_n}(t) = [R(d\boldsymbol{\varphi}) - 1]\ \mathbf{e'_n}(t) = -id\boldsymbol{\varphi} \bullet \mathbf{J}\ \mathbf{e'_n}(t) = -id\varphi_k\, J_k\ \mathbf{e'_n}(t)$ (4.3a)

$\Rightarrow (d\mathbf{e'_n})_i = -id\varphi_k\,(J_k)_{ij}\ (\mathbf{e'_n}(t))_j = -\,d\varphi_k \varepsilon_{kij}\ (\mathbf{e'_n}(t))_j = \varepsilon_{ikj}\, d\varphi_k\ (\mathbf{e'_n}(t))_j$

$\Rightarrow d\mathbf{e'_n} = d\boldsymbol{\varphi} \times \mathbf{e'_n}$ (4.3b)

$\Rightarrow (d\mathbf{e'_n}/dt)_S = \boldsymbol{\omega} \times \mathbf{e'_n}$ n = 1,2,3 where $\boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/ddt$ (4.4)

Thus $\boldsymbol{\omega}$ tells us how the unit vectors $\mathbf{e'_n}$ are instantaneously rotating relative to frame S. The three $\mathbf{e'_n}$ all rotate in the same manner because the entire frame S' is rotating as a rigid object.

In the picture above, then, frame S' and everything fixed within it as well as vector **b** all rotate together at $\boldsymbol{\omega}$ about the origin of frame S with instantaneous angular velocity $\boldsymbol{\omega}$. For example,

$(d\mathbf{e'_n}/dt)_{S'} = 0$ $(d\mathbf{b}/dt)_{S'} = 0$

Comments:

- Note that, in general,

$\mathbf{e'_n}(t+dt) = R(d\boldsymbol{\varphi})\ \mathbf{e'_n}(t) = R(d\boldsymbol{\varphi})\ R(\boldsymbol{\theta})\ \mathbf{e_n} = R(\boldsymbol{\theta} + d\boldsymbol{\Omega})\ \mathbf{e_n} \neq R(\boldsymbol{\theta} + d\boldsymbol{\varphi})\ \mathbf{e_n}$ (4.5)

where $|d\boldsymbol{\Omega}|$ is proportional to $|d\boldsymbol{\varphi}|$ but $d\boldsymbol{\Omega}$ is some complicated function of $\boldsymbol{\theta}$ and $d\boldsymbol{\varphi}$. This function is only simple of it happens that $\boldsymbol{\theta}$ and $d\boldsymbol{\varphi}$ (or $\boldsymbol{\omega}$) point in the same direction, in which case $d\boldsymbol{\Omega} = d\boldsymbol{\varphi}$.

- Often Frame S is taken to be a frame "at rest with respect to the stars" meaning it is an "inertial frame" in which Newton's non-relativistic law $\mathbf{F} = m\mathbf{a}$ applies (it does not apply in rotating frame S'). For the sections of this document which don't deal with $\mathbf{F} = m\mathbf{a}$, frame S need not be such an inertial frame.

- Goldstein refers to frame S as the **space** frame, and frame S' as the **body** frame. In his analysis, our Apparatus is just a rigid body at rest in frame S'.

- We could consider a different situation in which frame S' rotates at $\boldsymbol{\omega}$ about *its* origin and **b** stays fixed, but that is not the situation we wish to consider here. Another situation of possible interest might be either of these rotation senses combined with an active variation of the origin-spanning vector **b**(t).

- The limit $\mathbf{b} \rightarrow 0$ causes the origins of the two frames to coincide and this results in a simplification of some of the equations in sections below. We have then $\mathbf{b} = 0$ and $\dot{\mathbf{b}} \equiv (d\mathbf{b}/dt)_S = 0$, etc.

- Three applications of the above situation are:

(1) An ant (Particle) is moving around on a phonograph record rotating on its spindle with some $\boldsymbol{\omega}(t)$. The frame S' might be set up some distance $|\mathbf{b}(t)|$ from the spindle, and might have its $\mathbf{e'_1}$ axis pointing to the right and its $\mathbf{e'_2}$ axis pointing to the spindle. That is to say, $\mathbf{e'_1} = \hat{\boldsymbol{\theta}}'$ and $\mathbf{e'_1} = -\hat{\mathbf{r}}'$ if we think of r',θ' as polar coordinates for rotating frame S'.

