Note that page numbering is turned on in this template.

The first term is caused by the frame-S' components of **b** changing

The 2nd term on the RHS represents the contribution due to the rotation of the $\mathbf{e}'_i$ axes, while the 1st term is the contribution due to the fact that the S' components b'_i of **b** are changing. Something that causes these components to change would cause the origin of Frame S to move as viewed from Frame S'.

These are the components of vector **b** as projected onto the S' basis vectors.

As noted above, the velocity **ω** x **b** lies in the plane of paper and this term represents a motion of the S' origin (location **b**) in the upward direction in Fig 2, as shown.

Now, if **ω** is in the process of changing direction, the green circle is in the process of tipping out of the plane of paper, and this might contribute to the motion of the S' origin a component perpendicular to the plane of paper. Furthermore, it might be that the rotation axis indicated by vector **ω** is translating sideways in the above picture at some instantaneous velocity, and this would contribute to the motion of the S' origin another amount in the plane of paper.

Therefore, we can interpret $(d\mathbf{b}/dt)_{S'}$ in this manner

$$(d\mathbf{b}/dt)_{S'} = (d\mathbf{b}/dt)_{S}\,|^{\omega=0} \qquad (4.2)$$

where the notation means "evaluated at **ω** = 0".

Relative to Frame S, the motion of point **b** (the origin of frame S') has several contributions.

has a contribution from $\mathbf{v}_1$ and a contribution from the fact that the S' origin is rotating (this latter motion tends to move **b** in a direction tangent to the green circle.) If we were to freeze the instantaneous rotation of Frame S' shown in Figure 2, making ω= 0, then the origin of Frame S' (the tip of vector **b**) would be moving only with $\mathbf{v}_1$ (relative to frame S). Therefore,

$$(d\mathbf{b}/dt)_{S}\,|^{\omega=0} = \mathbf{v}_1 \qquad (4.3)$$

****

it regards this centripetal and "Euler" force as both being "fictional".

$(d\mathbf{r}'/dt)_S = \boldsymbol{\omega} \times \mathbf{r}'$

so in fact vector **r'** is also rotating. We now draw the

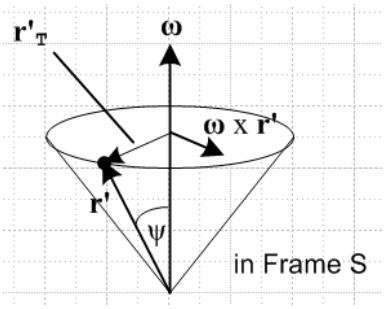
If we translate vector **r'** (which recall is not in the plane of paper) so its tail lies on the rotation axis,

****

Suppose in the above picture **b** is static, **v'** = 0 and **ω** = constant. Then (8.6) becomes

$\mathbf{F}_{fict} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$

Since **v'** = 0, as seen from Frame S' the vector **r'** shown above is *fixed* and so is our Particle. But in Frame S the vector **r'** is (instantaneously) going around on a circular path. If we translate vector **r'** (which recall is not in the plane of paper) so its tail lies on the rotation axis, we can draw this famous picture.



$(d\mathbf{r}'/dt)_S = \boldsymbol{\omega} \times \mathbf{r}'$

The equation says the change in **r'** is perpendicular to **r'**, so that is why we have circular motion. The picture shows the vector **ω** x **r'** from one use of the right hand rule and so, using the right hand rule again, one can see then that **ω** x (**ω** x **r'**) is a vector pointing in the opposite direction to the vector labeled $\mathbf{r}'_T$. From the location of the Particle at **r'**, this vector **ω** x (**ω** x **r'**) points toward the center of the circle on which the particle is instantaneously rotating at time t, and this is the correct direction for centripetal (center seeking) acceleration. The magnitude of the vector is $|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')| = \omega\,|\,\boldsymbol{\omega} \times \mathbf{r}'\,| = \omega^2 r'\sin\psi = \omega^2 r'_T$ and this is the correct magnitude for the centripetal acceleration. So the centripetal acceleration **ω** x **(ω** x **r')** points toward the rotation axis, while the centrifugal force – m**ω** x **(ω** x **r')** is directed away from the rotation axis, as all Merry-Go-Round (not to mention The Rotor) riders well know. Our Particle which is just sitting statically in Frame S' nevertheless feels this force.

[2**] Rotational acceleration due to $\dot{\boldsymbol{\omega}}$.**

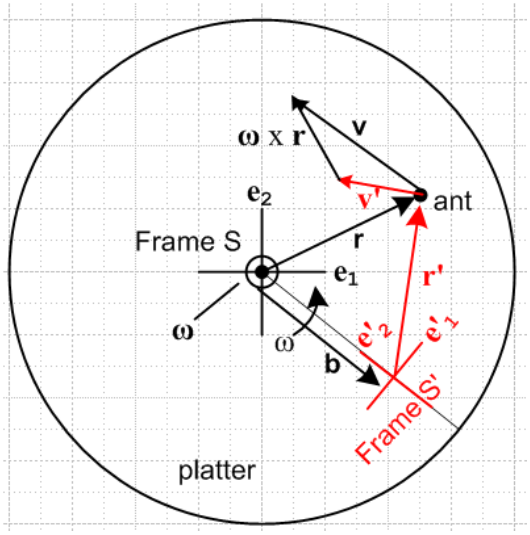
Again assume **b** is static and **v'** = 0 so (8.6) says

$\mathbf{F}_{fict} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$

In addition, assume $\boldsymbol{\omega}$ is 0 or at least very small so the centrifugal term can be ignored. Perhaps think of a turntable that is just starting up its rotation with some $\dot{\boldsymbol{\omega}}$. We then have

$$\mathbf{F}_{\mathtt{fict}} \approx \; - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

We shall address this fictional force with a simple example, our ant on a phonograph record as it spins up with some $\dot{\boldsymbol{\omega}}$ in the same direction as $\boldsymbol{\omega}$. The ant is just standing on the platter as it accelerates. The ant is accelerating in the counter clockwise (+ $\hat{\boldsymbol{\theta}}$) direction



Consider (7.6) for $\mathbf{a}_S$,

$$\mathbf{a}_S \;=\; \mathbf{a}'_S \;+ \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}'_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad (7.6)$$

$$\mathbf{a}_S \;=\; \mathbf{a}'_S + \ddot{\mathbf{b}}_S \quad . \qquad (7.5)$$

Consider the above picture and equation. The equation says

$$\mathbf{v}'_S = \boldsymbol{\omega} \times \mathbf{r}'$$

$$\mathbf{a}'_S = \dot{\boldsymbol{\omega}} \times \mathbf{r}' \; + \text{ second term}$$

$$\mathbf{a}_S \;=\; \mathbf{a}'_S + \ddot{\mathbf{b}}_S \quad . \qquad (7.5)$$

We start with our circular motion and apply $(d/dt)_S$ to get the second line above, and the third line shows how $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ then makes its way into the official Frame S acceleration of the Particle $\mathbf{a}_S$. Due to this contribution, the Particle accelerates in direction $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$

The term $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ contributes to acceleration $\mathbf{a}'_S$ of the particle.

****

Consider the above cone picture showing r' and ω. Suppose there is some $\dot{\omega}$ in the same direction as **ω** so the Particle (dot) is accelerating along its circular path. That acceleration is in the $\boldsymbol{\omega} \times \mathbf{r}'$ direction

Suppose in our general picture above we have some $\dot{\boldsymbol{\omega}}$ pointing in the out of the plane of paper, so the rotation speed is increasing. Assume again that $\mathbf{v}'_{S'} = 0$ so **r'** is glued to frame S' as it rotates. The tail of the vector **r'** (and the origin of Frame S' to which it is attached) is accelerating upward due to this $\dot{\boldsymbol{\omega}}$ which is in the $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ direction. Since **r'** is soldered to Frame S', *all* of **r'** is accelerating upward, including the tip where the Particle lies. The Particle sees no true force causing this acceleration and ascribes it to the fictitious force $\mathbf{F}_{fict} = -m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$. Notice that if **r'** were pointed directly to the viewer, thus parallel to **ω**, the vector **r'** would still be accelerating due to this effect, but in a direction parallel to its direction. It is then simply translating and therefore "the vector **r'** " is not really changing at all (see Section 1 (d)). This is why that there is an cross product in $\dot{\boldsymbol{\omega}} \times \mathbf{r}'$.

