old section 1 (j) PhL 8.17.12

(j) When do operations d/dt and taking a component "commute" ?

In a footnote at the end of this subsection, we do four derivative calculations of a generic vector **a**. Here we state the four results. We put subscripts on the derivatives of components even though we know from (i) that such subscripts, although correct, are superfluous :

1. $[(d\mathbf{a}/dt)_S]_j = (d(a)_j/dt)_S$

2. $[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_j/dt)_{S'} - \varepsilon_{jai}\,\omega_a\,(a)_i$

3. $[(d\mathbf{a}'/dt)_{S'}]_j = (d(a')_i/dt)_{S'}\,R_{ij}$

4. $[(d\mathbf{a}'/dt)_S]_j = (d(a')_i/dt)_S\,R_{ij} + \varepsilon_{jab}\omega_a R_{ib}(a')_i$ (1.38)

The lack of symmetry is due to the fact that our j^{th} component is always an S frame component, meaning that we have $A_j = \mathbf{A} \bullet \mathbf{e}_j$. We could write another four equations which have $[\ ....]'_j$ on the left and these would have R factors in the first two equations.

Now look at the first of the above four lines as LHS = RHS.
On the LHS we first apply $(d/dt)_S$ to **a**, and *then* we take the j^{th} component of the result.
On the RHS we first take the j^{th} component of **a**, then we apply $(d/dt)_S$ to the result.
The result is the same regardless of the order of doing these two operations.
The two operations "commute".

But:

On the second line this is *not true* because there is an extra term.
On the third line this is *not true* because there is an extra factor R_{ij}.
On the fourth line this is *not true* because there is an extra factor R_{ij} and an extra term.

Main Point: In general, the operations of doing d/dt and of taking a component *do not commute*.
One must never assume commutation in doing calculations.

Goldstein makes this point as a "word of caution" on the bottom of page 133 with an example on the top of page 134. In GPS the caution is stated on page 173, but the example has been removed.

Footnote: Here are details of the above four calculations which the reader is invited to ignore :

1. $(d\mathbf{a}/dt)_S = (d\,[(a)_i\mathbf{e}_i]/dt)_S = (d(a)_i/dt)_S\,\mathbf{e}_i + (a)_i\,(d\mathbf{e}_i/dt)_S = (d(a)_i/dt)\,\mathbf{e}_i$

   $\Rightarrow [(d\mathbf{a}/dt)_S]_j = \{(d(a)_i/dt)\,\mathbf{e}_i\} \bullet \mathbf{e}_j = (d(a)_i/dt)\,\delta_{i,j} = (d(a)_j/dt)$.

2. $(d\mathbf{a}/dt)_{S'} = (d\,[(a)_i\mathbf{e}_i]/dt)_{S'} = (d(a)_i/dt)_{S'}\,\mathbf{e}_i + (a)_i\,(d\mathbf{e}_i/dt)_{S'} = (d(a)_i/dt)\,\mathbf{e}_i - (a)_i\,\boldsymbol{\omega} \times \mathbf{e}_i$

In the above we used (1.27) and (1.28).

Now since $(\mathbf{e}_i)_j = \delta_{i,j}$ and $[\boldsymbol{\omega} \times \mathbf{e}_i]_j = \varepsilon_{jab}\omega_a(e_i)_b = \varepsilon_{jab}\omega_a\delta_{i,b} = \varepsilon_{jai}\omega_a$, this last line is

$[(d\mathbf{a}/dt)_{S'}]_j = (d(a)_j/dt) - \varepsilon_{jai}\,\omega_a(a)_i$.

For the other two derivatives we do copy paste and edit on the above:

3. $(d\mathbf{a}'/dt)_{S'} = (d\,[\,(a')_i\mathbf{e}'_i]/dt)_{S'} = (d(a')_i/dt)_{S'}\,\mathbf{e}'_i + (a')_i\,(d\mathbf{e}'_i/dt)_{S'} = (d(a')_i/dt)\,\mathbf{e}'_i$

   $\Rightarrow [(d\mathbf{a}'/dt)_{S'}]_j = (d(a')_i/dt)\,\mathbf{e}'_i \bullet \mathbf{e}_j = (d(a')_i/dt)\,R_{ij}$

4. $(d\mathbf{a}'/dt)_S = (d\,[\,(a')_i\mathbf{e}'_i]/dt)_S = (d(a')_i/dt)_S\,\mathbf{e}'_i + (a')_i\,(d\mathbf{e}'_i/dt)_S = (d(a')_i/dt)\,\mathbf{e}'_i + (a')_i\,\boldsymbol{\omega} \times \mathbf{e}'_i$

In the above we used (1.26) and (1.7) and (1.25).

Now since $(\mathbf{e}'_i)_j = R_{ij}$ (see 1.7) and $[\boldsymbol{\omega} \times \mathbf{e}'_i]_j = \varepsilon_{jab}\omega_a(e'_i)_b = \varepsilon_{jab}\omega_a R_{ib}$ this last line is

$[(d\mathbf{a}'/dt)_S]_j = (d(a')_i/dt)\,R_{ij} + \varepsilon_{jab}\omega_a R_{ib}(a')_i$.