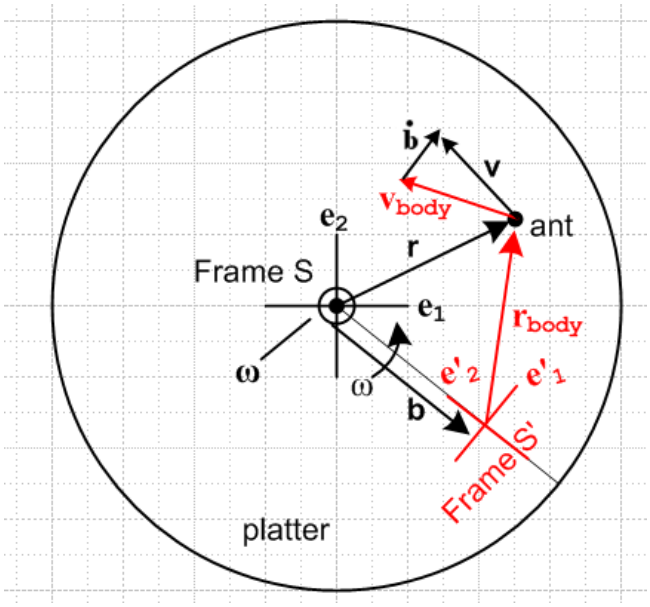


Fig 2

(2) On the Earth, Cartesian frame S might be an inertial frame located at Earth center with $\hat{\mathbf{z}}$ pointing to the North Pole. The $\hat{\mathbf{x}}$ axis points out through a point on the equator to some fixed distant star (the X Star), and then $\hat{\mathbf{y}}$ is the third Cartesian axis which points out through a different point on the equator to the Y Star. Meanwhile, Frame S' has its origin at some arbitrary point on the surface of the Earth. For this system, the $\hat{\mathbf{z}}'$ axis points "up", meaning in the $\hat{\mathbf{r}}$ direction for Frame S in sphericals. The $\hat{\mathbf{x}}'$ axis points to the East, and the $\hat{\mathbf{y}}'$ axis points North. The earth rotates at some $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ with $\omega > 0$.

(3) Some Apparatus is located in Frame S instead of S', and Frame S' is a "camera platform" which flies around in some complicated way and observes the activity in frame S. In this case, both $\boldsymbol{\omega}(t)$ and $\mathbf{b}(t)$ would be under the command of the pilot of the camera platform.

5. Determination of r(t).

Well, this is quite simple. We have, looking at Fig 1 above,

$$\mathbf{r}(t) = \mathbf{r}_{\mathbf{body}}(t) + \mathbf{b}(t) \qquad (5.1)$$

Since our Observer in frame S' determined $\mathbf{r}_{\mathbf{body}}(t)$ and since the above "setup" determines $\mathbf{b}(t)$. we immediately know $\mathbf{r}(t)$ and this part of our Purpose is achieved.

6. Determination of $\mathbf{v}(t) = (d\mathbf{r}/dt)_S$

(a) Compute $(d\mathbf{r}/dt)_S$ in a form suitable for G Rule Comparison

Start with our Section 2 result above, which was

$$\mathbf{r}(t) = \mathbf{r}_{body}(t) + \mathbf{b}(t) \qquad (5.1)$$

Differentiate this in frame S to get

$$(d\mathbf{r}/dt)_S = (d\mathbf{r}_{body}/dt)_S + (d\mathbf{b}/dt)_S \; . \qquad (6.1)$$

Now use the G Rule for vector $\mathbf{r}_{body}$ (equation (2.1) with $\mathbf{a} = \mathbf{r}_{body}$)

$$(d\mathbf{r}_{body}/dt)_S = \boldsymbol{\omega} \times \mathbf{r}_{body} + (d\mathbf{r}_{body}/dt)_{S'} \qquad (6.2)$$

Then we have

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r}_{body} + (d\mathbf{r}_{body}/dt)_{S'} + (d\mathbf{b}/dt)_S \qquad (6.3)$$

It is easy to see from the picture that (this can be derived by the same steps leading to (4.4) above)

$$(d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} \qquad (6.4)$$

so we then have

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r}_{body} + (d\mathbf{r}_{body}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{b} \qquad (6.5)$$

Combining the first and last terms and using the fact (5.1) that $\mathbf{r} = \mathbf{r}_{body} + \mathbf{b}$, we get