****

In its normal application to the Earth, one takes $\dot{\boldsymbol{\omega}} = 0$ since the Earth rotates at a constant angular momentum, and one takes $\mathbf{v}_1 = \dot{\mathbf{v}}_1 = 0$ since the center of our "green circle" (at the latitude of Frame S') is regarded as not moving (ignoring motion of the Earth around the Sun and possible other larger scale motions relative to the stars). So for the Earth situation one has

$$\mathbf{F}_{fict} = -2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (8.5)$$

Note that **r** refers to the distance to our Particle from the center of the Earth. Using $\mathbf{r} = \mathbf{r}' + \mathbf{b}$ we can write the above as

$$\mathbf{F}_{fict} = -2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \ . \qquad (8.6)$$

In a typical application one has $r' << b \sim 6400$ km and then one can ignore the second term to get

$$\mathbf{F}_{fict} \approx -2m\boldsymbol{\omega} \times \mathbf{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad (8.7)$$

The first fictitious force in (8.7),

$$- \mathbf{2}m\boldsymbol{\omega} \times \mathbf{v}' = + 2m\, \mathbf{v}' \times \boldsymbol{\omega}$$

is the famous **Coriolis** force which causes a ballistic Particle to deflect to the right in the northern hemisphere and to the left in the southern hemisphere.

The second fictitious force $- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the centrifugal (center-fleeing) force pushing our Particle away from the rotation axis, and $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the associated centripetal acceleration. To see that this is the case, consider this picture which shows vectors **r** and **ω**,

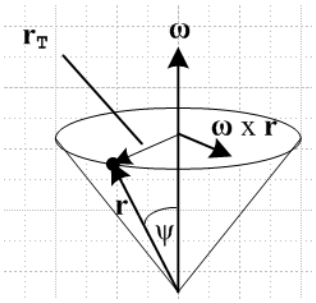


Fig 3

Using the right hand rule, one can see then that $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is a vector pointing in the opposite direction to the vector labeled $\mathbf{r}_T$. From the location of the Particle at **r**, this vector points toward the center of the circle on which the particle is instantaneously rotating at time t, and this is the correct direction for centripetal (center seeking) acceleration. The magnitude of the vector is $|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = \omega\, |\boldsymbol{\omega} \times \mathbf{r}| = \omega^2 r \sin\psi = \omega^2 r_T$ and this is the correct magnitude for the centripetal acceleration. So the centripetal acceleration $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points toward the rotation axis, while the centrifugal force $- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is directed away from the rotation axis, as all Merry-Go-Round (not to mention Rotor) riders well know.

In terms of the more general situation indicated by (8.5), one fictitious force there is $- m\, \dot{\boldsymbol{\omega}} \times \mathbf{r}$ which only exists only if $\boldsymbol{\omega} \neq$ constant. In the special case the $\dot{\boldsymbol{\omega}}$ happens to be in the direction of **ω**, meaning $\omega > 0$ (this would apply in our phonograph platter scenario if the platter is spinning up) then $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ is in the same direction as $\boldsymbol{\omega} \times \mathbf{r}$ shown in Fig 3 and that is the direction of the tangential acceleration of a point on the platter. Correspondingly, a Particle in Frame S' feels a fictitious force $- m\, \dot{\boldsymbol{\omega}} \times \mathbf{r}$ in the opposite direction.

Finally, the fictitious forces $- m\boldsymbol{\omega} \times \mathbf{v}_1 - m\, \dot{\mathbf{v}}_1$ arise if the center point of the rotation of

****

Now

$$\dot{\mathbf{r}}_S \equiv \partial_S\, \mathbf{r} = \mathbf{v}_S \equiv \mathbf{v} = \dot{\mathbf{r}} \qquad \text{// natural in frame S}$$

$$\dot{\mathbf{r}}_{S'} \equiv \partial_{S'} \mathbf{r} = \mathbf{v}_{S'}$$

$$\dot{\mathbf{r}}'_S \equiv \partial_S\, \mathbf{r}' = \mathbf{v}'_S \qquad \text{// Goldstein, see Section 10}$$

$$\dot{\mathbf{r}}'_{S'} \equiv \partial_{S'}\mathbf{r}' = \mathbf{v}'_{S'} \equiv \mathbf{v}' = \dot{\mathbf{r}}' \qquad \text{// natural in frame S'} \qquad (1.34)$$

To save space, we define ∂_S and $\partial_{S'}$ as shown in this list of definitions:

We now

$$\dot{\mathbf{r}} \equiv d\mathbf{r}/dt \qquad \ddot{\mathbf{r}} \equiv d^2\mathbf{r}/dt^2 \qquad \partial_S \equiv (d/dt)_S \qquad \partial_{S'} \equiv (d/dt)_{S'}$$

We now make six new definitions all of which should seem reasonable

$$\begin{aligned}
&\dot{\mathbf{a}}_S \equiv \partial_S\mathbf{a} = (d\mathbf{a}/dt)_S \equiv \dot{\mathbf{a}} \\
&\dot{\mathbf{a}}_S \equiv \partial_{S'}\mathbf{a} = (d\mathbf{a}/dt)_{S'} \\
&\dot{\mathbf{a}}'_S \equiv \partial_S\mathbf{a}' = (d\mathbf{a}'/dt)_S \\
&\dot{\mathbf{a}}'_S \equiv \partial'_S\mathbf{a}' = (d\mathbf{a}'/dt)_{S'} \equiv \dot{\mathbf{a}}' \qquad (1.31)
\end{aligned}$$

When a vector is associated with Frame S (such as vector **a**), and when all its derivatives (here only one) are taken *in* Frame S, then we drop the S subscript so that $\dot{\mathbf{a}}_S$ is abbreviated as $\dot{\mathbf{a}}$ shown on the first line. The same idea is used on the last line where "everything" is in Frame S'.

We can construct four second time derivatives of **a**, and four of **a'**, so here is the table of eight:

$$\ddot{\mathbf{a}}_{SS} \equiv \partial_S\partial_S\mathbf{a} = \partial_S\dot{\mathbf{a}}_S$$

What about *second* time derivatives? Using the $\partial_{\mathbf{x}}$ notation we find there are 8 such derivatives:

$$\begin{aligned}
&\partial_S\partial_S\mathbf{a} = \partial_S\,\dot{\mathbf{a}}_S = \ddot{\mathbf{a}}_{SS} && \text{// first four involve } \mathbf{a} \\
&\partial_S\partial_{S'}\mathbf{a} = \partial_S\,\dot{\mathbf{a}}_{S'} = \ddot{\mathbf{a}}_{SS'} \\
&\partial_{S'}\partial_S\mathbf{a} = \partial_{S'}\,\dot{\mathbf{a}}_S = \ddot{\mathbf{a}}_{S'S} \\
&\partial_{S'}\partial_{S'}\mathbf{a} = \partial_{S'}\,\dot{\mathbf{a}}_{S'} = \ddot{\mathbf{a}}_{S'S'} \\
&\partial_S\partial_S\mathbf{a}' = \partial_S\,\dot{\mathbf{a}}'_S = \ddot{\mathbf{a}}'_{SS} && \text{// second four involve } \mathbf{a}'
\end{aligned}$$

$$\partial_S \partial_{S'} \mathbf{a}' = \partial_S \dot{\mathbf{a}}'_{S'} = \ddot{\mathbf{a}}'_{SS'}$$
$$\partial_{S'} \partial_S \mathbf{a} = \partial_{S'} \dot{\mathbf{a}}'_S = \ddot{\mathbf{a}}'_{S'S}$$
$$\partial_{S'} \partial_{S'} \mathbf{a} = \partial_{S'} \dot{\mathbf{a}}'_{S'} = \ddot{\mathbf{a}}'_{S'S'} \qquad (1.32)$$