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}_{body}/dt)_{S'} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} \qquad (6.6)$$

(b) Compare with the G rule for $\mathbf{r}$ and get expression for $(d\mathbf{r}/dt)_{S'}$

The G Rule applied to $\mathbf{a} = \mathbf{r}$ says that

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + (d\mathbf{r}/dt)_{S'} \qquad (6.7)$$

So we may conclude that

$$(d\mathbf{r}/dt)_{S'} = (d\mathbf{r}_{body}/dt)_{S'} = \mathbf{v}_{body} \qquad (6.8)$$

Here is a much faster way obtain this same result:

$$\mathbf{r} = \mathbf{r}_{body} + \mathbf{b}$$

$$(d\mathbf{r}/dt)_{S'} = (d\mathbf{r}_{body}/dt)_{S'} + (d\mathbf{b}/dt)_{S'} = (d\mathbf{r}_{body}/dt)_{S'} + 0 = (d\mathbf{r}_{body}/dt)_{S'}$$

because vector **b** is constant when viewed from frame S' (just as $\mathbf{e'}_n$ are constants).

So first, we are consistent with the G Rule for **r**, and second, we know how to compute $\mathbf{v} \equiv (d\mathbf{r}/dt)_S$.

(c) Compute $(d\mathbf{r}/dt)_S$ in fully reduced form.

The bottom line then is (6.6) which we now write as

$$\mathbf{v}(t) = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} = \boldsymbol{\omega} \times \mathbf{r}_{body} + \boldsymbol{\omega} \times \mathbf{b} + \mathbf{v}_{body} \qquad (6.9)$$

Our Purpose was to find **v**(t) and there it is in terms of known quantities **ω**, **b**, $\mathbf{r}_{body}$ and $\mathbf{v}_{body}$

7. Determination of $\mathbf{a(t) = (dv/dt)_S}$

(a) Compute $(d\mathbf{v}/dt)_S$ in a form suitable for G Rule Comparison

Let's start then with (6.6) or (6.9),

$$\mathbf{v} = (d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} \qquad \textit{(6.6)}$$

Apply d/dt in frame S to get

$$\mathbf{a} = (d\mathbf{v}/dt)_S = (d\boldsymbol{\omega}/dt)_S \times \mathbf{r} + \boldsymbol{\omega} \times (d\mathbf{r}/dt)_S + (d\mathbf{v}_{body}/dt)_S$$

Define $\dot{\boldsymbol{\omega}} \equiv (d\boldsymbol{\omega}/dt)_S$ and $\mathbf{v} = (d\mathbf{r}/dt)_S$ so the above says (reordering terms)

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + (d\mathbf{v}_{body}/dt)_S \qquad (7.1)$$

For our Particle of interest, **w** x **v** is the centripetal acceleration toward the rotation axis, $\dot{\boldsymbol{\omega}}$ x **r** is the tangential acceleration due to varying **ω**, and $(d\mathbf{v}_{body}/dt)_S$ is the residual acceleration of the body in frame S', but measured in frame S. We can convert this to a frame S' derivative using the G rule for $\mathbf{v}_{body}$,

$$(d\mathbf{v}_{body}/dt)_S = \boldsymbol{\omega} \times \mathbf{v}_{body} + (d\mathbf{v}_{body}/dt)_{S'} = \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \qquad (7.2)$$

so that then

$$\mathbf{a}(t) = (d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \quad . \qquad (7.3)$$

(b) Compare with the G rule for **v** and get expression for $(d\mathbf{v}/dt)_{S'}$

Now the G Rule for **v** tells us that

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + (d\mathbf{v}/dt)_{S'} \quad . \tag{7.4}$$

Comparison of (7.3) and (7.4) shows that,

$$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \quad . \tag{7.5}$$

Recall that $\mathbf{a}_{body}$ is something known to our frame S' observer, so we can easily compute $(d\mathbf{v}/dt)_{S'}$ from (7.5), but this object $(d\mathbf{v}/dt)_{S'}$ is not of spectacular interest.

(c) Compute $(d\mathbf{r}/dt)_S$ in fully reduced form.