When both frame labels are the same, it means that both derivatives are computed in the same frame, so it is then appropriate to write such a derivative in these ways (here X is either S or S')

$$\partial_X \partial_X \mathbf{a} = (d\,(d\mathbf{a}/dt)_X / dt)_X = (d^2\mathbf{a}/dt^2)_X = (d^2/dt^2)_X\ \mathbf{a} = (d/dt)_X\,(d/dt)_X\ \mathbf{a} = \partial_X^2 \mathbf{a} \qquad (1.33)$$

Example 1 (the Big Example):

Suppose $\mathbf{a} = \mathbf{r}$ and $\mathbf{a}' = \mathbf{r}'$ where $\mathbf{r}$ and $\mathbf{r}'$ are the positions of our Particle in frames S and S'. We know that the relation between this pair $(\mathbf{a},\mathbf{a}') = (\mathbf{r},\mathbf{r}')$ is given by

$\mathbf{r} = \mathbf{r}' + \mathbf{b}$ // $\mathbf{r}$ and $\mathbf{r}'$ are "related in some way" and this is it *(1.24)*

In this case, we can talk about these four time derivatives:

$$(d\mathbf{r}/dt)_S = \partial_S\ \mathbf{r} = \dot{\mathbf{r}}_S = \mathbf{v}_S$$
$$(d\mathbf{r}/dt)_{S'} = \partial_{S'} \mathbf{r} = \dot{\mathbf{r}}_{S'} = \mathbf{v}_{S'}$$
$$(d\mathbf{r}'/dt)_S = \partial_S\ \mathbf{r}' = \dot{\mathbf{r}}'_S = \mathbf{v}'_S$$
$$(d\mathbf{r}'/dt)_{S'} = \partial_{S'} \mathbf{r}' = \dot{\mathbf{r}}'_{S'} = \mathbf{v}'_{S'}$$

where we define $\mathbf{v}_X \equiv \dot{\mathbf{r}}_X$ for frame X. This says that in our two-frame analysis, we have four velocities to worry about, and they are in general all different.

Velocity $\mathbf{v}_S$ is the "natural" velocity in frame S, while $\mathbf{v}'_{S'}$ is the natural velocity in frame S'. That is to say, an observer in frame S will naturally measure $(d\mathbf{r}/dt)_S$ whereas an observer in frame S' will naturally measure $(d\mathbf{r}'/dt)_{S'}$. For this reason, we are going to define

$$\mathbf{v} \equiv \dot{\mathbf{r}} \equiv \mathbf{v}_S = \dot{\mathbf{r}}_S = (d\mathbf{r}/dt)_S$$
$$\mathbf{v}' \equiv \dot{\mathbf{r}}' \equiv \mathbf{v}'_{S'} = \dot{\mathbf{r}}'_{S'} = (d\mathbf{r}'/dt)_{S'}$$

to have simpler names for the two natural velocities. The other two are "cross velocities". Since $\mathbf{r} = \mathbf{r}' + \mathbf{b}$, there is nothing really mysterious about these cross velocities, they are just the derivative of $\mathbf{r}'$ in frame S, and of $\mathbf{r}$ in frame S'. So we now rewrite the above list as

$\dot{\mathbf{r}}_S = (d\mathbf{r}/dt)_S = \partial_S\ \mathbf{r} = \mathbf{v}_S \equiv \mathbf{v} = \dot{\mathbf{r}}$ // natural in frame S

$\dot{\mathbf{r}}_{S'} = (d\mathbf{r}/dt)_{S'} = \partial_{S'} \mathbf{r} = \mathbf{v}_{S'}$

$\dot{\mathbf{r}}'_S = (d\mathbf{r}'/dt)_S = \partial_S\ \mathbf{r}' = \mathbf{v}'_S$ // Goldstein, see Section 10

$\dot{\mathbf{r}}'_{S'} = (d\mathbf{r}'/dt)_{S'} = \partial_{S'} \mathbf{r}' = \mathbf{v}'_{S'} \equiv \mathbf{v}' = \dot{\mathbf{r}}'$ // natural in frame S' (1.34)

As for second derivatives, we have (acceleration **a** is different from our generic **a** used above!)

$$\ddot{\mathbf{r}}_{ss} = \partial_s\partial_s\mathbf{r} = \partial_s\dot{\mathbf{r}}_s = \partial_s\mathbf{v}_s = \dot{\mathbf{v}}_{ss} = \mathbf{a}_{ss}$$
$$\ddot{\mathbf{r}}_{ss'} = \partial_s\partial_{s'}\mathbf{r} = \partial_s\dot{\mathbf{r}}_{s'} = \partial_s\mathbf{v}_{s'} = \dot{\mathbf{v}}_{ss'} = \mathbf{a}_{ss'}$$
$$\ddot{\mathbf{r}}_{s's} = \partial_{s'}\partial_s\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}_s = \partial_{s'}\mathbf{v}_s = \dot{\mathbf{v}}_{s's} = \mathbf{a}_{s's}$$
$$\ddot{\mathbf{r}}_{s's'} = \partial_{s'}\partial_{s'}\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}_{s'} = \partial_{s'}\mathbf{v}_{s'} = \dot{\mathbf{v}}_{s's'} = \mathbf{a}_{s's'}$$
$$\ddot{\mathbf{r}}'_{ss} = \partial_s\partial_s\mathbf{r}' = \partial_s\dot{\mathbf{r}}'_s = \partial_s\,\mathbf{v}'_s = \dot{\mathbf{v}}'_{ss} = \mathbf{a}'_{ss}$$
$$\ddot{\mathbf{r}}'_{ss'} = \partial_s\partial_{s'}\mathbf{r}' = \partial_s\dot{\mathbf{r}}'_{s'} = \partial_s\,\mathbf{v}'_{s'} = \dot{\mathbf{v}}'_{ss'} = \mathbf{a}'_{ss'}$$
$$\ddot{\mathbf{r}}'_{s's} = \partial_{s'}\partial_s\mathbf{r}' = \partial_{s'}\dot{\mathbf{r}}'_s = \partial_{s'}\,\mathbf{v}'_s = \dot{\mathbf{v}}'_{s's} = \mathbf{a}'_{s's}$$
$$'\ddot{\mathbf{r}}_{s's'} = \partial_{s'}\partial_{s'}\mathbf{r}' = \partial_{s'}\dot{\mathbf{r}}'_{s'} = \partial_{s'}\,\mathbf{v}'_{s'} = \dot{\mathbf{v}}'_{s's'} = \mathbf{a}'_{s's'} \qquad (1.35)$$

So in our two-frame analysis, we have <u>eight accelerations to worry about, they are in general all different</u>, and only two are "natural" and all the rest are "cross accelerations".

This time we shall make two kinds of abbreviations. First, if a vector has both subscripts the same, we suppress one of them just to save space. Second, if a vector is "natural", we suppress both subscripts. This way, things are compact, but nothing is ambiguous. Here then is our new list of second derivatives,

$$\partial_s\partial_s\mathbf{r} = \partial_s\dot{\mathbf{r}}_s = \partial_s\mathbf{v}_s = \dot{\mathbf{v}}_s = \mathbf{a}_s = \mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \qquad \text{// natural in frame S}$$
$$\partial_s\partial_{s'}\mathbf{r} = \partial_s\dot{\mathbf{r}}_{s'} = \partial_s\mathbf{v}_{s'} = \dot{\mathbf{v}}_{ss'} = \mathbf{a}_{ss'}$$
$$\partial_{s'}\partial_s\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}_s = \partial_{s'}\mathbf{v}_s = \dot{\mathbf{v}}_{s's} = \mathbf{a}_{s's}$$
$$\partial_{s'}\partial_{s'}\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}_{s'} = \partial_{s'}\mathbf{v}_{s'} = \dot{\mathbf{v}}_{s'} = \mathbf{a}_{s'}$$
$$\partial_s\partial_s\mathbf{r}' = \partial_s\dot{\mathbf{r}}'_s = \partial_s\,\mathbf{v}'_s = \dot{\mathbf{v}}'_s = \mathbf{a}'_s \qquad \text{// Goldstein, see Section 10}$$
$$\partial_s\partial_{s'}\mathbf{r}' = \partial_s\dot{\mathbf{r}}'_{s'} = \partial_s\,\mathbf{v}'_{s'} = \dot{\mathbf{v}}'_{ss'} = \mathbf{a}'_{ss'}$$
$$\partial_{s'}\partial_s\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}'_s = \partial_{s'}\,\mathbf{v}'_s = \dot{\mathbf{v}}'_{s's} = \mathbf{a}'_{s's}$$
$$\partial_{s'}\partial_{s'}\mathbf{r} = \partial_{s'}\dot{\mathbf{r}}'_{s'} = \partial_{s'}\,\mathbf{v}'_{s'} = \dot{\mathbf{v}}'_{s'} = \mathbf{a}'_{s'} = \mathbf{a}' = \dot{\mathbf{v}}' = \ddot{\mathbf{r}}' \qquad \text{// natural in frame S'} \qquad (1.36)$$

Notice that the two natural accelerations can be written in the manner one expects,

$$\mathbf{a} = \mathbf{a}_s = \partial_s\,\mathbf{v}_s = (d\mathbf{v}_s/dt)_s = (d^2\mathbf{r}/dt^2)_s$$
$$\mathbf{a}' = \mathbf{a}'_{s'} = \partial_{s'}\,\mathbf{v}_{s'} = (d\mathbf{v}'_s/dt)_{s'} = (d^2\mathbf{r}'/dt^2)_{s'} \qquad (1.37)$$

<u>Example 2:</u> In (1.24) we had $\mathbf{r} = \mathbf{r}' + \mathbf{b}$. The vector **b** is not a member of a pair of vectors (**b**, **b'**) because there is no vector **b'**, so we shall not attempt to apply the shortcut notations given above to this vector. For example, were we to use a vector named $\dot{\mathbf{b}}'$, based on the above it would be interpreted as $\partial_{s'}\mathbf{b}'$ which does not exist. So for the vector **b**, we shall use the following non-abbreviated notations,

$$\partial_s\mathbf{b} = \dot{\mathbf{b}}_s = (d\mathbf{b}/dt)_s$$

$\partial_{S'}\mathbf{b} = \dot{\mathbf{b}}_{S'} = (d\mathbf{b}/dt)_{S'}$

$\partial_S\partial_S\mathbf{b} = \partial_S\dot{\mathbf{b}}_S = \partial_S\mathbf{v}_S = \dot{\mathbf{v}}_{SS} = \mathbf{a}_{SS}$ // natural in frame S

$\partial_S\partial_{S'}\mathbf{b} = \partial_S\dot{\mathbf{r}}_{S'} = \partial_S\mathbf{v}_{S'} = \dot{\mathbf{v}}_{SS'} = \mathbf{a}_{SS'}$

$\partial_{S'}\partial_S\mathbf{b} = \partial_{S'}\dot{\mathbf{r}}_S = \partial_{S'}\mathbf{v}_S = \dot{\mathbf{v}}_{S'S} = \mathbf{a}_{S'S}$

$\partial_{S'}\partial_{S'}\mathbf{b} = \partial_{S'}\dot{\mathbf{r}}_{S'} = \partial_{S'}\mathbf{v}_{S'} = \dot{\mathbf{v}}_{S'S'} = \mathbf{a}_{S'S'}$

$\partial^2_S\,\mathbf{b} = \ddot{\mathbf{b}}_S = (d^2\mathbf{b}/dt^2)_S$

$\partial^2_{S'}\mathbf{b} = \ddot{\mathbf{b}}_{S'} = (d^2\mathbf{b}/dt^2)_{S'}$

$\partial_S\partial_{S'}\,\mathbf{b} = \ddot{\mathbf{b}}_{SS'}$

We shall have no need for the cross derivatives

(h) First and second time derivatives of vectors r, r' and b.

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[5] Notations used in this document.

The following notations will be used below :

$\mathbf{r}$	position vector of a Particle in frame S	
$\mathbf{r}'$	position vector of same Particle viewed from frame S'	
$\mathbf{b}$	a translation vector connecting the origins of frames S and S' (tail of **b** on S)	
$\mathbf{r} = \mathbf{r}' + \mathbf{b}$	see Fig 1 above	(1.38)
$\dot{\mathbf{b}}_S = (d\mathbf{b}/dt)_S$	rate of change of **b** as viewed from Frame S	
$\dot{\mathbf{b}}_{S'} = (d\mathbf{b}/dt)_{S'}$	rate of change of **b** as viewed from Frame S'	
$\ddot{\mathbf{b}}_S = (d^2\mathbf{b}/dt^2)_S$		
$\ddot{\mathbf{b}}_{S'} = (d^2\mathbf{b}/dt^2)_{S'}$		(1.41)

$\dot{\boldsymbol{\omega}} = (d\boldsymbol{\omega}/dt)_S = (d\boldsymbol{\omega}/dt)_S$ // that these are the same is shown in (2.7) below

From the 2 position vectors **r** and **r'** we can define 4 distinct velocity vectors because for each position vector we can differentiate with respect to either frame S or S' :

$$\begin{aligned}
\mathbf{v}_S &= \dot{\mathbf{r}}_S = (d\mathbf{r}/dt)_S = \partial_S\, \mathbf{r} \\
\mathbf{v}_{S'} &= \dot{\mathbf{r}}_{S'} = (d\mathbf{r}/dt)_{S'} = \partial_{S'}\mathbf{r} \\
\mathbf{v'}_S &= \dot{\mathbf{r}}'_S = (d\mathbf{r}'/dt)_S = \partial_S\, \mathbf{r}' \\
\mathbf{v'}_{S'} &= \dot{\mathbf{r}}'_{S'} = (d\mathbf{r}'/dt)_{S'} = \partial_{S'}\mathbf{r}'
\end{aligned} \tag{1.42}$$

Of the eight accelerations mentioned above, four are most frequently used,

$$\begin{aligned}
\mathbf{a}_S &= \dot{\mathbf{v}}_S = (d\mathbf{v}_S/dt)_S = (d^2\mathbf{r}/dt^2)_S = \partial_S{}^2\, \mathbf{r} \\
\mathbf{a}_{S'} &= \dot{\mathbf{v}}_{S'} = (d\mathbf{v}_{S'}/dt)_{S'} = (d^2\mathbf{r}/dt^2)_{S'} = \partial_{S'}{}^2\, \mathbf{r} \\
\mathbf{a'}_S &= \dot{\mathbf{v}}'_S = (d\mathbf{v}'_S/dt)_S = (d^2\mathbf{r}'/dt^2)_S = \partial_S{}^2\, \mathbf{r}' \\
\mathbf{a'}_{S'} &= \dot{\mathbf{v}}'_{S'} = (d\mathbf{v}'_{S'}/dt)_{S'} = (d^2\mathbf{r}'/dt^2)_{S'} = \partial_{S'}{}^2\, \mathbf{r}'
\end{aligned} \tag{1.43}$$

while the other four are these (cross accelerations)

$$\begin{aligned}
\mathbf{a}_{SS'} &= \dot{\mathbf{v}}_{SS'} = \partial_S \mathbf{v}_{S'} = (d\mathbf{v}_{S'}/dt)_S = (d[(d\mathbf{r}/dt)_{S'}]/dt)_S = \partial_S\partial_{S'}\mathbf{r} \\
\mathbf{a}_{S'S} &= \dot{\mathbf{v}}_{S'S} = \partial_{S'}\mathbf{v}_S = (d\mathbf{v}_S/dt)_{S'} = (d[(d\mathbf{r}/dt)_S]/dt)_{S'} = \partial_{S'}\partial_S\mathbf{r} \\
\mathbf{a'}_{SS'} &= \dot{\mathbf{v}}'_{SS'} = \partial_S\mathbf{v}'_{S'} = (d\mathbf{v}'_{S'}/dt)_S = (d[(d\mathbf{r}'/dt)_{S'}]/dt)_S = \partial_S\partial_{S'}\mathbf{r}' \\
\mathbf{a'}_{S'S} &= \dot{\mathbf{v}}'_{S'S} = \partial_{S'}\mathbf{v}'_S = (d\mathbf{v}'_S/dt)_{S'} = (d[(d\mathbf{r}'/dt)_S]/dt)_{S'} = \partial_{S'}\partial_S\mathbf{r}'
\end{aligned} \tag{1.44}$$