We found above that,

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \tag{7.3}$$

where recall that the $\boldsymbol{\omega} \times \mathbf{v}_{body}$ term came from the G rule for $\mathbf{v}_{body}$ which appears as (7.2) above. The last step is to replace **v** appearing in the $\boldsymbol{\omega} \times \mathbf{v}$ term by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}$ from (6.9) to get

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}] + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body}$$

$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times \mathbf{v}_{body} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body}$$

$$= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,\boldsymbol{\omega} \times \mathbf{v}_{body} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \mathbf{a}_{body}$$

and we see that our substitution has generated a *second* $\boldsymbol{\omega} \times \mathbf{v}_{body}$ term and we end up then with

$$\mathbf{a}(t) = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \tag{7.6}$$

Thus we have found **a**(t) in terms of known quantities. The only further reduction is $\mathbf{r} = \mathbf{r}_{body} + \mathbf{b}$.

8. The Fictitious Forces

If Frame S is an inertial frame, then Newton's 2nd Law in that frame, $\mathbf{F} = m\mathbf{a}$, is valid and we can write, using (7.6),

$$\mathbf{F} = m\mathbf{a} = m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2m\boldsymbol{\omega} \times \mathbf{v}_{body} + m\mathbf{a}_{body} \quad . \tag{8.1}$$

Our Observer in Frame S' would like to use his own personal "Newton's Law",

$$\mathbf{F}_{body} = m\,\mathbf{a}_{body} \quad . \tag{8.2}$$

Solving (8.1) for $m\mathbf{a}_{body}$ we get

$$\mathbf{F}_{body} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v}_{body}$$

$$= \mathbf{F} + \mathbf{F}_{fict} \qquad (8.3)$$

where

$$\mathbf{F}_{fict} = - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \mathbf{v}_{body} \qquad (8.4)$$

So our Observer in Frame S' is allowed to use his own version of Newton's Law, (7.8), *as long as* he includes in his sum of forces the "fictitious" forces shown in (7.10).

The **first fictitious force** $- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the centrifugal (center fleeing) force pushing our Particle away from the rotation axis, and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the associated centripetal acceleration. To see that this is the case, consider this picture which shows vectors **r** and **ω**,

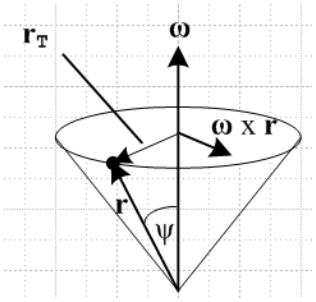


Fig 3

Using the right hand rule, one can see then that $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector pointing in the opposite direction to the vector labeled $\mathbf{r}_T$. From the location of the Particle at **r**, this vector points toward the center of the circle on which the particle is instantaneously rotating at time t, and this is the correct direction for centripetal (center seeking) acceleration. The magnitude of the vector is $|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = \omega \, | \, \boldsymbol{\omega} \times \mathbf{r} \, | = \omega^2 r \sin\psi = \omega^2 r_T$ and this is the correct magnitude for the centripetal acceleration. So the centripetal acceleration $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points toward the rotation axis, while the centrifugal force $- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is directed away from the rotation axis, as all Merry-Go-Round (not to mention Rotor) riders well know.

The **second fictitious force** $- m\,\dot{\boldsymbol{\omega}} \times \mathbf{r}$ in (7.10) exists only if $\boldsymbol{\omega} \neq$ constant. In the special case the $\dot{\boldsymbol{\omega}}$ happens to be in the direction of **ω,** meaning $\omega > 0$ (this would apply in our phonograph platter scenario if the platter is spinning up) then $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ is in the same direction as $\boldsymbol{\omega} \times \mathbf{r}$ shown in Fig 3 and that is the direction of the tangential acceleration of a point on the platter. Correspondingly, a Particle in Frame S' feels a fictitious force $- m\,\dot{\boldsymbol{\omega}} \times \mathbf{r}$ in the opposite direction.

The **third fictitious force**

$$- 2m\boldsymbol{\omega} \times \mathbf{v}_{body} = + 2m\, \mathbf{v}_{body} \times \boldsymbol{\omega}$$

is the famous **Coriolis** force which causes a ballistic Particle to deflect to the right in the northern hemisphere and to the left in the southern hemisphere.

9. Determination of L(t) and $(dL/dt)_S$

Our frame S' observer can measure (set m = 1)

$$\mathbf{L}_{body} = \mathbf{r}_{body} \times \mathbf{v}_{body} \tag{9.1}$$

and add this to his list of dutiful measurements made in frame S'. We then have

$$(d\mathbf{L}_{body}/dt)_{S'} = (d\mathbf{r}_{body}/dt)_{S'} \times \mathbf{v}_{body} + \mathbf{r}_{body} \times (d\mathbf{v}_{body}/dt)_{S'}$$

$$= \mathbf{v}_{body} \times \mathbf{v}_{body} + \mathbf{r}_{body} \times \mathbf{a}_{body} = \mathbf{r}_{body} \times \mathbf{a}_{body}$$

or

$$(d\mathbf{L}_{body}/dt)_{S'} = \mathbf{r}_{body} \times \mathbf{a}_{body} \ . \tag{9.2}$$

In frame S we have

$$\mathbf{L} = \mathbf{r} \times \mathbf{v} \tag{9.3}$$

Since we already have expressions for **r** and **v**,

$$\mathbf{r} = \mathbf{r}_{body} + \mathbf{b} \tag{5.1}$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} \tag{6.9}$$

we can insert to get

$$\mathbf{L} = [\mathbf{r}_{body} + \mathbf{b}] \times [\boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}]$$

$$= \mathbf{r}_{body} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{b} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r}_{body} \times \mathbf{v}_{body} + \mathbf{b} \times \mathbf{v}_{body}$$

$$= \mathbf{r}_{body} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{b} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{L}_{body} + \mathbf{b} \times \mathbf{v}_{body} \tag{9.4}$$

and we have found **L**(t) as a function of known quantities, one of our goals stated above.

(a) Compute $(d\mathbf{L}/dt)_S$ in a form suitable for G Rule Comparison

As for the derivative of **L**,

$$(d\mathbf{L}/dt)_S = (d\mathbf{r}/dt)_S \times \mathbf{v} + \mathbf{r} \times (d\mathbf{v}/dt)_S = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times (d\mathbf{v}/dt)_S = \mathbf{r} \times (d\mathbf{v}/dt)_S \tag{9.5}$$

We then use (7.3),

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \quad . \qquad (7.3)$$

to find that

$$(d\mathbf{L}/dt)_S = \mathbf{r} \times [\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body}]$$

$$= \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) + \mathbf{r} \times \mathbf{a}_{body} \quad . \qquad (9.6)$$

The vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \text{cyclic} = 0$ shows that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) - \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega})$$

$$= \boldsymbol{\omega} \times \mathbf{L} - \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega})$$

so that we get

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} - \mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) + \mathbf{r} \times \mathbf{a}_{body} \qquad (9.7)$$

Since $\mathbf{v}(t) = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}$ from (6.9), the second term on the right side may be written

$$-\mathbf{v} \times (\mathbf{r} \times \boldsymbol{\omega}) = [\boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}] \times (\mathbf{r} \times \boldsymbol{\omega}) = \mathbf{v}_{body} \times (\mathbf{r} \times \boldsymbol{\omega})$$

so that then

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} + \mathbf{v}_{body} \times (\mathbf{r} \times \boldsymbol{\omega}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) + \mathbf{r} \times \mathbf{a}_{body} \qquad (9.8)$$

But now the second and fourth terms can be combined using the identity above which we write as

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}_{body}) - \mathbf{v}_{body} \times (\mathbf{r} \times \boldsymbol{\omega})$$

and therefore

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} + \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}_{body}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times \mathbf{a}_{body} \qquad (9.9)$$

<u>(b) Compare with the G rule for **L** and get expression for $(d\mathbf{L}/dt)_{S'}$</u>

The G Rule for **L** states that

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega} \times \mathbf{L} + (d\mathbf{L}/dt)_{S'} \qquad (9.10)$$

so we conclude that

$$(d\mathbf{L}/dt)_{S'} = \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}_{body}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times \mathbf{a}_{body} \quad (9.11)$$

(c) Compute $(d\mathbf{L}/dt)_S$ in fully reduced form.