Similarly, we could define 8 angular momentum vectors, but the only ones of interest are the 4 for which the prime on position matches the prime on velocity

$$\begin{aligned}
\mathbf{L}_S &= \mathbf{r} \times m\mathbf{v}_S \\
\mathbf{L}_{S'} &= \mathbf{r} \times m\mathbf{v}_{S'} \\
\mathbf{L'}_S &= \mathbf{r}' \times m\mathbf{v}'_S \\
\mathbf{L'}_{S'} &= \mathbf{r}' \times m\mathbf{v}'_{S'}
\end{aligned} \tag{1.45}$$

We limit our interest to these four $\dot{\mathbf{L}}$ objects:

$$\begin{aligned}
\dot{\mathbf{L}}_S &= (d\mathbf{L}_S/dt)_S \\
\dot{\mathbf{L}}_{S'} &= (d\mathbf{L}_{S'}/dt)_{S'} \\
\dot{\mathbf{L}}'_S &= (d\mathbf{L}'_S/dt)_S \\
\dot{\mathbf{L}}'_{S'} &= (d\mathbf{L}'_{S'}/dt)_{S'}
\end{aligned} \tag{1.46}$$

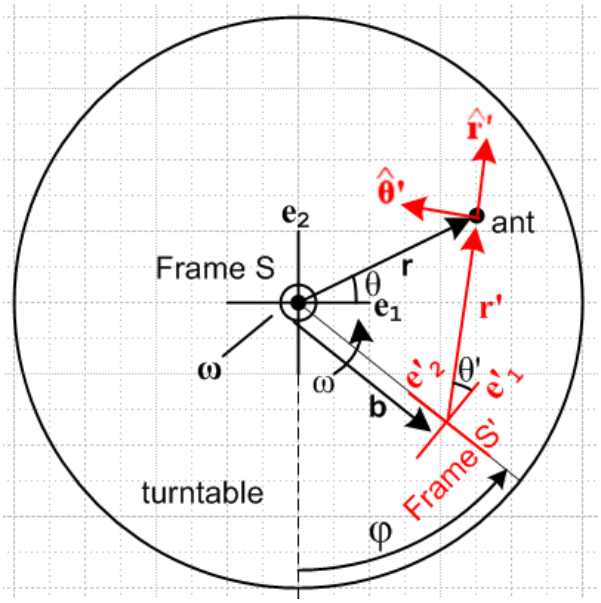
The **L** vector stuff plays a minimal role in this document, we just include it for completeness.

****

14. Example of Ant crawling on a Turntable

(a) Problem Setup

Consider our turntable occupied by an ant :



Suppose in the rotating frame S' the ant starts at some point and marches linearly inward at a constant velocity V toward the origin of Frame S'. What does one see in Frame S?

It seems a reasonable plan to use cylindrical coordinates for both the Frame S and Frame S' curvilinear coordinates,

$\xi_i = r,\theta,z$	for Frame S	basis vectors	$\hat{\boldsymbol{e}}_i = \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$
$\xi'_i = r',\theta',z'$	for Frame S'	basis vectors	$\hat{\boldsymbol{e}}'_i = \hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{z}}'$

(b) What do we know about all the basis vectors?

From the picture one sees that, for the Cartesian unit vectors,

$\mathbf{e}'_i = R_z(\varphi)\,\mathbf{e}_i$ for example $\mathbf{e}'_1 = R_z(\varphi)\,\mathbf{e}_1$

where φ is the instantaneous angle describing the orientation of Frame S' relative to Frame S.

The curvilinear unit vectors can be expressed in terms of the Cartesian ones as follows (stare at the picture)

$\hat{\boldsymbol{e}}_i = R_z(\theta)\,\mathbf{e}_i$ that is $\hat{\mathbf{r}} = R_z(\theta)\,\mathbf{e}_1$ $\hat{\boldsymbol{\theta}} = R_z(\theta)\,\mathbf{e}_2$ $\hat{\mathbf{z}} = R_z(\theta)\,\mathbf{e}_3 = \mathbf{e}_3$

$\hat{\boldsymbol{e}}'_i = R_z(\theta')\,\mathbf{e}'_i$ that is $\hat{\mathbf{r}}' = R_z(\theta')\,\mathbf{e}'_1$ $\hat{\boldsymbol{\theta}}' = R_z(\theta')\,\mathbf{e}'_2$ $\hat{\mathbf{z}}' = R_z(\theta')\,\mathbf{e}_3 = \mathbf{e}_3$

Since there are 4 kinds of unit vectors, there are 4*3/2 = 6 pairings for which we can write rotation connections. Three are stated above. The other three can be obtained as follows

$\hat{\boldsymbol{e}}_i = R_z(\theta)\,\mathbf{e}_i = R_z(\theta)[\,R_z(-\varphi)\,\mathbf{e}'_i] = R_z(\theta-\varphi)\,\mathbf{e}'_i$

$\hat{\boldsymbol{e}}'_i = R_z(\theta')\,\mathbf{e}'_i = R_z(\theta')\,[R_z(\varphi)\,\mathbf{e}_i] = R_z(\theta'+\varphi)\,\mathbf{e}_i$

$\hat{\boldsymbol{e}}'_i = R_z(\theta'+\varphi)\,\mathbf{e}_i = R_z(\theta'+\varphi)\,[R_z(-\theta)]\,\hat{\boldsymbol{e}}_i = R_z(\theta'+\varphi-\theta)\,\hat{\boldsymbol{e}}_i$

So we can now summarize all 6 pairings:

$\mathbf{e}'_i = R_z(\varphi)\,\mathbf{e}_i$	$\mathbf{e}'_i$	$\mathbf{e}_i$
$\hat{\boldsymbol{e}}_i = R_z(\theta)\,\mathbf{e}_i$	$\hat{\boldsymbol{e}}_i$	$\mathbf{e}_i$
$\hat{\boldsymbol{e}}'_i = R_z(\theta')\,\mathbf{e}'_i$	$\hat{\boldsymbol{e}}'_i$	$\mathbf{e}'_i$
$\hat{\boldsymbol{e}}_i = R_z(\theta-\varphi)\,\mathbf{e}'_i$	$\hat{\boldsymbol{e}}_i$	$\mathbf{e}'_i$
$\hat{\boldsymbol{e}}'_i = R_z(\theta'+\varphi)\,\mathbf{e}_i$	$\hat{\boldsymbol{e}}'_i$	$\mathbf{e}_i$
$\hat{\boldsymbol{e}}'_i = R_z(\theta'+\varphi-\theta)\,\hat{\boldsymbol{e}}_i$	$\hat{\boldsymbol{e}}'_i$	$\hat{\boldsymbol{e}}_i$

Since we know that

$$R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we really know everything we could possibly want to know about all the basis vectors. For example, from the connection $\hat{\boldsymbol{e}}'_i = R_z(\theta'+\varphi-\theta)\,\hat{\boldsymbol{e}}_i$ we can write

$$\begin{pmatrix} \hat{\boldsymbol{e}}'_1 \\ \hat{\boldsymbol{e}}'_2 \\ \hat{\boldsymbol{e}}'_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} \qquad \psi = \theta'+\varphi-\theta$$

which tells us that

$\hat{\boldsymbol{e}}'_1 = \cos\psi\,\hat{\boldsymbol{e}}_1 - \sin\psi\,\hat{\boldsymbol{e}}_2 \quad \Rightarrow \quad \hat{\mathbf{r}}' = \cos(\theta'+\varphi-\theta)\,\hat{\mathbf{r}} - \sin(\theta'+\varphi-\theta)\,\hat{\boldsymbol{\theta}}$

$\hat{\boldsymbol{e}}'_2 = \sin\psi\,\hat{\boldsymbol{e}}_1 + \cos\psi\,\hat{\boldsymbol{e}}_2 \quad \Rightarrow \quad \hat{\boldsymbol{\theta}}' = \sin(\theta'+\varphi-\theta)\,\hat{\mathbf{r}} + \cos(\theta'+\varphi-\theta)\,\hat{\boldsymbol{\theta}}$

Another example is $\hat{\boldsymbol{e}}_i = R_z(\theta-\varphi)\,\mathbf{e}'_i$ so $\mathbf{e}'_i = R_z(\varphi-\theta)\,\hat{\boldsymbol{e}}_i$:

$$\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \mathbf{e'}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} \qquad \psi = \varphi - \theta$$

which tells us that

$$\mathbf{e'}_1 = \cos\psi\ \hat{\boldsymbol{e}}_1 - \sin\psi\ \hat{\boldsymbol{e}}_2 \quad \Rightarrow \quad \mathbf{e'}_1 = \cos(\varphi-\theta)\ \hat{\mathbf{r}} - \sin(\varphi-\theta)\ \hat{\boldsymbol{\theta}}$$

$$\mathbf{e'}_2 = \sin\psi\ \hat{\boldsymbol{e}}_1 + \cos\psi\ \hat{\boldsymbol{e}}_2 \quad \Rightarrow \quad \mathbf{e'}_2 = \sin(\varphi-\theta)\ \hat{\mathbf{r}} + \cos(\varphi-\theta)\ \hat{\boldsymbol{\theta}}$$

(c) Solving the problem for velocity

Ant's Motion in Frame S.

Assume the ant starts at some (r'_0,θ'_0) at t = 0 and then has $\mathbf{v}' = -V\ \hat{\mathbf{r}}'$. We find then that

$$\mathbf{r}'(t) = \mathbf{r}'_0 - Vt\ \hat{\mathbf{r}}'$$

or

$$r' = r'_0 - Vt$$
$$\theta' = \theta'_0$$

and we shall only be interested in times small enough so $r' > 0$.

Relation between Frame S and Frame S'

Assume at time t= 0 frame S' we have $\varphi = \varphi_0$.

Assume the rotation follows some angular velocity profile $\omega = \omega(t)$. Since $\omega = d\varphi/dt$, we have

$$d\varphi/dt = \omega(t) \quad \Rightarrow \quad \varphi(t) = \varphi_0 + \int_0^t \omega(\tau)d\tau$$

So from now on, we assume that we know the quantity $\varphi(t)$.

Motion of vector **b**.

From the picture we see that

$$\mathbf{b}(t) = -b\ R_z(\varphi)\ \mathbf{e'}_2 = -b\ R_z(\varphi)R_z(-\theta')\ \hat{\boldsymbol{e}}'_2 = -b\ R_z(\varphi-\theta')$$

Relation between r and r'

Vector **r** is given by

$$\mathbf{r}(t) = \mathbf{b} + \mathbf{r}' = -b\ R_z(\varphi)\ \mathbf{e'}_2 + \mathbf{r}'_0 - Vt\ \hat{\mathbf{r}}'$$

$= \quad -b\, R_z(\varphi)\, \mathbf{e}_2 + \mathbf{r}'_0 - Vt\, \hat{\mathbf{r}}'$

The triangle formed by **r**, **r'** and **b** has an internal angle of $(\pi/2-\theta')$ at the vertex opposite vector **r**. The law of cosines then says

$$r^2 = r'^2 + b^2 - 2rb' \cos(\pi/2-\theta') = r'^2 + b^2 - 2r'b \sin(\theta')$$

But in our problem $\theta' = \theta'_0$ and $r' = r'_0 - Vt$ so this becomes

$$r^2 = (r'_0 - Vt)^2 + b^2 - 2(r'_0 - Vt)b \sin(\theta'_0)$$

$$r = [(r'_0 - Vt)^2 + b^2 - 2(r'_0 - Vt)b \sin(\theta'_0)]^{1/2}$$

$$\hat{\mathbf{r}} = R_z(\theta)\, \mathbf{e}_1$$

$$\mathbf{r}(t) = r\, \hat{\mathbf{r}}$$

This then is the ant trajectory in Frame S

Since the turntable falls into our Special Case #1 pigeonhole of Section 4 (d) (**ω** through origin of Frame S), we know that $\dot{\mathbf{b}}_S = \boldsymbol{\omega} \times \mathbf{b}$ and $\dot{\mathbf{b}}_{S'} = 0$ (vector **b** is soldered to the Frame S' unit vectors). From the summary in Section 12, we select (6.6c) which then says (setting $\dot{\mathbf{b}}_{S'} = 0$)

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$$

Assume the ant starts at some (r'_0,θ'_0) at t = 0 and then has $\mathbf{v}' = -V\, \hat{\mathbf{r}}'$. We find then that

$$r' = r'_0 - Vt$$
$$\theta' = \theta'_0$$

and we shall only be interested in times small enough so $r' > 0$.

Since $\mathbf{r}' + \mathbf{b} = \mathbf{r}$

The Frame S' velocity of the ant is this, where V is that ant's constant inward radial speed,

$\mathbf{v}' = -V\, \hat{\mathbf{r}}' = -V \cos\psi\, \hat{\mathbf{r}} + V \sin\psi\, \hat{\boldsymbol{\theta}}$ $\qquad \psi = \theta - \varphi - \theta'$ $\qquad$ // using **** above

$$\mathbf{r} = r\, \hat{\mathbf{r}}$$

$$\boldsymbol{\omega} = \omega\, \hat{\mathbf{z}}$$

$\boldsymbol{\omega} \times \mathbf{r} = \omega\, \hat{\mathbf{z}} \times r\, \hat{\mathbf{r}} = \omega r \hat{\boldsymbol{\theta}}$

$\mathbf{v} = v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}}$

Therefore *** says

$v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} = -V \cos\psi\, \hat{\mathbf{r}} + V \sin\psi\, \hat{\boldsymbol{\theta}} + \omega r \hat{\boldsymbol{\theta}}$

which in turn says

$v_r = -V \cos\psi$
$v_\theta = V \sin\psi + \omega r$

This to me is not a very satisfying solution . I want the velocity components as functions of time with some initial conditions etc.

****

We note in passing that

$\omega = d\varphi/dt$
$\mathbf{b} = R_z(\varphi)\, [\, -b\mathbf{e}_2]$.

****

Recall from Section 1 (a) that, for *any* two sets of unit vectors related by some rotation R,

$\mathbf{e}_i = R\, \mathbf{e}'_i \quad \Leftrightarrow \quad \mathbf{e}_i = (R^{-1})_{ij}\, \mathbf{e}'_j$.

We can apply this rule to any of our equations above relating two sets of orthogonal coordinates. For example

$\mathbf{e}'_i = R_z(\varphi)\, \mathbf{e}_i \quad \Leftrightarrow \quad \mathbf{e}'_i = [R_z(-\varphi)]_{ij}\, \mathbf{e}_j$
$\hat{\boldsymbol{e}}'_i = R_z(\theta')\, \mathbf{e}'_i \quad \Leftrightarrow \quad \hat{\boldsymbol{e}}'_i = [R_z(-\theta')]_{ij}\, \mathbf{e}'_j$

If follows that

$\hat{\boldsymbol{e}}'_i = R_z(\theta')\, \mathbf{e}'_i = R_z(\theta')\, R_z(\varphi)\, \mathbf{e}_i = R_z(\theta'+\varphi)\, \mathbf{e}_i$

****

13. Rotating Frames in Curvilinear Coordinates

The solution equations to our Original Problem are summarized in Section 12 above, and those to the Inverse Problem are summarized in Section 13 (d).