We had

$$(d\mathbf{L}/dt)_S = \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) + \mathbf{r} \times \mathbf{a}_{body} \quad . \quad (9.6)$$

Since $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body}$, the first term may be written

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = \mathbf{r} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body})$$

The triple cross term may be simplified by using this vector identity

$$\mathbf{A} \times [\, \mathbf{B} \times (\mathbf{C} \times \mathbf{D}) \,] = \mathbf{B}\,[\mathbf{A},\mathbf{C},\mathbf{D}] - (\mathbf{A} \bullet \mathbf{B})\,(\mathbf{C} \times \mathbf{D})$$
$$\mathbf{r} \times [\, \boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) \,] = \boldsymbol{\omega}\,[\mathbf{r},\mathbf{r},\mathbf{D}] - (\mathbf{r} \bullet \boldsymbol{\omega})\,(\mathbf{r} \times \boldsymbol{\omega}) = -(\mathbf{r} \bullet \boldsymbol{\omega})\,(\mathbf{r} \times \boldsymbol{\omega}) \quad (9.12)$$

so that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) = -(\mathbf{r} \bullet \boldsymbol{\omega})\,(\mathbf{r} \times \boldsymbol{\omega}) + \mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body})$$

Then we have

$$(d\mathbf{L}/dt)_S = -(\mathbf{r} \bullet \boldsymbol{\omega})\,(\mathbf{r} \times \boldsymbol{\omega}) - \mathbf{r} \times (\mathbf{r} \times \dot{\boldsymbol{\omega}}) + 2\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}_{body}) + \mathbf{r} \times \mathbf{a}_{body} \quad . \quad (9.13)$$

into which one could insert $\mathbf{r} = \mathbf{r}_{body} + \mathbf{b}$ to get a result expressed entirely in terms of "body" objects and $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$ and $\mathbf{b}$.

10. Conclusions

We seem to have accomplished our Purpose, and here are the results for frame S quantities in terms of $\boldsymbol{\omega}$, $\dot{\boldsymbol{\omega}}$, $\mathbf{b}$ (see Fig 1) and frame S' quantities

$$\mathbf{r} = \mathbf{b} + \mathbf{r}_{body} \quad (5.1)$$

$$\mathbf{v} = (d\mathbf{r}/dt)_S = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{body} = \boldsymbol{\omega} \times \mathbf{r}_{body} + \boldsymbol{\omega} \times \mathbf{b} + \mathbf{v}_{body} \quad (6.9)$$

$$\mathbf{a} = (d\mathbf{v}/dt)_S = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_{body} + \mathbf{a}_{body} \quad (7.6)$$

$$\mathbf{L} = \mathbf{r}_{body} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{b} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{r}_{body} \times \mathbf{v}_{body} + \mathbf{b} \times \mathbf{v}_{body} \quad (9.4)$$

$$(d\mathbf{L}/dt)_S = -(\mathbf{r}\bullet\boldsymbol{\omega})(\mathbf{r}\times\boldsymbol{\omega}) - \mathbf{r}\times(\mathbf{r}\times\dot{\boldsymbol{\omega}}) + 2\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{v}_{body}) + \mathbf{r}\times\mathbf{a}_{body} \qquad (9.13)$$

There are three G rules, those for **r**, **v**, and **L** and they are of course all valid with expressions as shown:

$$(d\mathbf{r}/dt)_S = \boldsymbol{\omega}\times\mathbf{r} + (d\mathbf{r}/dt)_{S'} \qquad (6.7)$$

$$(d\mathbf{r}/dt)_{S'} = \mathbf{v}_{body} \qquad (6.8)$$

$$(d\mathbf{v}/dt)_S = \boldsymbol{\omega}\times\mathbf{v} + (d\mathbf{v}/dt)_{S'} \quad . \qquad (7.4)$$

$$(d\mathbf{v}/dt)_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times\mathbf{v}_{body} + \mathbf{a}_{body} \qquad (7.5)$$

$$(d\mathbf{L}/dt)_S = \boldsymbol{\omega}\times\mathbf{L} + (d\mathbf{L}/dt)_{S'} \qquad (9.10)$$

$$(d\mathbf{L}/dt)_{S'} = \boldsymbol{\omega}\times(\mathbf{r}\times\mathbf{v}_{body}) - \mathbf{r}\times(\mathbf{r}\times\dot{\boldsymbol{\omega}}) + \mathbf{r}\times\mathbf{a}_{body} \qquad (9.11)$$

Newton's 2nd Law for an Observer on the rotating frame S' has this form

$\mathbf{F}_{body} = m\,\mathbf{a}_{body}$ where $\mathbf{F}_{body} = \mathbf{F} + \mathbf{F}_{fict}$ where

$\mathbf{F}_{fict} = -\,m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r} - 2m\boldsymbol{\omega}\times\mathbf{v}_{body}$ = the fictitious forces

F = any other forces (eg, gravity).