All equations are stated in bolded vector notation. Such equations may be evaluated in any orthogonal coordinate system one wants. Any set of orthogonal curvilinear coordinates provides such an orthogonal coordinate system. In general the curvilinear basis vectors like $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ for spherical coordinates "move" as the vector they describe moves, unlike the Cartesian basis vectors, but that is fine. For any particular vector, they form a viable set of orthogonal basis vectors.

In theory, we might want to use one curvilinear system of coordinates ξ_i with basis unit vectors $\hat{\boldsymbol{e}}_i$ for frame S, and an entirely different system ξ'_i with basis unit vectors $\hat{\boldsymbol{e}}'_i$ for frame S'. If **V** is an arbitrary vector, we then have these four expansions of interest :

$$\mathbf{V} = V_i \mathbf{e}_i = (V)'_i \mathbf{e}'_i = (V)_i \, \hat{\boldsymbol{e}}_i = (V)'_i \, \hat{\boldsymbol{e}}'_i$$

where we use italics to denote curvilinear vector components (in spherical coordinates, $\hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{\theta}}$) .

Comments:

• The *unit* vectors $\hat{\boldsymbol{e}}_i$, are normalized versions of the "tangent base vectors" which are tangent to coordinate lines (curves) arising in Cartesian space when $\xi_i(\mathbf{r})$ = constant is plotted in the Cartesian **r** space. Because the unit vectors have been normalized, our components V_i and V'_i are neither contravariant nor covariant vector components with respect to the non-linear transformation $\boldsymbol{\xi} = \mathbf{F}(\mathbf{r})$ which defines the curvilinear coordinates.

• In practice, in place of something like V_i one can use V_{ξ_i}. For example, in cylindrical coordinates one would write

$$V_1 = V_r$$
$$V_2 = V_\theta$$
$$V_3 = V_z$$

and then the need for italicized components goes away.

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (6.6c)$$

If we take the ith component of this equation in the $\mathbf{e}_i$ basis we get

$$(v)_i = (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i \qquad \text{components in basis } \mathbf{e}_i$$

$$(v)'_i = (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i \qquad \text{components in basis } \mathbf{e}'_i$$

$(v)_i = (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i$ components in basis $\hat{e}_i$

$(v)'_i = (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i$ components in basis $\hat{e}'_i$

For example, in r,θ,z cylindrical coordinates if we have $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, then $(\omega)_j = \delta_{j3}\omega$ so, in the third equation above

$\varepsilon_{ijk}(\omega)_j(r)_k = \varepsilon_{ijk}\,\delta_{j3}\,(r)_k = \omega\,\varepsilon_{i3k}(r)_k = -\,\omega\,\varepsilon_{ik3}(r)_k$

Then the third line above becomes

$(v)_i = (v')_i - \omega\,\varepsilon_{ik3}(r)_k + (\dot{b}_{S'})_i$ components in basis $\hat{e}_i$

$(v)_1 = (v')_1 - \omega\,\varepsilon_{123}(r)_2 + (\dot{b}_{S'})_i$

$(v)_2 = (v')_2 - \omega\,\varepsilon_{213}(r)_1 + (\dot{b}_{S'})_2$

$(v)_3 = (v')_3 - \omega\,\varepsilon_{3k3}(r)_k + (\dot{b}_{S'})_3 = (v')_3 + (\dot{b}_{S'})_3$

which translates into

$v_r = (v')_r - \omega\, r_\theta + (\dot{b}_{S'})_r = (v')_r + (\dot{b}_{S'})_r$

$v_\theta = (v')_\theta + \omega\, r_r + (\dot{b}_{S'})_\theta = (v')_\theta + \omega\, r + (\dot{b}_{S'})_\theta$

$v_z = (v')_z + (\dot{b}_{S'})_z$

I just don't see how this could possibly be useful! You are never given $(v')_r$ in Original Problem. Therefore, I think this whole section is not useful.

****

<u>Ant's Motion in Frame S'.</u>

Assume the ant starts at some (r'_0,θ'_0) at t = 0 and moves in circular motion about the S' origin at constant angular velocity Ω. Ant's linear velocity is then $v' = r'_0\Omega$ and we have

$\mathbf{v}' = r'_0\Omega\,\hat{\boldsymbol{\theta}}'$

The ant's position and acceleration are given by

$\mathbf{r}' = r'_0\,\hat{\mathbf{r}}'$ $\quad a' = -\Omega^2\, r'_0\hat{\mathbf{r}}'$

The angular position of the ant is given by

$$\theta' = \theta'_0 + \Omega t$$

Thus, in line with our Original Problem statement, these are the given quantities in Frame S' ,

$$\mathbf{r}' = r'_0\,\hat{\mathbf{r}}'$$
$$\mathbf{v}' = r'_0\Omega\,\hat{\boldsymbol{\theta}}'$$
$$\mathbf{a}' = -\Omega^2\, r'_0\hat{\mathbf{r}}'$$

Our goal is to compute **r**, **v** and **a** as seen in Frame S.

Using (***), we can express **v**' and **a**' as follows,

$$\mathbf{v}' = r'_0\Omega \sin(\theta'+\varphi)\,\hat{\mathbf{x}} + r'_0\Omega\cos(\theta'+\varphi)\,\hat{\mathbf{y}}$$

$$\mathbf{a}' = -\Omega^2\, r'_0\cos(\theta'+\varphi)\,\hat{\mathbf{x}} + \Omega^2\, r'_0 \sin(\theta'+\varphi)\,\hat{\mathbf{y}}$$

Sections about φ(t) and b are unchanged. I think **r**(t) still has the same form to this point

$$\mathbf{r}(t) = [-b\sin\varphi + r'\cos(\theta'+\varphi)]\,\hat{\mathbf{x}} + [-b\cos\varphi - r'\sin(\theta'+\varphi)]\,\hat{\mathbf{y}}$$

or

$$\mathbf{r}(t) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \quad \text{where}$$
$$x = -b\sin\varphi + r'_0\cos(\theta'+\varphi)$$
$$y = -b\cos\varphi - r'_0\sin(\theta'+\varphi)$$

where *now* θ' and φ are known functions of t. θ' in particular is different! We now examine,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \ .$$

We still have $\boldsymbol{\omega} \times \mathbf{r} = \omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}$ so this says

$$\mathbf{v} = r'_0\Omega \sin(\theta'+\varphi)\,\hat{\mathbf{x}} + r'_0\Omega\cos(\theta'+\varphi)\,\hat{\mathbf{y}} + \omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}$$

so we then have

$$\mathbf{v} = v_{\mathbf{x}}\,\hat{\mathbf{x}} + v_{\mathbf{y}}\,\hat{\mathbf{y}} \quad \text{where}$$
$$v_{\mathbf{x}} = r'_0\Omega \sin(\theta'+\varphi) - \omega y$$
$$v_{\mathbf{y}} = r'_0\Omega\cos(\theta'+\varphi) + \omega x$$

The acceleration terms are all the same except for the first one a' and we end up with

$$\mathbf{a} = a_{\mathbf{x}}\,\hat{\mathbf{x}} + a_{\mathbf{y}}\,\hat{\mathbf{y}} \quad \text{where}$$

$a_x = -\dot{\omega}y - 2\omega v_y - \omega^2 x - \Omega^2 r'_0 \cos(\theta'+\varphi)$

$a_y = \dot{\omega}x + 2\omega v_x + \omega^2 y + \Omega^2 r'_0 \sin(\theta'+\varphi)$

In the special case that ω = constant we can say

$\mathbf{r}(t) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ where

$x = -b\sin\varphi + r'_0\cos(\theta'+\varphi)$

$y = -b\cos\varphi - r'_0\sin(\theta'+\varphi)$

$\theta' = \theta'_0 + \Omega t$

$\varphi = \varphi_0 + \omega t$

Assume ant starts at $\theta'_0 = 0$ and that frame S' starts at $\varphi_0 = 0$. Then we have

$x = -b\sin(\omega t) + r'_0\cos(\omega t+\Omega t)$

$y = -b\cos(\omega t) - r'_0\sin(\omega t+\Omega t)$

****

Side Problem I don't know how to do.

$\mathbf{v}' = -V\,\hat{\mathbf{r}}' + r\Omega\,\hat{\boldsymbol{\theta}}'$

$\hat{\mathbf{r}}' = \cos(\theta'+\varphi)\,\hat{\mathbf{x}} - \sin(\theta'+\varphi)\,\hat{\mathbf{y}}$

$\hat{\boldsymbol{\theta}}' = \sin(\theta'+\varphi)\,\hat{\mathbf{x}} + \cos(\theta'+\varphi)\,\hat{\mathbf{y}}$

$\mathbf{v}' = -V\,[\cos(\theta'+\varphi)\,\hat{\mathbf{x}} - \sin(\theta'+\varphi)\,\hat{\mathbf{y}}] + r\Omega\,[\sin(\theta'+\varphi)\,\hat{\mathbf{x}} + \cos(\theta'+\varphi)\,\hat{\mathbf{y}}]$

$= v'_x\hat{\mathbf{x}} + v'_y\hat{\mathbf{y}}$

where

$v'_x = -V\cos(\theta'+\varphi) + r\Omega\sin(\theta'+\varphi) = dx'/dt$

$v'_y = V\sin(\theta'+\varphi) + r\Omega\cos(\theta'+\varphi) = dy'/dt$

****

$\mathbf{v} = -V\,\hat{\mathbf{r}} + r\Omega\,\hat{\boldsymbol{\theta}}$

This says

$v_r = -V$

$v_\theta = r\Omega$

How would a computer solve this?

$\mathbf{r} = \mathbf{r}_0$

$d\mathbf{r} = [-V\,\hat{\mathbf{r}} + r\Omega\,\hat{\boldsymbol{\theta}}]\,dt$
$\mathbf{r} = \mathbf{r} + d\mathbf{r}$
$r = |\mathbf{r}|$
$d\hat{\mathbf{r}} = \Omega \times \hat{\mathbf{r}}\,dt$

****

From the Inverse Problem equations in section 13 (d) we have

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_S, \qquad \textit{(13.2c)}$$

From (14.37) we then get

$$\mathbf{v}' = V\hat{\mathbf{n}} - \boldsymbol{\omega} \times [Vt\,\hat{\mathbf{n}} + \mathbf{r}_0] \qquad (14.44)$$

and using (14.39) this becomes.. $R_z(\theta-\varphi)\,\mathbf{e}'_1$

$$\mathbf{v}' = V\,R_z(\theta-\varphi)\,\mathbf{e}'_1 - \boldsymbol{\omega} \times [R_z(\theta-\varphi)\,\mathbf{e}'_1 + \mathbf{r}_0] \qquad (14.45)$$

Using the theorem given in (1.1) and (1.2) we can write

$$R_z(\theta-\varphi)\,\mathbf{e}'_1 = [R_z(\varphi-\theta)]_{1i}\,\mathbf{e}'_i \qquad (14.46)$$

so the above becomes

$$\mathbf{v}' = V\,R_z(\theta-\varphi)\,\mathbf{e}'_1 - [R_z(\varphi-\theta)]_{1i}\,\boldsymbol{\omega} \times \mathbf{e}'_i + \boldsymbol{\omega} \times \mathbf{r}_0 \qquad (14.47)$$

Letting $R \equiv R_z(\varphi-\theta)$, the middle term (without the minus sign) is

$R_{11}\,\boldsymbol{\omega} \times \mathbf{e}'_1 + R_{12}\,\boldsymbol{\omega} \times \mathbf{e}'_2 + R_{13}\,\boldsymbol{\omega} \times \mathbf{e}'_3$

$= \omega R_{11}\,\mathbf{e}'_2 - \omega R_{12}\mathbf{e}'_1 + 0$

$= \omega \cos(\varphi-\theta)\,\mathbf{e}'_2 + \omega \sin(\varphi-\theta)\,\mathbf{e}'_1$

$$= \omega \cos(\theta-\varphi)\,\mathbf{e}'_2 - \omega \sin(\theta-\varphi)\,\mathbf{e}'_1 \qquad (14.48)$$

Also

$$\boldsymbol{\omega} \times \mathbf{r}_0 = \boldsymbol{\omega} \times [\, x_0 \mathbf{e}_1 + y_0 \mathbf{e}_2] = \omega x_0 \mathbf{e}_2 - \omega y_0 \mathbf{e}_1 = \omega x_0 R_z(-\varphi)\mathbf{e}'_2 - \omega y_0 R_z(-\varphi)\mathbf{e}'_1$$

$$= R_z(-\varphi) \{ \omega x_0 \mathbf{e}'_2 - \omega y_0 \mathbf{e}'_1\} \qquad (14.49)$$

Inserting (14.48) and (14.49) into (14.47) then gives

$$\mathbf{v}' = V R_z(\theta-\varphi) \mathbf{e}'_1 - \omega \cos(\theta-\varphi) \mathbf{e}'_2 + \omega \sin(\theta-\varphi) \mathbf{e}'_1 + R_z(-\varphi) \{ \omega x_0 \mathbf{e}'_2 - \omega y_0 \mathbf{e}'_1\}$$

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$$\mathbf{r}'(t) = Vt [R_z(\theta-\varphi)]^{-1}_{1m} \mathbf{e}'_m + (r_0)_i [R_z(-\varphi)]^{-1}_{ij} \mathbf{e}'_j + b \mathbf{e}'_2$$

$$\mathbf{r}'(t) \bullet \mathbf{e}'_1 = (r')'_1 = Vt [R_z(\theta-\varphi)]^{-1}_{1m} \delta_{m,1} + (r_0)_i [R_z(-\varphi)]^{-1}_{ij} \delta_{j,1} + b \delta_{1,2}$$

$$= Vt [R_z(\theta-\varphi)]^{-1}_{11} + (r_0)_i [R_z(-\varphi)]^{-1}_{i1}$$

$$= Vt [R_z(-\theta+\varphi)]_{11} + (r_0)_i [R_z(\varphi)]_{i1}$$

$$= Vt\cos(-\theta+\varphi) + (r_0)_1 [R_z(\varphi)]_{11} + (r_0)_2 [R_z(\varphi)]_{21}$$

$$= Vt\cos(-\theta+\varphi) + (r_0)_1 \cos(\varphi) + (r_0)_2 \sin(\varphi)$$

$$\mathbf{r}'(t) \bullet \mathbf{e}'_2 = (r')'_2 = Vt [R_z(\theta-\varphi)]^{-1}_{1m} \delta_{m,2} + (r_0)_i [R_z(-\varphi)]^{-1}_{ij} \delta_{j,2} + b \delta_{2,2}$$

$$= Vt [R_z(\theta-\varphi)]^{-1}_{12} + (r_0)_i [R_z(-\varphi)]^{-1}_{i2} + b$$

$$= Vt [R_z(-\theta+\varphi)]_{12} + (r_0)_i [R_z(\varphi)]_{i2} + b$$

$$= Vt\sin(\theta-\varphi) + (r_0)_1 [R_z(\varphi)]_{12} + (r_0)_2 [R_z(\varphi)]_{22} + b$$

$$= Vt\sin(\theta-\varphi) - (r_0)_1 \sin\varphi + (r_0)_2 \cos\varphi + b$$